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CARLEMAN INTEGRAL OPERATORS AS MULTIPLICATION OPERATORS AND PERTURBATION THEORY

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Abstract. In this paper we introduce a multiplication operation that allows us to give to the Carleman integral operator of second class [3, 8] the form of a multiplication operator. Also we establish the formaly theory of perturbation of such operators.

1. Introduction

It is well known that the multiplication operators [1,2] possess a very rich structure theory, such that these operators played an important role in the study of operators on Hilbert Spaces.

In this paper, we introduce a multiplication operation that allows us to give to the Carleman integral operator of second class [3, 8] the form of a multiplication operator.

In what follows, we shall assume that the reader is familiar with the fundamental results and the standard notation of the Integral operators theory [8–12]. Let X be an arbitrary set, μ a $\sigma-$finite measure on X (μ is defined on a $\sigma-$algebra of subsets of X, we don't indicate this $\sigma-$algebra), $L_2(X,\mu)$ the Hilbert space of square integrable functions with respect to μ. Instead of writing "$\mu-$measurable", "$\mu-$almost everywhere" and "$(d\mu(x))$" we write "measurable", "a.e." and "dx".

A linear operator $A : D(A) \longrightarrow L_2(X,\mu)$, where the domain $D(A)$ is a dense linear manifold in $L_2(X,\mu)$, is said to be integral if there exists a measurable function K on $X \times X$, a kernel, such that, for every $f \in D(A)$,

$$Af(x) = \int_X K(x,y) f(y)\, dy \quad \text{a.e.} \tag{1.1}$$

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A kernel K on $X \times X$ is said to be Carleman if $K(x,y) \in L_2(X,\mu)$ for almost every fixed x, that is to say

$$\int_X |K(x,y)|^2 \, dy \ < \infty \ \text{ a.e.}$$

An integral operator A with a kernel K is called Carleman operator if K is a Carleman kernel. Every Carleman kernel K defines a Carleman function k from X to $L_2(X,\mu)$ by $k(x) = \overline{K(x,\cdot)}$ for all x in X for which $K(x,\cdot) \in L_2(X,\mu)$.

Now we consider the Carleman integral operator (1.1) of second class [3,8] generated by the following symmetric kernel

$$K(x,y) = \sum_{n=0}^{\infty} a_n \psi_n(x) \overline{\psi_n(y)},$$

where the overbar denotes the complex conjugation, $(\psi_n(x))_{n=0}^{\infty}$ is an orthonormal sequence in $L^2(X,\mu)$ such that

$$\sum_{n=0}^{\infty} |\psi_n(x)|^2 < \infty \text{ a.e.},$$

and $(a_n)_{n=0}^{\infty}$ is a real number sequence verifying

$$\sum_{n=0}^{\infty} a_n^2 |\psi_n(x)|^2 < \infty \text{ a.e.}$$

We call $(\psi_n(x))_{n=0}^{\infty}$ a Carleman sequence.

Moreover, we assume that there exists a numeric sequence $(\gamma_n)_{n=0}^{\infty}$ such that

$$\sum_{n=0}^{\infty} \gamma_n \psi_n(x) = 0 \ \text{ a.e.,} \tag{1.2}$$

and

$$\sum_{n=0}^{\infty} \left| \frac{\gamma_n}{a_n - \lambda} \right|^2 < \infty. \tag{1.3}$$

With the conditions (1.2) and (1.3), the symmetric operator $A = (A^*)^*$ admits the defect indices $(1,1)$ (see [3–6]), and its adjoint operator is given by

$$A^* f(x) = \sum_{n=0}^{\infty} a_n (f,\psi_n) \psi_n(x),$$
$$D(A^*) = \left\{ f \in L^2(X,\mu) : \sum_{n=0}^{\infty} a_n (f,\psi_n) \psi_n(x) \in L^2(X,\mu) \right\}.$$

Moreover, we have

$$\begin{cases} \varphi_\lambda(x) = \sum\limits_{n=0}^{\infty} \frac{\gamma_n}{a_n - \lambda} \psi_n(x) \in \mathfrak{N}_{\overline{\lambda}}, & \lambda \in \mathbb{C}, \lambda \neq a_n, n = 1,2,\dots, \\ \varphi_{a_n}(x) = \psi_n(x), \end{cases}$$

$\mathfrak{N}_{\overline{\lambda}}$ being the defect space associated to λ (see [3, 4]).

2. Position Operator

Let $\psi = (\psi_n)_{n=0}^{\infty}$ be a fixed Carleman sequence in $L^2(X, \mu)$. It is clear from the foregoing that ψ is not a complete sequence in $L^2(X, \mu)$. We set $\mathfrak{L}_\psi$ the closure of the linear span of the sequence $(\psi_n(x))_{n=0}^{\infty}$

$$\mathfrak{L}_\psi = \overline{\operatorname{span}\{\psi_n : n \in \mathbb{N}\}}.$$

We start this section by defining some formaly spaces.

2.1. **Formal Elements.**

Definition 2.1 ([7])**.** We call formal element any expression of the form

$$f = \sum_{n \in \mathbb{N}} a_n \psi_n, \tag{2.1}$$

where the coefficients a_n $(n \in \mathbb{N})$ are scalars.

The sequence $(a_n)_n$ is said to generate the formal element f.

Definition 2.2. We say that f is the zero formal element and we note $f = 0$ if $a_n = 0$ for all $n \in \mathbb{N}$.

We say that two formal elements $f = \sum_{n \in \mathbb{N}} a_n \psi_n$ and $g = \sum_{n \in \mathbb{N}} b_n \psi_n$ are equal if $a_n = b_n$ for all $n \in \mathbb{N}$.

If φ is a scalar function defined for each a_n, we set

$$\varphi\left(\sum_n a_n \psi_n\right) = \sum_n \varphi(a_n)\,\psi_n,$$

or in another form,

$$\varphi(a_1, a_2, \ldots, a_n, \ldots) = (\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n), \ldots).$$

For example let

$$\varphi(x) = \frac{1}{x}, \quad x \neq 0.$$

If $a_n \neq 0$ for all $n \in \mathbb{N}$, then the formal element

$$\varphi\left(\sum_n a_n \psi_n\right) = \sum_n \frac{1}{a_n} \psi_n$$

is called inverse of the formal element$f = \sum_n a_n \psi_n$.

Furthermore, we define the conjugate of a formal element f by

$$\overline{f} = \sum_n \overline{a_n} \psi_n.$$

Denotes by $\mathcal{F}_\psi$ the set of all formal elements (2.1). On $\mathcal{F}_\psi$, we define the following algebraic operations:

(a) the sum

$$\begin{array}{cccc} + & : & \mathcal{F}_\psi \times \mathcal{F}_\psi & \to & \mathcal{F}_\psi \\ & & (\sum\limits_n a_n\psi_n) + (\sum\limits_n b_n\psi_n) & = & \sum\limits_n (a_n + b_n)\,\psi_n \end{array},$$

(b) and the product

$$\begin{array}{cccc} \cdot & : & \mathbb{C} \times \mathcal{F}_\psi & \to & \mathcal{F}_\psi \\ & & \lambda \cdot (\sum\limits_n a_n\psi_n) & = & \sum\limits_n (\lambda \cdot a_n)\,\psi_n. \end{array}$$

Hence, we obtain a complex vector space structure for $\mathcal{F}_\psi$.

2.2. **Bounded Formal Elements.**

Definition 2.3. A formal element $f = \sum\limits_{n\in\mathbb{N}} a_n\psi_n$ is bounded if its sequence $(a_n)_n$ is bounded.

We denote by $\mathcal{B}_\psi$ the set of all bounded formal elements. It's clear that $\mathcal{B}_\psi$ is a subspace of $\mathcal{F}_\psi$. We claim that:

(a) $\mathcal{L}_\psi$ is a subspace of $\mathcal{B}_\psi$.
(b) Furthermore we have the strict inclusions:

$$\mathcal{L}_\psi \subset \mathcal{B}_\psi \subset \mathcal{F}_\psi.$$

We define a linear form $\langle\cdot,\cdot\rangle$ on $\mathcal{F}_\psi$ by setting

$$\left\langle \sum_n a_n\psi_n, \sum_n b_n\psi_n \right\rangle = \sum_n a_n\overline{b_n}, \tag{2.2}$$

with the series converging on the right side.

Proposition 2.1. *The form* (2.2) *verifies the properties of scalar product.*

Proof. Indeed, let

$$f = \sum_n a_n\psi_n, \quad g = \sum_n b_n\psi_n, \quad f_1 = \sum_n a_n^1\psi_n \quad \text{and } f_2 = \sum_n a_n^2\psi_n,$$

in $\mathcal{F}_\psi$.

We have then:

(a)

$$\langle f, g\rangle = \sum_n a_n\overline{b_n} = \overline{\sum_n b_n\overline{a_n}} = \overline{\langle g, f\rangle},$$

(b)

$$\begin{aligned}\langle \lambda f, g\rangle &= \left\langle \lambda\left(\sum_n a_n\psi_n\right), \sum_n a_n\psi_n \right\rangle = \left\langle \sum_n (\lambda a_n)\,\psi_n, \sum_n b_n\psi_n \right\rangle \\ &= \sum_n (\lambda a_n)\,\overline{b_n} = \lambda\langle \sum_n a_n\psi_n, \sum_n b_n\psi_n\rangle = \lambda\,\langle f, g\rangle\,,\end{aligned}$$

(c)

$$\begin{aligned}\langle f_1+f_2,g\rangle &= \left\langle \sum_n \left(a_n^1+a_n^2\right)\psi_n, \sum_n b_n\psi_n \right\rangle \\ &= \sum_n \left(a_n^1+a_n^2\right)\overline{b_n} = \sum_n a_n^1\overline{b_n} + \sum_n a_n^2\overline{b_n} = \langle f_1,g\rangle + \langle f_2,g\rangle\,,\end{aligned}$$

(d)

$$\langle f,f\rangle = \sum_n |a_n|^2 \geq 0 \text{ and } \langle f,f\rangle > 0, \quad \text{if } f\neq 0.$$

□

Remark 2.1. On $\mathcal{L}_\psi$, the scalar product $\langle .\,,.\rangle$ coincides with the scalar product $(.\,,.)$ of $L^2(X,\mu)$.

2.3. **The Multiplication Operation.** Here, we introduce the crucial tool of our work.

Definition 2.4. We call multiplication with respect to the Carleman sequence $(\psi_n)_n$, the operation denoted "$\circ$" and defined by

$$f\circ g = \sum_n \langle f,\psi_n\rangle \langle g,\psi_n\rangle \psi_n = \sum_n a_n b_n \psi_n, \quad \text{for all } (f,g)\in \mathcal{F}_\psi^2.$$

Definition 2.5. We call position operator in $\mathcal{L}_\psi$ any unitary selfadjoint operator satisfying

$$U(f\circ g) = Uf\circ Ug, \quad \text{for all } f,g\in\mathcal{L}_\psi.$$

The term "position operator" comes from the fact (as it will be shown in the following theorem) that for the elements of the sequence $\psi=(\psi_n)_n$, the operator U acts as operator of change of position of these elements.

2.4. **Main Results.**

Theorem 2.1. *A linear operator defined on $\mathcal{L}_\psi$ is a position operator if and only if there exists an involution j (i.e. $j^2=Id$)of the set $\mathbb{N}$ such that for all $n\in\mathbb{N}$*

$$U\psi_n = \psi_{j(n)}. \tag{2.3}$$

Proof.

(a) It is easy to see that if (2.3) holds, then U is a position operator.

(b) Let U be a position operator. According to Proposition 2.1 we can write:

$$U\psi_n = \sum_k \alpha_{n,k}\,\psi_k, \quad \text{with } \sum_k |\alpha_{n,k}|^2 = 1$$

since $U\psi_n\in\mathcal{L}_\psi$.

On the other hand, we have

$$\sum_k \alpha_{n,k}\,\psi_k = \sum_k \alpha_{n,k}^2\,\psi_k \tag{2.4}$$

as

$$U\psi_n = U\left(\psi_n \circ \psi_n\right) = U\psi_n \circ U\psi_n.$$

The equalities (2.4) lead to the resolution of the system:

$$\begin{cases} \sum_n \left|\alpha_{n,k}\right|^2 = 1, \\ \alpha_{n,k}^2 = \alpha_{n,k}, \end{cases} \qquad k \in \mathbb{N}. \tag{2.5}$$

We get then

$$\text{for all } n \in \mathbb{N}, \text{ there exists only one } k_n \in \mathbb{N} : \begin{cases} \alpha_{n,k_n} = 1, \\ \alpha_{n,k} = 0, \end{cases} \qquad \text{for all } k \neq k_n.$$

Let us now consider the following application

$$\begin{array}{cccc} j & : & \mathbb{N} & \to & \mathbb{N}, \\ & & n & \mapsto & j\left(n\right) = k_n. \end{array}$$

It's clear that j is injective.

Now let $m \in \mathbb{N}$. Since $U^2 = I$, then

$$U\left(U\psi_m\right) = U\psi_{j(m)} = \psi_{j(j(m))} = \psi_m.$$

Hence

$$j(j(m)) = m.$$

Finally j is well defined as involution. □

Remark 2.2.

(a) We emphasize that involution j depends of the operator U, i.e. $j = j_U$. We then write

$$U\psi_n = \psi_{j(n)} = \psi_{j_U(n)}$$

and

$$Uf = U\left(\sum_n a_n \psi_n\right) = \sum_n a_n \psi_{j(n)} = f_U.$$

(b) The position operator U can be extended over $\mathcal{F}_\psi$ as follows. If $f = \sum_n a_n \psi_n \in \mathcal{F}_\psi$, then

$$Uf = f_U = \sum_n a_n \psi_{j_U(n)}.$$

3. Carleman Operator in $\mathcal{F}_\psi$

3.1. **Case of Defect Indices $(1,1)$.** Let $\alpha = \sum_n \alpha_p \psi_p \in \mathcal{F}_\psi$, we introduce the operator $\overset{\circ}{A}_\alpha$ defined in $\mathcal{L}_\psi$ by

$$\overset{\circ}{A}_\alpha f = \alpha \circ f = \sum_n \left\langle \alpha, \psi_n\right\rangle \left\langle f, \psi_n\right\rangle \psi_n.$$

It is clear that $\overset{\circ}{A}_\alpha$ is a Carleman operator induced by the kernel

$$K(x,y) = \sum_n \alpha_n \psi_n(x) \overline{\psi_n(y)},$$

with domain

$$D(\overset{\circ}{A}_\alpha) = \left\{ f \in \mathcal{L}_\psi : \sum_n |\alpha_n (f, \psi_n)|^2 < \infty \right\}.$$

Moreover, if $\alpha = \overline{\alpha}$, then $\overset{\circ}{A}_\alpha$ is selfadjoint.

Now let $\Theta = \sum_n \gamma_n \psi_n \in \mathcal{F}_\psi$ and $\Theta \notin \mathcal{L}_\psi$ (i.e., $\sum_n |\gamma_n|^2 = \infty$). We introduce the following set

$$\mathcal{H}_\Theta = \{ f + \mu\Theta : f \in \mathcal{L}_\psi, \mu \in \mathbb{C} \}, \tag{3.1}$$

which verify the following properties.

Proposition 3.1.

(a) $\mathcal{H}_\Theta$ *is a subset of* $\mathcal{F}_\psi$.
(b) $\mathcal{H}_\theta = \mathcal{L}_\psi \oplus \mathbb{C}\theta$, *i.e. direct sum of* $\mathcal{L}_\psi$ *with* $\mathbb{C}\theta = \{\mu\theta : \mu \in \mathbb{C}\}$.

Proof. The first property is easy to establish. We show the uniqueness for the second.

Let $g_1 = f_1 + \mu_1\theta$ and $g_2 = f_2 + \mu_2\theta$ two formal elements in $\mathcal{H}_\theta$. Then

$$g_1 = g_2 \Leftrightarrow f_1 - f_2 = (\mu_2 - \mu_1)\,\theta.$$

This last equality is verified only if $\mu_2 = \mu_1$. Therefore $f_1 = f_2$. □

Denote by Q the projector of $\mathcal{H}_\Theta$ on $\mathcal{L}_\psi$, that is to say: if $g \in \mathcal{H}_\Theta$, $g = f + \mu\Theta$ with $f \in \mathcal{L}_\psi$ and $\mu \in \mathbb{C}$ then $Qg = f$.

We define the operator B_α by

$$B_\alpha f = Q(\alpha \circ f), \quad f \in \mathcal{L}_\psi.$$

It is clear that,

$$D(B_\alpha) = \{ f \in \mathcal{L}_\psi : \alpha \circ f \in \mathcal{H}_\Theta \}.$$

Theorem 3.1. B_α *is a densely defined and closed operator.*

Proof.

(a) Since

$$\operatorname{span}\{\psi_n : n \in \mathbb{N}\} \subset D(B_\alpha)$$

and that $(\psi_n)_n$is complete in $\mathcal{L}_\psi$, then

$$\overline{D(B_\alpha)} = \mathcal{L}_\psi.$$

(b) Let $(f_n)_n$, be a sequence of elements in $D(B_\alpha)$. Checking

$$\begin{cases} f_n & \to & f, \\ B_\alpha f_n & \to & g, \end{cases} \quad \text{(convergence in the } L^2 \text{ sense).}$$

We have then

$$B_\alpha f_n = Q(\alpha \circ f_n),$$

with

$$\alpha \circ f_n = g_n + \mu\Theta, g_n \in \mathcal{L}_\psi.$$

Then

$$g_n = \alpha \circ f_n - \mu_n\Theta \in \mathcal{L}_\psi,$$

This implies that:

$$\langle g_n, \psi_m\rangle = \alpha_m \langle f_n, \psi_m\rangle - \mu_n\gamma_m\psi_m, \quad \text{for all } m \in \mathbb{N}.$$

Or, when n tends to ∞, we have:

$$g_n \to g \quad \text{and} \quad f_n \to f.$$

Therefore, there exist $\mu \in \mathbb{C}$ such that:

$$\lim_{n\to\infty} \mu_n = \mu.$$

And as Q is a closed operator, then we can write

$$\alpha \circ f \in \mathcal{H}_\Theta \quad \text{and} \quad g = Q(\alpha \circ f).$$

Finally $f \in D(B_\alpha)$ and $g = B_\alpha f$. ☐

It follows from this theorem that the adjoint operator B_α^* exists and $B_\alpha^{**} = B_\alpha$. Let us denote by A_α, the operator adjoint of B_α

$$A_\alpha = B_\alpha^*.$$

In the case $\alpha = \overline{\alpha}$, the operator A_α is symmetric and we have the following results.

Theorem 3.2. *A_α admits defect indices $(1,1)$ if and only if*

$$\varphi_\lambda = (\alpha - \lambda)^{-1} \circ \Theta \in \mathcal{L}_\psi.$$

In this case $\varphi_\lambda \in \mathcal{N}_{\overline{\lambda}}$ (defect space associated with λ, [3]).

Proof. We know (see [3]) that A_α has the defect indices $(1,1)$ if and only if, its defect subspaces $\mathcal{N}_{\overline{\lambda}}$ and $\mathcal{N}_\lambda$ are unidimensional.

We have

$$\mathcal{N}_{\overline{\lambda}} = \ker\left(A_\alpha^* - \lambda I\right) = \ker\left(B_\alpha - \lambda I\right).$$

So it suffices to solve the system

$$\begin{cases} B_\alpha \varphi_\lambda = \lambda \varphi_\lambda, \\ \varphi_\lambda \in \mathcal{L}_\psi, \end{cases}$$

i.e.,

$$\begin{aligned} \begin{cases} Q\left(\alpha \circ \varphi_\lambda\right) = \lambda \varphi_\lambda, \\ \varphi_\lambda \in \mathcal{L}_\psi, \end{cases} \quad &\Longleftrightarrow \quad \begin{cases} \left(\alpha \circ \varphi_\lambda\right) = \lambda \varphi_\lambda + \mu \Theta, \mu \in \mathbb{C}, \\ \varphi_\lambda \in \mathcal{L}_\psi, \end{cases} \\ &\Longleftrightarrow \quad \begin{cases} \left(\alpha - \lambda\right) \circ \varphi_\lambda = \Theta, \\ \varphi_\lambda \in \mathcal{L}_\psi, \end{cases} \\ &\Longleftrightarrow \quad \begin{cases} \varphi_\lambda = \left(\alpha - \lambda\right)^{-1} \circ \Theta, \\ \varphi_\lambda \in \mathcal{L}_\psi. \end{cases} \end{aligned}$$

□

3.2. **Case of Defect Indices $(\mathbf{m}, \mathbf{m})$.** In this section we give the generalization for the case of defect indices (m, m), where $m > 1$.

Let $\Theta_1, \Theta_2, \ldots, \Theta_m$, (where $m \in \mathbb{N}$) formal elements not belonging to $\mathcal{L}_\psi$ and let

$$\mathcal{H}_\Theta = \left\{ f + \sum_{k=1}^{m} \mu_k \Theta_k, \quad f \in \mathcal{L}_\psi, \mu_k \in \mathbb{C}, \quad k = 1, \ldots, m \right\}.$$

We consider the operator B_α defined by

$$\begin{aligned} B_\alpha f &= Q\left(\alpha \circ f\right), \quad \text{for } f \in D\left(B_\alpha\right), \\ D\left(B_\alpha\right) &= \left\{ f \in \mathcal{L}_\psi : \alpha \circ f \in \mathcal{H}_\Theta \right\}. \end{aligned}$$

We assume that $\alpha = \overline{\alpha}$ and we set

$$A_\alpha = B_\alpha^*.$$

By analogy to the case of defect indices $(1,1)$ we also have the following.

Theorem 3.3. *The operator B_α is densely defined and closed.*

Theorem 3.4. *The operator A_α admits defect indices (m, m) if and only if*

$$\varphi_\lambda^{(k)} = \left(\alpha - \lambda\right) \circ \Theta_k \in \mathcal{L}_\psi, \quad k = 1, \ldots, m.$$

In this case the functions $\varphi_\lambda^{(k)} (k = 1, \ldots, m)$ are linearly independent and generate the defect space $\mathcal{N}_{\overline{\lambda}}$.

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