SUPER MEAN LABELING OF SOME SUBDIVISION GRAPHS

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Abstract. Let $G$ be a graph and $f : V(G) \rightarrow \{1, 2, 3, \ldots, p + q\}$ be an injection. For each edge $e = uv$, the induced edge labeling $f^*$ is defined as follows:

$$f^*(e) = \begin{cases} f(u) + f(v), & \text{if } f(u) + f(v) \text{ is even,} \\ \frac{f(u) + f(v)}{2}, & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

Then $f$ is called super mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p + q\}$. A graph that admits a super mean labeling is called super mean graph. In this paper, we have studied the super meanness property of the subdivision of the $H$-graph $H_n$, $H_n \circ K_1$, $H_n \circ S_2$, slanting ladder, $T_n \circ K_1$, $C_n \circ K_1$ and $C_n \circ C_m$.

1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with $p$ vertices and $q$ edges. For notations and terminology we follow [2].

The path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$. A triangular snake is obtained from a path by identifying each edge of the path with an edge of the cycle $C_3$. The graph $C_m \circ C_n$ is obtained by identifying an edge of $C_m$ with an edge of $C_n$. The slanting ladder $SL_n$ is a graph obtained from two paths $u_1u_2 \ldots u_n$ and $v_1v_2 \ldots v_n$ by joining each $u_i$ with $v_{i+1}$, $1 \leq i \leq n - 1$. The $H$-graph of a path $P_n$, denoted by $H_n$ is the graph obtained from two copies of $P_n$ with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{n+1}$ and $u_{n+1}$ if $n$ is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if $n$ is even. The corona of a graph $G$ on $p$ vertices $v_1, v_2, \ldots, v_p$ is the graph obtained from $G$ by adding $p$ new vertices $u_1, u_2, \ldots, u_p$.
and the new edges $u_i v_i$ for $1 \leq i \leq p$. The corona of $G$ is denoted by $G \odot K_1$. The 2-corona of a graph $G$, denoted by $G \odot S_2$ is a graph obtained from $G$ by identifying the center vertex of the star $S_2$ at each vertex of $G$. A graph which can be obtained from a given graph by breaking up each edge into one or more segments by inserting intermediate vertices between its two ends. If each edge of a graph $G$ is broken into two by exactly one vertex, then the resultant graph is taken as $S(G)$.

A vertex labeling of $G$ is an assignment $f : V(G) \rightarrow \{1, 2, \ldots, p+q\}$ be an injection. For a vertex labeling $f$, the induced edge labeling $f^*(e = uv)$ is defined by

$$f^*(e) = \begin{cases} \frac{f(u) + f(v)}{2}, & \text{if } f(u) + f(v) \text{ is even,} \\ \frac{f(u) + f(v) + 1}{2}, & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

Then $f$ is called super mean labeling if

$$f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p + q\}.$$ 

Clearly $f^*$ is injective. A graph that admits a super mean labeling is called super mean graph.

A super mean labeling of the graph $P_7^2$ is shown in Figure 1.

![Figure 1](image)

The concept of mean labeling was introduced and studied by S. Somasundaram and R. Ponraj [5]. Some new families of mean graphs are discussed in [10, 11].

The concept of super mean labeling was introduced and studied by D. Ramya et al. [4]. Further some more results on super mean graphs are discussed in [1, 3, 6–9].

In this paper, we have studied the super meanness of the subdivision of the graphs $H$-graph $H_n$, $H_n \odot K_1$, $H_n \odot S_2$, slanting ladder, $T_n \odot K_1$, $C_n \odot K_1$ and $C_n @ C_m$.

### 2. Super Mean Graphs

**Theorem 2.1.** The graph $S(H_n)$ is a super mean graph, for $n \geq 3$.

**Proof.** Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the paths of length $n - 1$. Each edge $u_i u_{i+1}$ is subdivided by a vertex $x_i$, $1 \leq i \leq n - 1$ and each edge $v_i v_{i+1}$ is subdivided by a vertex $y_i$, $1 \leq i \leq n - 1$. The edge $u_{n+1} v_{n+1}$ is divided by a vertex $z$ when $n$ is odd. The edge $u_{n+2} v_n$ is divided by a vertex $z$ when $n$ is even. The graph $S(H_n)$ has $4n - 1$ vertices and $4n - 2$ edges.

Define $f : V(S(H_n)) \rightarrow \{1, 2, 3, \ldots, p + q = 8n - 3\}$ as follows:

$$f(u_i) = 4i - 3, \quad 1 \leq i \leq n,$$
Theorem 2.2. The graph \( S(H_n \odot K_1) \) is a super mean graph, for \( n \geq 3 \).

Proof. Let \( u_1, u_2, \ldots, u_n \) and \( v_1, v_2, \ldots, v_n \) be the vertices of the paths of length \( n - 1 \). Let \( a_1, a_2, u_i \) be the path attached at each \( u_i, 1 \leq i \leq n \) and \( b_1, b_2, v_i \) be the path attached at each \( v_i, 1 \leq i \leq n \). Each edge \( u_i u_{i+1} \) is subdivided by a vertex \( x_i \), \( 1 \leq i \leq n - 1 \) and each edge \( v_i v_{i+1} \) is subdivided by a vertex \( y_i \), \( 1 \leq i \leq n - 1 \). The edge \( u_{n+1} v_{n+1} \) is divided by a vertex \( z \) when \( n \) is odd. The edge \( u_{n+2} v_2 \) is divided by a vertex \( z \) when \( n \) is even. The graph \( S(H_n \odot K_1) \) has \( 8n - 1 \) vertices and \( 8n - 2 \) edges.

Proof. Let \( u_i \) be a vertex of the path attached at each \( u_i, 1 \leq i \leq n \) and \( v_i \) be the path attached at each \( v_i, 1 \leq i \leq n \). Each edge \( u_i u_{i+1} \) is subdivided by a vertex \( x_i \), \( 1 \leq i \leq n - 1 \) and each edge \( v_i v_{i+1} \) is subdivided by a vertex \( y_i \), \( 1 \leq i \leq n - 1 \). The edge \( u_{n+1} v_{n+1} \) is divided by a vertex \( z \) when \( n \) is odd. The edge \( u_{n+2} v_2 \) is divided by a vertex \( z \) when \( n \) is even. The graph \( S(H_n \odot K_1) \) has \( 8n - 1 \) vertices and \( 8n - 2 \) edges.
Define $f : V(S(H_n \odot K_1)) \rightarrow \{1, 2, 3, \ldots, p + q = 16n - 3\}$ as follows:

\[
\begin{align*}
    f(u_i) &= \begin{cases} 
        5, & i = 1, \\
        8i - 7, & 2 \leq i \leq n,
    \end{cases} \\
    f(v_i) &= \begin{cases} 
        8n + 3, & i = 1, \\
        8(n + i) - 9, & 2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\
        8(n + i) - 7, & \left\lfloor \frac{n+1}{2} \right\rfloor \leq i \leq n,
    \end{cases} \\
    f(a_{1,i}) &= \begin{cases} 
        1, & i = 1, \\
        8i - 2, & 2 \leq i \leq n,
    \end{cases} \\
    f(a_{2,i}) &= 8i - 5, \quad 1 \leq i \leq n.
\end{align*}
\]
The induced edge labeling is obtained as follows:

\[ f^*(u_i x_i) = \begin{cases} 6, & i = 1, \\ 8i - 4, & 2 \leq i \leq n - 1, \end{cases} \]

\[ f^*(x_i u_{i+1}) = 8i, \quad 1 \leq i \leq n - 1, \]

\[ f^*(v_i y_i) = \begin{cases} 8n + 4, & i = 1, \\ 8(n + i) - 6, & 2 \leq i \leq \lfloor \frac{n-1}{2} \rfloor, \\ 8(n + i) - 4, & \lfloor \frac{n+1}{2} \rfloor \leq i \leq n - 1, \end{cases} \]

\[ f^*(y_i v_{i+1}) = \begin{cases} 8(n + i) - 2, & 1 \leq i \leq \lfloor \frac{n-3}{2} \rfloor, \\ 12n - 5, & i = \frac{n-1}{2} \text{ and } n \text{ is odd}, \\ 12n - 9, & i = \frac{n-2}{2} \text{ and } n \text{ is even}, \\ 8(n + i), & \lfloor \frac{n+1}{2} \rfloor \leq i \leq n - 1, \end{cases} \]

\[ f^*(a_{1,i} a_{2,i}) = \begin{cases} 2, & i = 1, \\ 8i - 3, & 2 \leq i \leq n, \end{cases} \]

\[ f^*(a_{2,i} u_i) = \begin{cases} 4, & i = 1, \\ 8i - 6, & 2 \leq i \leq n, \end{cases} \]

\[ f^*(b_{1,i} b_{2,i}) = \begin{cases} 8n, & i = 1, \\ 8(n + i) - 5, & 2 \leq i \leq \lfloor \frac{n-1}{2} \rfloor, \\ 8(n + i) - 3, & \lfloor \frac{n+1}{2} \rfloor \leq i \leq n - 1, \\ 16n - 4, & i = n, \end{cases} \]
Theorem 2.3. The graph \( S(H_n \odot K_1) \) is a super mean graph, for \( n \geq 3 \).

Proof. Let \( u_1, u_2, \ldots, u_n \) and \( v_1, v_2, \ldots, v_n \) be the vertices of the paths of length \( n - 1 \). Let \( a_1, a_2, a_3, a_4 \) be the paths attached at each \( u_i, 1 \leq i \leq n \) and \( b_1, b_2, b_3, b_4 \) be the paths attached at each \( v_i, 1 \leq i \leq n \). Each edge \( u_iu_{i+1} \) is subdivided by a vertex \( x_i, 1 \leq i \leq n - 1 \) and each edge \( v_iv_{i+1} \) is subdivided by a vertex \( y_i, 1 \leq i \leq n - 1 \). The edge \( u_{n+1}v_{n+1} \) is divided by a vertex \( z \) when \( n \) is odd. The edge \( u_{n+2}v_{n+2} \) is divided by a vertex \( z \) when \( n \) is even. The graph \( S(H_n \odot S_2) \) has \( 12n - 1 \) vertices and \( 12n - 2 \) edges.

Define \( f : V(S(H_n \odot S_2)) \rightarrow \{1, 2, 3, \ldots, p + q = 24n - 3\} \) as follows:

\[
\begin{align*}
  f(u_i) &= 12i - 7, & 1 \leq i \leq n, \\
  f(v_i) &= \begin{cases} 
    12(n + i) - 9, & 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\
    12(n + i) - 7, & \left\lfloor \frac{n+1}{2} \right\rfloor \leq i \leq n,
  \end{cases} \\
  f(a_{1,i}) &= \begin{cases} 
    1, & i = 1, \\
    12i - 13, & 2 \leq i \leq n,
  \end{cases} \\
  f(a_{2,i}) &= \begin{cases} 
    3, & i = 1, \\
    12i - 11, & 2 \leq i \leq n,
  \end{cases} \\
  f(a_{3,i}) &= 12i - 3, & 1 \leq i \leq n, \\
  f(a_{4,i}) &= 12i - 5, & 1 \leq i \leq n, \\
  f(x_i) &= 12i + 2, & 1 \leq i \leq n - 1,
\end{align*}
\]

Thus, \( f \) is a super mean labeling and hence \( S(H_n \odot K_1) \) is a super mean graph.

For example, a super mean labeling of \( S(H_9 \odot K_1) \) and \( S(H_{10} \odot K_1) \) are shown in Figure 3. \(\square\)

Theorem 2.3. The graph \( S(H_n \odot S_2) \) is a super mean graph, for \( n \geq 3 \).
Figure 3

\[ f(b_{1,i}) = \begin{cases} 
12n - 1, & \text{if } i = 1, \\
12(n + i) - 15, & 2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\
18n - 8, & i = \frac{n+1}{2} \text{ and } n \text{ is odd}, \\
18n - 14, & i = \frac{n}{2} \text{ and } n \text{ is even}, \\
12(n + i) - 13, & \left\lceil \frac{n+3}{2} \right\rceil \leq i \leq n,
\end{cases} \]
For the vertex labeling $f$, the induced edge labels are obtained as follows:

$$f(b_{2,i}) = \begin{cases} 
12n + 1, & i = 1, \\
12(n + i) - 13, & 2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\
18n - 6, & i = \frac{n+1}{2} \text{ and } n \text{ is odd}, \\
18n - 12, & i = \frac{n}{2} \text{ and } n \text{ is even}, \\
12(n + i) - 11, & \left\lceil \frac{n+3}{2} \right\rceil \leq i \leq n,
\end{cases}$$

$$f(b_{3,i}) = \begin{cases} 
12(n + i) - 5, & 1 \leq i \leq \left\lfloor \frac{n-3}{2} \right\rfloor, \\
18n - 10, & i = \frac{n-1}{2} \text{ and } n \text{ is odd}, \\
18n - 16, & i = \frac{n}{2} \text{ and } n \text{ is even}, \\
12(n + i) - 3, & \left\lceil \frac{n+1}{2} \right\rceil \leq i \leq n,
\end{cases}$$

$$f(b_{4,i}) = \begin{cases} 
12(n + i) - 7, & 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\
12(n + i) - 5, & \left\lceil \frac{n+1}{2} \right\rceil \leq i \leq n,
\end{cases}$$

$$f(z) = \begin{cases} 
18n - 4, & \text{if } n \text{ is odd}, \\
18n - 10, & \text{if } n \text{ is even},
\end{cases}$$

and $f(y_i) = \begin{cases} 
12(n + i), & 1 \leq i \leq \left\lfloor \frac{n-3}{2} \right\rfloor, \\
18n - 9, & i = \frac{n-1}{2} \text{ and } n \text{ is odd}, \\
18n - 15, & i = \frac{n}{2} \text{ and } n \text{ is even}, \\
12(n + i) + 2, & \left\lceil \frac{n+1}{2} \right\rceil \leq i \leq n - 1.
\end{cases}$

$$f^*(a_{1,2}) = \begin{cases} 
2, & i = 1, \\
12(i - 1), & 2 \leq i \leq n,
\end{cases}$$

$$f^*(a_{2,3}) = \begin{cases} 
4, & i = 1, \\
12i - 9, & 2 \leq i \leq n,
\end{cases}$$

$$f^*(a_{3,4}) = 12i - 4, \quad 1 \leq i \leq n,$$

$$f^*(a_{4,1}) = 12i - 6, \quad 1 \leq i \leq n,$$

$$f^*(u_i x_i) = 12i - 2, \quad 1 \leq i \leq n - 1,$$

$$f^*(x_i u_{i+1}) = 12i + 4, \quad 1 \leq i \leq n - 1,$$

$$f^*(b_{1,2}) = \begin{cases} 
12n, & i = 1, \\
12(n + i) - 14, & 2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\
18n - 7, & i = \frac{n+1}{2} \text{ and } n \text{ is odd}, \\
18n - 13, & i = \frac{n}{2} \text{ and } n \text{ is even}, \\
12(n + i) - 12, & \left\lceil \frac{n+3}{2} \right\rceil \leq i \leq n,
\end{cases}$$
Case (i):

Theorem 2.4. The graph \( S(L_n) \) is a super mean graph, for \( n \geq 2 \).

Proof. Let \( u_1, u_2, \ldots, u_n \) and \( v_1, v_2, \ldots, v_n \) be the vertices on the paths of length \( n - 1 \). Let \( x_i, y_i \) and \( z_i \) be the vertices subdivided the edges \( u_iu_{i+1}, v_iv_{i+1} \) and \( v_ii+1 \) respectively for each \( i, 1 \leq i \leq n - 1 \). The graph \( S(SL_n) \) has \( 5n - 3 \) vertices and \( 6n - 6 \) edges.

Case (i): \( n \) is odd.
Define $f : V(SL_n) \to \{1, 2, \ldots, p + q = 11n - 9\}$ as follows:

$$f(u_i) = \begin{cases} 1, & i = 1, \\ 5, & i = 2, \\ 13, & i = 3, \\ 11i - 13, & 4 \leq i \leq n \text{ and } i \text{ is even}, \\ 11i - 19, & 4 \leq i \leq n \text{ and } i \text{ is odd}, \\ 11i - 13, & 2 \leq i \leq n - 1 \text{ and } i \text{ is even}, \\ 11i - 8, & 2 \leq i \leq n - 1 \text{ and } i \text{ is odd}, \\ 11n - 9, & i = n, \end{cases}$$

$$f(v_i) = \begin{cases} 11, & i = 1, \\ 11i - 2, & 2 \leq i \leq n - 1 \text{ and } i \text{ is even}, \\ 11i - 8, & 2 \leq i \leq n - 1 \text{ and } i \text{ is odd}, \\ 11n - 9, & i = n, \end{cases}$$
Case (ii): \( n \) is even, \( n \geq 4 \).

\[
\begin{align*}
  f(x_i) &= \begin{cases} 
    3, & i = 1, \\
    10, & i = 2, \\
    11i - 5, & 3 \leq i \leq n - 1 \text{ and } i \text{ is odd}, \\
    11i - 10, & 3 \leq i \leq n - 1 \text{ and } i \text{ is even},
  \end{cases} \\
  f(y_i) &= \begin{cases} 
    11i + 6, & 1 \leq i \leq n - 2 \text{ and } i \text{ is odd}, \\
    11i + 1, & 1 \leq i \leq n - 2 \text{ and } i \text{ is even},
  \end{cases} \\
  f(y_{n-1}) &= 11(n - 1), \\
  f(z_i) &= \begin{cases} 
    7, & i = 1, \\
    11i - 6, & 2 \leq i \leq n - 1.
  \end{cases}
\]

For the vertex labeling \( f \), the induced edge labeling \( f^* \) is given follows:

\[
\begin{align*}
  f^*(u_ix_i) &= \begin{cases} 
    2, & i = 1, \\
    8, & i = 2, \\
    11i - 12, & 3 \leq i \leq n - 2 \text{ and } i \text{ is odd}, \\
    11i - 11, & 3 \leq i \leq n - 2 \text{ and } i \text{ is even},
  \end{cases} \\
  f^*(x_iu_{i+1}) &= \begin{cases} 
    4, & i = 1, \\
    12, & i = 2, \\
    11i - 3, & 3 \leq i \leq n - 1 \text{ and } i \text{ is odd}, \\
    11i - 9, & 3 \leq i \leq n - 1 \text{ and } i \text{ is even},
  \end{cases} \\
  f^*(v_iy_i) &= \begin{cases} 
    14, & i = 1, \\
    11i, & 2 \leq i \leq n - 2 \text{ and } i \text{ is even}, \\
    11i - 1, & 2 \leq i \leq n - 2 \text{ and } i \text{ is odd}, \\
    11n - 12, & i = n - 1,
  \end{cases} \\
  f^*(y_{i+1}v_i) &= \begin{cases} 
    11i + 8, & 1 \leq i \leq n - 2 \text{ and } i \text{ is odd}, \\
    11i + 2, & 1 \leq i \leq n - 2 \text{ and } i \text{ is even}, \\
    11n - 10, & i = n - 1,
  \end{cases} \\
  f^*(v_iz_i) &= \begin{cases} 
    9, & i = 1, \\
    11i - 4, & 2 \leq i \leq n - 1 \text{ and } i \text{ is even}, \\
    11i - 6, & 2 \leq i \leq n - 1 \text{ and } i \text{ is odd},
  \end{cases} \\
  f^*(z_{i+1}u_i) &= \begin{cases} 
    6, & i = 1, \\
    11i - 7, & 2 \leq i \leq n - 1 \text{ and } i \text{ is even}, \\
    11i - 4, & 2 \leq i \leq n - 1 \text{ and } i \text{ is odd}.
  \end{cases}
\]
Define \( f: V(SL_n) \rightarrow \{1, 2, \ldots, p + q = 11n - 9\} \) as follows:

\[
f(u_i) = \begin{cases}
  1, & i = 1, \\
  5, & i = 2, \\
  13, & i = 3, \\
  11i - 13, & 4 \leq i \leq n - 1 \text{ and } i \text{ is even}, \\
  11i - 19, & 4 \leq i \leq n - 1 \text{ and } i \text{ is odd}, \\
  11n - 11, & i = n,
\end{cases}
\]

\[
f(v_i) = \begin{cases}
  11, & i = 1, \\
  11i - 2, & 2 \leq i \leq n - 1 \text{ and } i \text{ is even}, \\
  11i - 8, & 2 \leq i \leq n - 1 \text{ and } i \text{ is odd}, \\
  11n - 9, & i = n,
\end{cases}
\]

\[
f(x_i) = \begin{cases}
  3, & i = 1, \\
  10, & i = 2, \\
  11i - 5, & 3 \leq i \leq n \text{ and } i \text{ is odd}, \\
  11i - 10, & 3 \leq i \leq n \text{ and } i \text{ is even},
\end{cases}
\]

\[
f(y_i) = \begin{cases}
  11i + 6, & 1 \leq i \leq n - 2 \text{ and } i \text{ is odd}, \\
  11i + 1, & 1 \leq i \leq n - 2 \text{ and } i \text{ is even}, \\
  11n - 12, & i = n - 1,
\end{cases}
\]

\[
f(z_i) = \begin{cases}
  7, & i = 1, \\
  11i - 6, & 2 \leq i \leq n - 1.
\end{cases}
\]

The induced edge labeling is obtained as follows:

\[
f^*(u,x_i) = \begin{cases}
  2, & i = 1, \\
  8, & i = 2, \\
  11i - 12, & 3 \leq i \leq n - 1 \text{ and } i \text{ is odd}, \\
  11i - 10, & 3 \leq i \leq n - 1 \text{ and } i \text{ is even},
\end{cases}
\]

\[
f^*(x,x_{i+1}) = \begin{cases}
  4, & i = 1, \\
  12, & i = 2, \\
  11i - 3, & 3 \leq i \leq n - 2 \text{ and } i \text{ is odd}, \\
  11i - 9, & 3 \leq i \leq n - 2 \text{ and } i \text{ is even}, \\
  11n - 13, & i = n - 1,
\end{cases}
\]

\[
f^*(v,y_i) = \begin{cases}
  14, & i = 1, \\
  11i, & 2 \leq i \leq n - 2 \text{ and } i \text{ is even}, \\
  11i - 1, & 2 \leq i \leq n - 2 \text{ and } i \text{ is odd}, \\
  11n - 15, & i = n - 1,
\end{cases}
\]
Thus, $f$ is a super mean labeling of $S(SL_n)$ and hence $S(SL_n)$ is a super mean graph. For example, a super mean labeling of $S(SL_7)$ and $S(SL_8)$ are shown in Figure 5.

When $n = 2$, a super mean labeling of the graph is shown in Figure 6.

**Theorem 2.5.** The graph $S(T_n \odot K_1)$ is a super mean graph for any $n$.

**Proof.** Let $u_1, u_2, \ldots, u_n, u_{n+1}$ be the vertices on the path of length $n$ in $T_n$ and let $v_i, 1 \leq i \leq n$ be the vertices of $T_n$ in which $v_i$ is adjacent to $u_i$ and $u_{i+1}$. Let $v'_i a_i v_i$ be the path attached at each $v_i, 1 \leq i \leq n$ and $u'_i b_i u_i$ be the path attached at each $u_i, 1 \leq i \leq n + 1$. Let $x_i, y_i$ and $z_i$ be the vertices which subdivided the edges $u_i u_{i+1}, u_i v_i$.
and \( v_iu_{i+1} \) respectively for each \( i, 1 \leq i \leq n \). The graph \( S(T_n \odot K_1) \) has \( 9n + 3 \) vertices and \( 10n + 2 \) edges.

Define \( f : V(S(T_n \odot K_1)) \to \{1, 2, \ldots, p + q = 19n + 5\} \) as follows:

\[
\begin{align*}
f(u_i) &= 19i - 14, \\ f(v_i) &= 19i - 8, \\ f(v'_i) &= 19i - 4, \\ f(a_i) &= 19i - 6, \\ f(u'_i) &= \begin{cases} 1, & i = 1 \\ 19i - 20, & 2 \leq i \leq n + 1, \end{cases} \\ f(b_i) &= \begin{cases} 3, & i = 1 \\ 19i - 18, & 2 \leq i \leq n + 1, \end{cases} \\ f(x_i) &= 19i - 9, \\ f(y_i) &= 19i - 12, \\ f(z_i) &= 19i + 2, \end{align*}
\]

The induced edge labeling is defined as follows:

\[
\begin{align*}
f^*(u_ix_i) &= 19i - 11, \\ f^*(x_iu_{i+1}) &= 19i - 2, \\ f^*(u_iy_i) &= 19i - 13, \\ f^*(y_iv_i) &= 19i - 10, \\ f^*(v_iz_i) &= 19i - 3, \\ f^*(z_iu_{i+1}) &= 19i + 4, \\ f^*(v_iy'_i) &= 19i - 7, \\ f^*(a_iy'_i) &= 19i - 5, \\ f^*(u_ib_i) &= \begin{cases} 4, & i = 1 \\ 19i - 16, & 2 \leq i \leq n + 1, \end{cases} \end{align*}
\]
Thus, $f$ is a super mean labeling of $S(T_n \odot K_1)$.

For example, a super mean labeling of $S(T_6 \odot K_1)$ is shown in Figure 7.

---

**Theorem 2.6.** The graph $S(C_n \odot K_1)$ is a super mean graph, for $n \geq 3$.

**Proof.** Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle $C_n$. Let $v_i, y_i, u_i$ be the path attached at each $u_i$, $1 \leq i \leq n$. Each edge $u_iu_{i+1}$ is subdivided by a vertex $x_i$, $1 \leq i \leq n - 1$ and the edge $u_nu_1$ is subdivided by a vertex $x_n$.

**Case(i):** $n$ is odd.

Define $f : V(S(C_n \odot K_1)) \to \{1, 2, \ldots, 8n\}$ as follows:

$$
\begin{align*}
  f(u_i) &= \begin{cases} 
    5, & i = 1, \\
    16i - 21, & 2 \leq i \leq \frac{n+1}{2}, \\
    8n, & i = \frac{n+3}{2}, \\
    16(n - i) + 22, & \frac{n+5}{2} \leq i \leq n,
  \end{cases} \\
  f(u_i) &= \begin{cases} 
    1, & i = 1, \\
    16i - 17, & 2 \leq i \leq \frac{n+1}{2}, \\
    16(n - i) + 18, & \frac{n+3}{2} \leq i \leq n,
  \end{cases} \\
  f(x_i) &= \begin{cases} 
    16i - 9, & 1 \leq i \leq \frac{n-1}{2}, \\
    8n - 3, & i = \frac{n+1}{2}, \\
    16(n - i) + 10, & \frac{n+3}{2} \leq i \leq n,
  \end{cases}
\end{align*}
$$

---

*Figure 7. $S(T_6 \odot K_1)$*
\[
f(y_i) = \begin{cases} 
3, & i = 1, \\
16i - 19, & 2 \leq i \leq \frac{n+1}{2}, \\
16(n - i) + 20, & \frac{n+3}{2} \leq i \leq n.
\end{cases}
\]

The induced edge labeling is defined as follows:

\[
f^*(u_i) = \begin{cases} 
6, & i = 1, \\
16i - 15, & 2 \leq i \leq \frac{n-1}{2}, \\
8(n - 1), & i = \frac{n+1}{2}, \\
8n - 7, & i = \frac{n+3}{2}, \\
16(n - i) + 16, & \frac{n+3}{2} \leq i \leq n,
\end{cases}
\]

\[
f^*(x_iu_i) = \begin{cases} 
16i - 7, & 1 \leq i \leq \frac{n-1}{2}, \\
8n - 1, & i = \frac{n+1}{2}, \\
16(n - i) + 8, & \frac{n+3}{2} \leq i \leq n - 1,
\end{cases}
\]

\[
f^*(x_au_1) = 8,
\]

\[
f^*(v_i) = \begin{cases} 
2, & i = 1, \\
16i - 17, & 2 \leq i \leq \frac{n+1}{2}, \\
8n, & i = \frac{n+3}{2}, \\
16(n - i) + 18, & \frac{n+3}{2} \leq i \leq n,
\end{cases}
\]

and

\[
f^*(y_iu_i) = \begin{cases} 
4, & i = 1, \\
16i - 20, & 2 \leq i \leq \frac{n+1}{2}, \\
8n - 2, & i = \frac{n+3}{2}, \\
16(n - i) + 21, & \frac{n+5}{2} \leq i \leq n.
\end{cases}
\]

**Case (ii):** \(n\) is even.

\[
f(u_i) = \begin{cases} 
5, & i = 1, \\
16i - 21, & 2 \leq i \leq \frac{n}{2}, \\
8n - 4, & i = \frac{n+2}{2}, \\
16(n - i) + 22, & \frac{n+4}{2} \leq i \leq n,
\end{cases}
\]

\[
f(u_i) = \begin{cases} 
1, & i = 1, \\
16i - 17, & 2 \leq i \leq \frac{n}{2}, \\
8n, & i = \frac{n+2}{2}, \\
16(n - i) + 18, & \frac{n+4}{2} \leq i \leq n,
\end{cases}
\]

\[
f(x_i) = \begin{cases} 
16i - 9, & 1 \leq i \leq \frac{n}{2}, \\
8n - 7, & i = \frac{n+2}{2}, \\
16(n - i) + 10, & \frac{n+4}{2} \leq i \leq n,
\end{cases}
\]
SUPER MEAN LABELING OF SOME SUBDIVISION GRAPHS

\[ f(y_i) = \begin{cases} 
3, & i = 1, \\
16i - 19, & 2 \leq i \leq \frac{n}{2}, \\
8n - 2, & i = \frac{n+2}{2}, \\
16(n - i) + 20, & \frac{n+4}{2} \leq i \leq n.
\end{cases} \]

For the vertex labeling \( f \), the induced edge labeling \( f^* \) is given as follows:

\[ f^*(u_i x_i) = \begin{cases} 
6, & i = 1, \\
16i - 15, & 2 \leq i \leq \frac{n}{2}, \\
8n - 5, & i = \frac{n+2}{2}, \\
16(n - i + 1), & \frac{n+4}{2} \leq i \leq n,
\end{cases} \]

\[ f^*(x_i u_{i+1}) = \begin{cases} 
16i - 7, & 1 \leq i \leq \frac{n-2}{2}, \\
8n - 6, & i = \frac{n}{2}, \\
16(n - i) + 8, & \frac{n+2}{2} \leq i \leq n - 1,
\end{cases} \]

\[ f^*(x_n u_1) = 8, \]

\[ f^*(v_i y_i) = \begin{cases} 
2, & i = 1, \\
16i - 18, & 2 \leq i \leq \frac{n}{2}, \\
8n - 1, & i = \frac{n+2}{2}, \\
16(n - i) + 19, & \frac{n+4}{2} \leq i \leq n,
\end{cases} \]

and \( f^*(y_i u_i) = \begin{cases} 
4, & i = 1, \\
16i - 20, & 2 \leq i \leq \frac{n}{2}, \\
8n - 3, & i = \frac{n+2}{2}, \\
16(n - i) + 21, & \frac{n+4}{2} \leq i \leq n.
\end{cases} \]

Thus, \( f \) is a super mean labeling and hence \( S(C_n \odot K_1) \) is a super mean graph. \( \square \)

For example, a super mean labeling of \( S(C_{11} \odot K_1) \) and \( S(C_{12} \odot K_1) \) are shown in Figure 8.

Theorem 2.7. The graph \( S(C_m \odot C_n) \) is a super mean graph for \( m, n \geq 3 \).

Proof. \( C_m \odot C_n \) is a graph obtained by identifying an edge of two cycles \( C_m \) and \( C_n \). \( C_m \odot C_n \) has \( m + n - 2 \) vertices and \( m + n - 1 \) edges. In \( S(C_m \odot C_n) \), \( 2(m + n - 2) \) vertices lies on the circle and one vertex lies on a chord. Then, the graph \( S(C_m \odot C_n) \) has \( 2m + 2n - 3 \) vertices and \( 2(m + n - 1) \) edges.

Let us assume that \( m \leq n \).
Case (i): \( m \) is odd and \( n \) is odd.

Let \( m = 2k + 1, \ k \geq 1 \) and \( n = 2l + 1, \ l \geq 1 \). We denote the vertices of \( S(C_m \odot C_n) \) is shown in Figure 9.
Define \( f : V(S(C_m @ C_n)) \to \{ 1, 2, 3, \ldots, p + q = 4(m + n) - 5 \} \) as follows:

\[
f(u_i) = \begin{cases} 
1, & i = 1, \\
8i - 9, & 2 \leq i \leq k, \\
4m - 6, & i = k + 1, \\
8i, & k + 2 \leq i \leq k + l, \\
8m + n - i - 9, & k + l + 1 \leq i \leq k + 2l - 1, \\
4m + 5, & i = k + 2l, \\
4m, & i = k + 2l + 1, \\
8m + n - i - 6, & k + 2l + 2 \leq i \leq 2k + 2l,
\end{cases}
\]

and \( f(z) = 4m - 3 \).

The induced edge labeling \( f^* \) is obtained as follows:

\[
f^*(u_i x_i) = \begin{cases} 
2, & i = 1, \\
8i - 7, & 2 \leq i \leq k, \\
4m + 1, & i = k + 1, \\
8i + 2, & k + 2 \leq i \leq k + l, \\
8m + n - i - 11, & k + l + 1 \leq i \leq k + 2l - 1, \\
4m + 4, & i = k + 2l, \\
4m - 2, & i = k + 2l + 1, \\
8m + n - 1 - i, & k + 2l + 2 \leq i \leq 2k + 2l - 2,
\end{cases}
\]

\[
f^*(x_i u_{i+1}) = \begin{cases} 
8i - 3, & 1 \leq i \leq k, \\
8i + 6, & k + 1 \leq i \leq k + l - 1, \\
8m + n - i - 15, & k + l \leq i \leq k + 2l - 2, \\
4m + 6, & i = k + 2l - 1, \\
4m + 2, & i = k + 2l, \\
8m + n - i - 12, & k + 2l + 1 \leq i \leq 2k + 2l - 1,
\end{cases}
\]

\[
f^*(x_{2k+2l} u_1) = 4, \\
f^*(u_{k+1} z) = 4m - 4, \\
\text{and } f^*(z u_{k+2l+1}) = 4m - 1.
\]
Thus, \( f \) is a super mean labeling. A super mean labeling of \( S(C_7@C_9) \) is shown in Figure 10.

**Figure 10**

**Case (ii):** \( m \) is odd and \( n \) is even.

Let \( m = 2k + 1, k \geq 1 \) and \( n = 2l, l \geq 2 \). Define \( f : V(S(C_m@C_n)) \to \{1, 2, 3, \ldots, p + q = 4(m + n) - 5\} \) as follows:

\[
\begin{align*}
f(u_i) &= \begin{cases} 
1, & i = 1, \\
8i - 9, & 2 \leq i \leq k, \\
4m - 6, & i = k + 1, \\
8i, & k + 2 \leq i \leq k + l - 1, \\
8(m + n - i) - 9, & k + l \leq i \leq k + 2l - 2, \\
4m + 5, & i = k + 2l - 1, \\
4m, & i = k + 2l, \\
8(m + n - i) - 6, & k + 2l + 1 \leq i \leq 2k + 2l - 1, 
\end{cases} \\
f(x_i) &= \begin{cases} 
8i - 5, & 1 \leq i \leq k, \\
8i + 4, & k + 1 \leq i \leq k + l - 1, \\
8(m + n - i) - 13, & k + l \leq i \leq k + 2l - 2, \\
4m + 3, & i = k + 2l - 1, \\
4m - 5, & i = k + 2l, \\
8(m + n - i) - 10, & k + 2l + 1 \leq i \leq 2k + 2l - 1, 
\end{cases}
\]

and \( f(z) = 4m - 3 \).
For the vertex labeling $f$, the induced edge labeling $f^*$ is given as follows:

$$f^*(u_i x_i) = \begin{cases} 
2, & i = 1, \\
8i - 7, & 2 \leq i \leq k, \\
4m + 1, & i = k + 1, \\
8i + 2, & k + 2 \leq i \leq k + l - 1, \\
8(m + n - i) - 11, & k + l \leq i \leq k + 2l - 2, \\
4m + 4, & i = k + 2l - 1, \\
4m - 2, & i = k + 2l, \\
8(m + n - i) - 8, & k + 2l + 1 \leq i \leq 2k + 2l - 1,
\end{cases}$$

$$f^*(x_i u_{i+1}) = \begin{cases} 
8i - 3, & 1 \leq i \leq k, \\
8i + 6, & k + 1 \leq i \leq k + l - 1, \\
8(m + n - i) - 15, & k + l \leq i \leq k + 2l - 3, \\
4m + 6, & i = k + 2l - 2, \\
4m + 2, & i = k + 2l - 1, \\
8(m + n - i) - 12, & k + 2l \leq i \leq 2k + 2l - 2,
\end{cases}$$

$$f^*(x_{2k+2l-1} u_1) = 4,$$
$$f^*(u_{k+1} z) = 4m - 4,$$
and $$f^*(z u_{k+2l}) = 4m - 1.$$

Thus, $f$ is a super mean labeling. A super mean labeling of $S(C_7 \circ C_{10})$ is shown in Figure 11.

**Case (iii):** $m$ is even and $n$ is even.

Let $m = 2k$, $k \geq 2$ and $n = 2l$, $l \geq 2$. 
Define $f : V(S(C_m \@ C_n)) \to \{1, 2, 3, \ldots, p + q = 4(m + n) - 5\}$ as follows:

$$f(u_i) = \begin{cases} 
1, & i = 1, \\
8i - 9, & 2 \leq i \leq k, \\
4m, & i = k + 1, \\
4m + 5, & i = k + 2, \\
8i - 13, & k + 3 \leq i \leq k + l + 1, \\
8(m + n - i) + 4, & k + l + 2 \leq i \leq k + 2l - 1, \\
8(m + n - i) - 6, & k + 2l \leq i \leq 2k + 2l - 2, \\
\end{cases}$$

$$f(x_i) = \begin{cases} 
8i - 5, & 1 \leq i \leq k + 1, \\
8i - 9, & k + 2 \leq i \leq k + l, \\
8(m + n - i), & k + l + 1 \leq i \leq k + 2l - 1, \\
8(m + n - i) - 10, & k + 2l \leq i \leq 2k + 2l - 2, \\
\end{cases}$$

and $f(z) = 4m - 3$.

For the vertex labeling $f$, the induced edge labeling $f^*$ is obtained as follows:

$$f^*(u_i x_i) = \begin{cases} 
2, & i = 1, \\
8i - 7, & 2 \leq i \leq k, \\
4m + 2, & i = k + 1, \\
4m + 6, & i = k + 2, \\
8i - 11, & k + 3 \leq i \leq k + l, \\
8(m + n - i) + 2, & k + l + 1 \leq i \leq k + 2l - 1, \\
8(m + n - i) - 8, & k + 2l \leq i \leq 2k + 2l - 2, \\
\end{cases}$$

$$f^*(x_i u_{i+1}) = \begin{cases} 
8i - 3, & 1 \leq i \leq k - 1, \\
4m - 2, & i = k, \\
4m + 4, & i = k + 1, \\
8i - 7, & k + 2 \leq i \leq k + l, \\
8(m + n - i) - 2, & k + l + 1 \leq i \leq k + 2l - 2, \\
4m + 1, & i = k + 2l - 1, \\
8(m + n - i) - 12, & k + 2l \leq i \leq 2k + 2l - 3, \\
\end{cases}$$

$$f^*(x_{2k+2l-2} u_1) = 4,$$

$$f^*(u_{k+1} z) = 4m - 1,$$

and $$f^*(z u_{k+2l}) = 4m - 4.$$ 

Thus, $f$ is a super mean labeling. A super mean labeling of $S(C_6 \@ C_8)$ is shown in Figure 12.

Hence, the graph $S(C_m \@ C_n)$ is a super mean graph for $m, n \geq 3$. \qed
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