CARLEMAN INTEGRAL OPERATORS AS MULTIPLICATION OPERATORS AND PERTURBATION THEORY

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ABSTRACT. In this paper we introduce a multiplication operation that allows us to give to the Carleman integral operator of second class [3,8] the form of a multiplication operator. Also we establish the formal theory of perturbation of such operators.

1. INTRODUCTION

It is well known that the multiplication operators [1,2] possess a very rich structure theory, such that these operators played an important role in the study of operators on Hilbert Spaces.

In this paper, we introduce a multiplication operation that allows us to give to the Carleman integral operator of second class [3,8] the form of a multiplication operator.

In what follows, we shall assume that the reader is familiar with the fundamental results and the standard notation of the Integral operators theory [8–12]. Let \( X \) be an arbitrary set, \( \mu \) a \( \sigma \)-finite measure on \( X \) (\( \mu \) is defined on a \( \sigma \)-algebra of subsets of \( X \), we don’t indicate this \( \sigma \)-algebra), \( L_2 (X, \mu) \) the Hilbert space of square integrable functions with respect to \( \mu \). Instead of writing “\( \mu \)-measurable”, “\( \mu \)-almost everywhere” and “\( (d\mu(x)) \)” we write “measurable”, “a.e.” and “\( dx \)”.

A linear operator \( A : D(A) \rightarrow L_2 (X, \mu) \), where the domain \( D(A) \) is a dense linear manifold in \( L_2 (X, \mu) \), is said to be integral if there exists a measurable function \( K \) on \( X \times X \), a kernel, such that, for every \( f \in D(A) \),

\[
Af(x) = \int_X K(x,y) f(y) \, dy \quad \text{a.e.}
\]
A kernel $K$ on $X \times X$ is said to be Carleman if $K(x, y) \in L_2(X, \mu)$ for almost every fixed $x$, that is to say
\[ \int_X |K(x, y)|^2 \, dy < \infty \text{ a.e.} \]

An integral operator $A$ with a kernel $K$ is called Carleman operator if $K$ is a Carleman kernel. Every Carleman kernel $K$ defines a Carleman function $k$ from $X$ to $L_2(X, \mu)$ by
\[ k(x) = K(x, \cdot) \text{ for all } x \text{ in } X \text{ for which } K(x, \cdot) \in L_2(X, \mu). \]

Now we consider the Carleman integral operator (1.1) of second class [3,8] generated by the following symmetric kernel
\[ K(x, y) = \sum_{n=0}^{\infty} a_n \psi_n(x) \overline{\psi_n(y)}, \]

where the overbar denotes the complex conjugation, $(\psi_n(x))_{n=0}^{\infty}$ is an orthonormal sequence in $L^2(X, \mu)$ such that
\[ \sum_{n=0}^{\infty} |\psi_n(x)|^2 < \infty \text{ a.e.}, \]

and $(a_n)_{n=0}^{\infty}$ is a real number sequence verifying
\[ \sum_{n=0}^{\infty} a_n^2 |\psi_n(x)|^2 < \infty \text{ a.e.}. \]

We call $(\psi_n(x))_{n=0}^{\infty}$ a Carleman sequence.

Moreover, we assume that there exists a numeric sequence $(\gamma_n)_{n=0}^{\infty}$ such that
\[ \sum_{n=0}^{\infty} \gamma_n \psi_n(x) = 0 \text{ a.e.,} \]

and
\[ \sum_{n=0}^{\infty} \left| \frac{\gamma_n}{a_n - \lambda} \right|^2 < \infty. \]

With the conditions (1.2) and (1.3), the symmetric operator $A = (A^*)^*$ admits the defect indices (1, 1) (see [3–6]), and its adjoint operator is given by
\[ A^*f(x) = \sum_{n=0}^{\infty} a_n (f, \psi_n) \psi_n(x), \]
\[ D(A^*) = \left\{ f \in L^2(X, \mu) : \sum_{n=0}^{\infty} a_n (f, \psi_n) \psi_n(x) \in L^2(X, \mu) \right\}. \]

Moreover, we have
\[ \varphi_\lambda(x) = \sum_{n=0}^{\infty} \frac{\gamma_n}{a_n - \lambda} \psi_n(x) \in \mathcal{M}_X, \quad \lambda \in \mathbb{C}, \lambda \neq a_n, n = 1, 2, \ldots, \]
\[ \varphi_{a_n}(x) = \psi_n(x), \]
\( M_{\lambda} \) being the defect space associated to \( \lambda \) (see [3, 4]).

2. Position Operator

Let \( \psi = (\psi_n)_{n=0}^{\infty} \) be a fixed Carleman sequence in \( L^2 (X, \mu) \). It is clear from the foregoing that \( \psi \) is not a complete sequence in \( L^2 (X, \mu) \). We set \( \mathcal{L}_\psi \) the closure of the linear span of the sequence \( (\psi_n (x))_{n=0}^{\infty} \)

\[ \mathcal{L}_\psi = \text{span} \{ \psi_n : n \in \mathbb{N} \} . \]

We start this section by defining some formal spaces.

2.1. Formal Elements.

**Definition 2.1 ([7]).** We call formal element any expression of the form

\[ f = \sum_{n \in \mathbb{N}} a_n \psi_n, \]

where the coefficients \( a_n (n \in \mathbb{N}) \) are scalars.

The sequence \( (a_n)_n \) is said to generate the formal element \( f \).

**Definition 2.2.** We say that \( f \) is the zero formal element and we note \( f = 0 \) if \( a_n = 0 \) for all \( n \in \mathbb{N} \).

We say that two formal elements \( f = \sum_{n \in \mathbb{N}} a_n \psi_n \) and \( g = \sum_{n \in \mathbb{N}} b_n \psi_n \) are equal if \( a_n = b_n \) for all \( n \in \mathbb{N} \).

If \( \varphi \) is a scalar function defined for each \( a_n \), we set

\[ \varphi \left( \sum_n a_n \psi_n \right) = \sum_n \varphi (a_n) \psi_n, \]

or in another form,

\[ \varphi (a_1, a_2, \ldots, a_n, \ldots) = (\varphi (a_1), \varphi (a_2), \ldots, \varphi (a_n), \ldots). \]

For example let

\[ \varphi (x) = \frac{1}{x}, \quad x \neq 0. \]

If \( a_n \neq 0 \) for all \( n \in \mathbb{N} \), then the formal element

\[ \varphi \left( \sum_n a_n \psi_n \right) = \sum_n \frac{1}{a_n} \psi_n \]

is called inverse of the formal element \( f = \sum_n a_n \psi_n \).

Furthermore, we define the conjugate of a formal element \( f \) by

\[ \bar{f} = \sum_n \overline{a_n} \psi_n. \]

Denotes by \( \mathcal{F}_\psi \) the set of all formal elements (2.1). On \( \mathcal{F}_\psi \), we define the following algebraic operations:
(a) the sum
\[ F \psi \times F \psi \rightarrow F \psi \]
\[ (\sum_n a_n \psi_n) + (\sum_n b_n \psi_n) = \sum_n (a_n + b_n) \psi_n, \]
(b) and the product
\[ \cdot : \mathbb{C} \times F \psi \rightarrow F \psi \]
\[ \lambda \cdot (\sum_n a_n \psi_n) = \sum_n (\lambda \cdot a_n) \psi_n. \]
Hence, we obtain a complex vector space structure for \( F \psi \).

2.2. Bounded Formal Elements.

**Definition 2.3.** A formal element \( f = \sum_{n \in \mathbb{N}} a_n \psi_n \) is bounded if its sequence \((a_n)\) is bounded.

We denote by \( \mathcal{B}_\psi \) the set of all bounded formal elements. It’s clear that \( \mathcal{B}_\psi \) is a subspace of \( F \psi \). We claim that:

(a) \( \mathcal{L}_\psi \) is a subspace of \( \mathcal{B}_\psi \).

(b) Furthermore we have the strict inclusions:
\[ \mathcal{L}_\psi \subset \mathcal{B}_\psi \subset F \psi. \]

We define a linear form \( \langle \cdot , \cdot \rangle \) on \( F \psi \) by setting
\[ \langle \sum_n a_n \psi_n, \sum_n b_n \psi_n \rangle = \sum_n a_n \overline{b_n}, \]
with the series converging on the right side.

**Proposition 2.1.** The form (2.2) verifies the properties of scalar product.

**Proof.** Indeed, let
\[ f = \sum_n a_n \psi_n, \quad g = \sum_n b_n \psi_n, \quad f_1 = \sum_n a_n^1 \psi_n \quad \text{and} \quad f_2 = \sum_n a_n^2 \psi_n, \]
in \( F \psi \).
We have then:

(a)
\[ \langle f, g \rangle = \sum_n a_n \overline{b_n} = \sum_n b_n \overline{a_n} = \langle g, f \rangle, \]

(b)
\[ \langle \lambda f, g \rangle = \lambda \langle \sum_n a_n \psi_n, \sum_n \psi_n \rangle = \lambda \langle \sum_n (\lambda a_n) \psi_n, \sum_n \psi_n \rangle \\
= \sum_n (\lambda a_n) \overline{b_n} = \lambda \langle \sum_n a_n \psi_n, \sum_n b_n \psi_n \rangle = \lambda \langle f, g \rangle, \]
(c) \[
\langle f_1 + f_2, g \rangle = \left\langle \sum_n (a_n^1 + a_n^2) \psi_n, \sum_n b_n \psi_n \right\rangle \\
= \sum_n (a_n^1 + a_n^2) \overline{b_n} = \sum_n a_n^1 \overline{b_n} + \sum_n a_n^2 \overline{b_n} = \langle f_1, g \rangle + \langle f_2, g \rangle,
\]

(d) \[
\langle f, f \rangle = \sum_n |a_n|^2 \geq 0 \text{ and } \langle f, f \rangle > 0, \quad \text{if } f \neq 0.
\]

□

Remark 2.1. On \( L_\psi \), the scalar product \( \langle ., . \rangle \) coincides with the scalar product \( (., .) \) of \( L^2(X, \mu) \).

2.3. The Multiplication Operation. Here, we introduce the crucial tool of our work.

Definition 2.4. We call multiplication with respect to the Carleman sequence \( (\psi_n)_n \), the operation denoted “\( \circ \)” and defined by
\[
f \circ g = \sum_n \langle f, \psi_n \rangle \langle g, \psi_n \rangle \psi_n = \sum_n a_n b_n \psi_n, \quad \text{for all } (f, g) \in F^2_\psi.
\]

Definition 2.5. We call position operator in \( L_\psi \) any unitary selfadjoint operator satisfying
\[
U (f \circ g) = Uf \circ Ug, \quad \text{for all } f, g \in L_\psi.
\]

The term “position operator” comes from the fact (as it will be shown in the following theorem) that for the elements of the sequence \( \psi = (\psi_n)_n \), the operator \( U \) acts as operator of change of position of these elements.

2.4. Main Results.

Theorem 2.1. A linear operator defined on \( L_\psi \) is a position operator if and only if there exists an involution \( j \) (i.e. \( j^2 = Id \)) of the set \( N \) such that for all \( n \in N \)
\[
U \psi_n = \psi_{j(n)}.
\]

Proof.

(a) It is easy to see that if (2.3) holds, then \( U \) is a position operator.

(b) Let \( U \) be a position operator. According to Proposition 2.1 we can write:
\[
U \psi_n = \sum_k \alpha_{n,k} \psi_k, \quad \text{with } \sum_k |\alpha_{n,k}|^2 = 1
\]
since \( U \psi_n \in L_\psi \).

On the other hand, we have
\[
\sum_k \alpha_{n,k} \psi_k = \sum_k \alpha_{n,k}^2 \psi_k
\]
as
\[ U\psi_n = U(\psi_n \circ \psi_n) = U\psi_n \circ U\psi_n. \]

The equalities (2.4) lead to the resolution of the system:
\[
\begin{align*}
\sum_n |\alpha_{n,k}|^2 &= 1, \\
\alpha_{n,k}^2 &= \alpha_{n,k},
\end{align*}
\]
(2.5)
for all \( n \in \mathbb{N} \), there exists only one \( k_n \in \mathbb{N} \): \[
\begin{align*}
\alpha_{n,k_n} &= 1, \\
\alpha_{n,k} &= 0, \text{ for all } k \neq k_n.
\end{align*}
\]

Let us now consider the following application
\[ j : \mathbb{N} \to \mathbb{N}, \quad n \mapsto j(n) = k_n. \]

It’s clear that \( j \) is injective.

Now let \( m \in \mathbb{N} \). Since \( U^2 = I \), then
\[ U(U\psi_m) = U\psi_{j(m)} = \psi_{j(j(m))} = \psi_m. \]

Hence
\[ j(j(m)) = m. \]

Finally \( j \) is well defined as involution. \( \square \)

**Remark 2.2.**

(a) We emphasize that involution \( j \) depends of the operator \( U \), i.e. \( j = j_U \). We then write
\[ U\psi_n = \psi_{j(n)} = \psi_{j_U(n)} \]
and
\[ Uf = U \left( \sum_n a_n \psi_n \right) = \sum_n a_n \psi_{j(n)} = f_U. \]

(b) The position operator \( U \) can be extended over \( \mathcal{F}_\psi \) as follows. If \( f = \sum_n a_n \psi_n \in \mathcal{F}_\psi \), then
\[ Uf = f_U = \sum_n a_n \psi_{j_U(n)}. \]

3. **Carleman Operator in \( \mathcal{F}_\psi \)**

3.1. **Case of Defect Indices \((1,1)\).** Let \( \alpha = \sum_n \alpha_p \psi_p \in \mathcal{F}_\psi \), we introduce the operator \( \hat{A}_\alpha \) defined in \( \mathcal{L}_\psi \) by
\[ \hat{A}_\alpha f = \alpha \circ f = \sum_n \langle \alpha, \psi_n \rangle \langle f, \psi_n \rangle \psi_n. \]
It is clear that $\hat{A}_\alpha$ is a Carleman operator induced by the kernel
$$K(x, y) = \sum_n \alpha_n \psi_n(x) \overline{\psi_n(y)},$$
with domain
$$D(\hat{A}_\alpha) = \left\{ f \in \mathcal{L}_\psi : \sum_n |\alpha_n (f, \psi_n)|^2 < \infty \right\}.$$ 

Moreover, if $\alpha = \overline{\pi}$, then $\hat{A}_\alpha$ is selfadjoint.

Now let $\Theta = \sum_n \gamma_n \psi_n \in \mathcal{F}_\psi$ and $\Theta \notin \mathcal{L}_\psi$ (i.e., $\sum_n |\gamma_n|^2 = \infty$). We introduce the following set
\begin{equation}
(3.1) \quad \mathcal{H}_\Theta = \left\{ f + \mu \Theta : f \in \mathcal{L}_\psi, \mu \in \mathbb{C} \right\},
\end{equation}
which verify the following properties.

**Proposition 3.1.**
\begin{enumerate}
\item $\mathcal{H}_\Theta$ is a subset of $\mathcal{F}_\psi$.
\item $\mathcal{H}_\Theta = \mathcal{L}_\psi \oplus \mathbb{C}\Theta$, i.e. direct sum of $\mathcal{L}_\psi$ with $\mathbb{C}\Theta = \{ \mu \Theta : \mu \in \mathbb{C} \}$.
\end{enumerate}

**Proof.** The first property is easy to establish. We show the uniqueness for the second.

Let $g_1 = f_1 + \mu_1 \Theta$ and $g_2 = f_2 + \mu_2 \Theta$ two formal elements in $\mathcal{H}_\Theta$. Then
$$g_1 = g_2 \Leftrightarrow f_1 - f_2 = (\mu_2 - \mu_1) \Theta.$$ 

This last equality is verified only if $\mu_2 = \mu_1$. Therefore $f_1 = f_2$. \hfill \Box

Denote by $Q$ the projector of $\mathcal{H}_\Theta$ on $\mathcal{L}_\psi$, that is to say: if $g \in \mathcal{H}_\Theta$, $g = f + \mu \Theta$ with $f \in \mathcal{L}_\psi$ and $\mu \in \mathbb{C}$ then $Qg = f$.

We define the operator $B_\alpha$ by
$$B_\alpha f = Q(\alpha \circ f), \quad f \in \mathcal{L}_\psi.$$ 

It is clear that,
$$D(B_\alpha) = \{ f \in \mathcal{L}_\psi : \alpha \circ f \in \mathcal{H}_\Theta \}.$$ 

**Theorem 3.1.** $B_\alpha$ is a densely defined and closed operator.
Proof.
(a) Since \( \text{span} \{ \psi_n : n \in \mathbb{N} \} \subset D(B_\alpha) \)
and that \((\psi_n)_n\) is complete in \(L_\psi\), then
\[
D(B_\alpha) = L_\psi.
\]
(b) Let \((f_n)_n\) be a sequence of elements in \(D(B_\alpha)\). Checking
\[
\begin{align*}
  f_n &\to f, \\
  B_\alpha f_n &\to g,
\end{align*}
\]
(convergence in the \(L^2\) sense).

We have then
\[
B_\alpha f_n = Q(\alpha \circ f_n),
\]
with
\[
\alpha \circ f_n = g_n + \mu \Theta, g_n \in L_\psi.
\]

Then
\[
g_n = \alpha \circ f_n - \mu_n \Theta \in L_\psi,
\]
This implies that:
\[
\langle g_n, \psi_m \rangle = \alpha_m \langle f_n, \psi_m \rangle - \mu_n \gamma_m \psi_m, \quad \text{for all } m \in \mathbb{N}.
\]
Or, when \(n\) tends to \(\infty\), we have:
\[
g_n \to g \quad \text{and} \quad f_n \to f.
\]
Therefore, there exist \(\mu \in \mathbb{C}\) such that:
\[
\lim_{n \to \infty} \mu_n = \mu.
\]
And as \(Q\) is a closed operator, then we can write
\[
\alpha \circ f \in \mathcal{H}_\Theta \quad \text{and} \quad g = Q(\alpha \circ f).
\]
Finally \(f \in D(B_\alpha)\) and \(g = B_\alpha f\). \(\square\)

It follows from this theorem that the adjoint operator \(B_\alpha^*\) exists and \(B_\alpha^{**} = B_\alpha\).

Let us denote by \(A_\alpha\), the operator adjoint of \(B_\alpha\)
\[
A_\alpha = B_\alpha^*.
\]

In the case \(\alpha = \overline{\alpha}\), the operator \(A_\alpha\) is symmetric and we have the following results.

**Theorem 3.2.** \(A_\alpha\) admits defect indices \((1, 1)\) if and only if
\[
\varphi_\lambda = (\alpha - \lambda)^{-1} \circ \Theta \in L_\psi.
\]
In this case \(\varphi_\lambda \in \mathcal{N}_\mathcal{F}\) (defect space associated with \(\lambda\), [3]).
**Proof.** We know (see [3]) that $A_\alpha$ has the defect indices $(1,1)$ if and only if, its defect subspaces $N_\infty$ and $N_\lambda$ are unidimensional.

We have

$$N_\infty = \ker (A_\alpha^* - \lambda I) = \ker (B_\alpha - \lambda I).$$

So it suffices to solve the system

$$\begin{cases}
B_\alpha \varphi_\lambda = \lambda \varphi_\lambda, \\
\varphi_\lambda \in \mathcal{L}_\psi,
\end{cases}$$

i.e.,

$$\begin{cases}
Q (\alpha \circ \varphi_\lambda) = \lambda \varphi_\lambda, \\
\varphi_\lambda \in \mathcal{L}_\psi,
\end{cases} \iff \begin{cases}
(\alpha \circ \varphi_\lambda) = \lambda \varphi_\lambda + \mu \Theta, \mu \in \mathbb{C}, \\
\varphi_\lambda \in \mathcal{L}_\psi,
\end{cases}$$

$$\iff \begin{cases}
(\alpha - \lambda) \circ \varphi_\lambda = \Theta, \\
\varphi_\lambda \in \mathcal{L}_\psi,
\end{cases} \iff \begin{cases}
\varphi_\lambda = (\alpha - \lambda)^{-1} \circ \Theta, \\
\varphi_\lambda \in \mathcal{L}_\psi.
\end{cases}$$

3.2. **Case of Defect Indices** $(m,m)$. In this section we give the generalization for the case of defect indices $(m,m)$, where $m > 1$.

Let $\Theta_1, \Theta_2, \ldots, \Theta_m$, (where $m \in \mathbb{N}$) formal elements not belonging to $\mathcal{L}_\psi$ and let

$$\mathcal{H}_\Theta = \left\{ f + \sum_{k=1}^{m} \mu_k \Theta_k, \ f \in \mathcal{L}_\psi, \mu_k \in \mathbb{C}, \ k = 1, \ldots, m \right\}.$$  

We consider the operator $B_\alpha$ defined by

$$B_\alpha f = Q (\alpha \circ f), \quad \text{for } f \in D (B_\alpha),$$

$$D (B_\alpha) = \left\{ f \in \mathcal{L}_\psi : \alpha \circ f \in \mathcal{H}_\Theta \right\}.$$  

We assume that $\alpha = \overline{\alpha}$ and we set

$$A_\alpha = B_\alpha^*.$$  

By analogy to the case of defect indices $(1,1)$ we also have the following.

**Theorem 3.3.** The operator $B_\alpha$ is densely defined and closed.

**Theorem 3.4.** The operator $A_\alpha$ admits defect indices $(m,m)$ if and only if

$$\varphi^{(k)}_\lambda = (\alpha - \lambda) \circ \Theta_k \in \mathcal{L}_\psi, \quad k = 1, \ldots, m.$$  

In this case the functions $\varphi^{(k)}_\lambda (k = 1, \ldots, m)$ are linearly independent and generate the defect space $N_\infty$.
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