SOLUTION OF A PARTIAL DIFFERENTIAL EQUATION RELATED TO THE OPERATOR $\bigodot^k_B$

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ABSTRACT. In this paper, we consider the equation

$$\bigodot^k_B u(x) = \sum_{r=0}^{m} c_r \bigodot^r_B \delta,$$

where $\bigodot^k_B$ is the operator iterated $k$-time and is defined by

$$\bigodot^k_B = \left[ (B_{x_1} + B_{x_2} + \cdots + B_{x_p})^4 - (B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}})^4 \right]^k,$$

where $p + q = n, x = (x_1, \ldots, x_n) \in \mathbb{R}^+_n$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha_i}{x_i} \frac{\partial}{\partial x_i}, v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}$, $x_i > 0, i = 1, 2, \ldots, n, c_r$ is a constant, $k$ is a nonnegative integer, $\delta$ is the Dirac-delta distribution, $\bigodot^0_B \delta = \delta$ and $n$ is the dimension of $\mathbb{R}^+_n$. It is shown that, depending on the relationship between $k$ and $m$, the solution to this equation can be ordinary functions, tempered distributions, or singular distributions.

1. INTRODUCTION

Bupasiri [5] has first introduced the elementary solution of the $n$-dimensional $\bigodot^k_B$ operator and showed that the solution of the convolution form $(-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x))^{*1}$ is a unique elementary solution of the $\bigodot^k_B u(x) = \delta$.
Yildirim, Sarikaya and Ozturk [3] studied the Bessel diamond operator, iterated $k$-times,

\[
\diamond^k_B = \left[ \left( \sum_{i=1}^{p} B_{x_i} \right)^2 \right]^{k} - \left[ \left( \sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^{k}
= \left[ \sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j} \right]^k \left[ \sum_{i=1}^{p} B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j} \right]^k.
\]  

(1.1)

Yildirim, Sarikaya and Ozturk [3] showed that the function $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution for the operator $\diamond^k_B$, where $*$ indicates convolution, and $R_{2k}(x), S_{2k}(x)$ are defined by (1.4) and (1.5) respectively, that is,

\[
\diamond^k_B ((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta(x).
\]

We consider the equation

\[
\square^k_B u(x) = \sum_{r=0}^{m} c_r \square^r_B \delta,
\]

where $\square^k_B$ is the operator iterated $k$-time and is defined by

\[
\square^k_B = \left[ \left( \sum_{i=1}^{p} B_{x_i} \right)^4 \right]^{k} - \left[ \left( \sum_{j=p+1}^{p+q} B_{x_j} \right)^4 \right]^{k}
= \left[ \left( \sum_{i=1}^{p} B_{x_i} \right)^2 \right]^{k} - \left[ \left( \sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^{k} \left[ \left( \sum_{i=1}^{p} B_{x_i} \right)^2 + \left( \sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^{k}
= \diamond^k_B \circ^k_B,
\]

where

\[
\circ^k_B = \left[ \left( \sum_{i=1}^{p} B_{x_i} \right)^2 + \left( \sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^{k}
= \left( \frac{\Delta_B}{2} + \frac{\square_B}{2} \right)^2 \left( \frac{\Delta_B}{2} - \frac{\square_B}{2} \right)^2 \left( \frac{\Delta_B}{2} + \frac{\square_B}{2} \right)^k
= \left( \frac{\Delta_B}{2} + \frac{\square_B}{2} \right)^k.
\]

The purpose of this article, is finding the solution to the equation

\[
\square^k_B u(x) = \sum_{r=0}^{m} c_r \square^r_B \delta
\]

(1.3)
by using convolutions of the generalized function. It is also shown that the type of solution to (1.3) depends on the relationship between \( k \) and \( m \), according to the following cases:

1. If \( m < k \) and \( m = 0 \), then (1.3) has the solution
   \[
   u(x) = c_0 \left( (-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^*k(x))^{* -1} \right),
   \]
   which is an elementary solution of the \( R_B \) operator in Theorem 2.2, is an ordinary function when \( 6k \geq n \), and is a tempered distribution when \( 6k < n \).

2. If \( m < k \) then the solution of (1.3) is
   \[
   u(x) = \sum_{r=1}^{m} c_r c_0 \left( (-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*k-r}(x))^{* -1} \right),
   \]
   which is an ordinary function when \( 6k - 6r \geq n \) and is tempered distribution when \( 6k - 6r < n \).

3. If \( m \geq k \) and \( k \leq m \leq M \), then (1.3) has the solution
   \[
   u(x) = \sum_{r=k}^{M} c_r \oplus_{B}^{r-k} \delta,
   \]
   which is only a singular distribution. Before going that point, the following definitions and some concepts are needed.

**Lemma 1.1.** Given the equation \( \Delta_B^k u(x) = \delta(x) \) for \( x \in \mathbb{R}^+_n = \{ x : x = (x_1, \ldots, x_n), \ x_1 > 0, \ldots, x_n > 0 \} \), where \( \Delta_B^k \) is the Bessel-ultra hyperbolic operator iterated \( k \)-times. Then \( u(x) = R_{2k}(x) \) is an elementary solution of the operator \( \Delta_B^k \), where

\[

\Delta_B^k = \left[ \sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j} \right]^k,
\]

\[

R_{2k}(x) = \frac{V^{\frac{2k-n-|v|}{2}}}{K_n(2k)}
\]

\[

= \frac{\left( x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \right)^{\frac{2k-n-|v|}{2}}}{K_n(2k)}.
\]

for

\[

V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2,
\]

and

\[

K_n(2k) = \frac{\pi^n \cdot 2^{n+2k+n-2k|v|} \cdot \Gamma \left( \frac{2k+n-2|v|}{2} \right) \cdot \Gamma \left( \frac{n+2k-2|v|}{2} \right) \cdot \Gamma \left( \frac{1-k}{2} \right)}{\Gamma \left( \frac{2n+2k-2|v|}{2} \right) \cdot \Gamma \left( \frac{p+2|v|-2k}{2} \right) \cdot \Gamma \left( \frac{1-p}{2} \right)}.
\]

**Lemma 1.2.** Given the equation \( \Delta_B^k u(x) = \delta(x) \) for \( x \in \mathbb{R}^+_n \), where \( \Delta_B^k \) is the Laplace Bessel operator iterated \( k \)-times. Then \( u(x) = (-1)^k S_{2k}(x) \) is an elementary solution of the operator \( \Delta_B^k \), where
\[ \Delta^k_B = \left[ \sum_{i=1}^{p} B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j} \right]^k, \]

(1.5)

\[ S_{2k}(x) = \frac{|x|^{2k-n-2|v|}}{w_n(2k)}, \quad p + q = n, \]

\[ |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}, \]

and

\[ w_n(2k) = \frac{\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma(v_i + \frac{1}{2}) \Gamma(k)}{2^{n+2|v|-4k} \Gamma\left(\frac{n+2|v|-2k}{2}\right)}. \]

**Lemma 1.3.** The convolution \( R_{2k}(x) \ast (-1)^k S_{2k}(x) \) is an elementary solution for the operator \( \Diamond^k_B \) iterated \( k \)-times and is defined by (1.1).

**Lemma 1.4.** \( R_{2k}(x) \) and \( S_{2k}(x) \) are homogeneous distributions of order \( (2k-n-2|v|) \).

We need to show that \( R_{2k}(x) \) and \( (-1)^k S_{2k}(x) \) satisfy the Euler equation; that is,

\[ (2k-n-2|v|) R_{2k}(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_{2k}(x), \]

\[ (2k-n-2|v|) S_{2k}(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} S_{2k}(x). \]

**Lemma 1.5** (The \( B \)-convolution of tempered distribution). \( R_{2k}(x) \ast S_{2k}(x) \) exists and is a tempered distribution.

*Proof.* For the proofs of Lemmas 1.1–1.5, see [3, p. 378–383]. \( \square \)

**Lemma 1.6** (The \( B \)-convolution of \( R_{2k}(x) \) and \( S_{2k}(x) \)). Let \( R_{2k}(x) \) and \( S_{2k}(x) \) defined by (1.4) and (1.5) respectively, then we obtain:

1. \( S_{2k}(x) \ast S_{2m}(x) = S_{2k+2m}(x) \), where \( k \) and \( m \) are nonnegative integers.
2. \( R_{2k}(x) \ast R_{2m}(x) = R_{2k+2m}(x) \), where \( k \) and \( m \) are nonnegative integers.

**Lemma 1.7.** The function \( R_{-2k}(x) \) and \( (-1)^k S_{-2k}(x) \) are the inverse in the convolution algebra of \( R_{2k}(x) \) and \( (-1)^k S_{2k}(x) \), respectively. That is,

\[ R_{-2k}(x) \ast R_{2k}(x) = R_{-2k+2k}(x) = R_0(x) = \delta(x), \]

\[ (-1)^k S_{-2k}(x) \ast (-1)^k S_{2k}(x) = S_{-2k+2k}(x) = S_0(x) = \delta(x). \]

*Proof.* For the proofs of Lemma 1.7 and Lemma 1.6, see [4]. \( \square \)

**Lemma 1.8.** Given the equation

\[ \oplus_B^k u(x) = \delta(x), \]

(1.6)
where $\oplus_{B}^{k}$ is the operator iterated $k$-times defined by (1.2), $\delta(x)$ is the Dirac-delta distribution, $x \in \mathbb{R}^{n}_{+}$ and $k$ is a nonnegative integer. Then we obtain

\[(1.7) \quad u(x) = (R_{6k}(x) * (-1)^{3k} S_{6k}(x)) * (C^{*k}(x))^{-1}\]

is a Green’s function or an elementary solution for the operator $\oplus_{B}^{k}$ iterated $k$-times where $\oplus_{B}^{k}$ is defined by (1.2), and

\[(1.8) \quad C(x) = \frac{1}{2} R_{d}(x) + \frac{1}{2} (-1)^{2} S_{4}(x),\]

where $C^{*k}(x)$ denotes the convolution of $C$ with itself $k$ times, $(C^{*k}(x))^{-1}$ denotes the inverse of $C^{*k}(x)$ in the convolution algebra. Moreover $u(x)$ is a tempered distribution.

Proof. For a proof of the above lemma, see [5].

2. Main Results

**Theorem 2.1.** For $0 < r < k$,

\[\oplus_{B}^{k} (c_{0} ((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{-1})) = ((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*k}(x))^{-1})\]

and for $k \leq m$,

\[\oplus_{B}^{m} (c_{0} ((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{-1})) = \oplus_{B}^{m-k} \delta.\]

Proof. For $0 < r < k$, from (1.6),

\[\oplus_{B}^{k} (c_{0}((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{-1}) = \delta.\]

Thus,

\[\oplus_{B}^{k-r} \oplus_{B}^{r} (c_{0}((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{-1}) = \delta\]

or

\[\oplus_{B}^{k-r} \delta * \oplus_{B}^{r} (c_{0}((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{-1}) = \delta.\]

Convolving both sides by $((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*k}(k-r)(x))^{-1})$, we obtain

\[\oplus_{B}^{k-r} ((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*k}(k-r)(x))^{-1}) \]

\[* \oplus_{B}^{r} (c_{0}((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{-1}) \]

\[= ((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*k}(k-r)(x))^{-1}) * \delta.\]

By Lemma 1.8,

\[\delta * \oplus_{B}^{r} (c_{0}((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{-1}) \]

\[= ((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*k}(k-r)(x))^{-1}) * \delta.\]

It follows that

\[\oplus_{B}^{k} (c_{0}((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{-1}) \]

\[= ((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*k}(k-r)(x))^{-1}),\]
as required. For \( k \leq m \)
\[
\oplus_B^m \left( c_0 \left( (-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*1} \right) \right) \\
= \oplus_B^{m-k} \oplus_B^k \left( (-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*1} \right).
\]

It follows that

\[
\oplus_B^m \left( c_0 \left( (-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*1} \right) \right) = \oplus_B^{m-k} \delta
\]

by Lemma 1.8. This completes the proof. \( \square \)

**Theorem 2.2.** Consider the linear differential equation

\[ (2.1) \]
\[
\oplus_B^k u(x) = \sum_{r=0}^m c_r \oplus_B^r \delta,
\]

where \( p + q = n \), \( n \) is odd with \( p \) odd and \( q \) even, or \( n \) is even with \( p \) odd and \( q \) odd, \( x \in \mathbb{R}_n^+ = \{ x : x = (x_1, \ldots, x_n), x_i > 0, \ldots, x_n > 0 \} \), \( c_r \) is a constant, \( \delta \) is the Dirac-delta distribution, and \( \oplus_B^0 \delta = \delta \). Then the type of solution to (2.1) depends on the relationship between \( k \) and \( m \), according to the following cases:

1. If \( m < k \) and \( m = 0 \), then (2.1) has the solution
   \[
   u(x) = c_0 \left( (-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*1} \right),
   \]
   which is an elementary solution of the \( \oplus_B^k \) operator in Theorem 2.1, is an ordinary function when \( 6k \geq n \), and is a temper distribution when \( 6k < n \).

2. If \( m < k \) then the solution of (2.1) is
   \[
   u(x) = \sum_{r=1}^m c_r \left( (-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*1} \right),
   \]
   which is an ordinary function when \( 6k - 6r \geq n \) and is tempered distribution when \( 6k - 6r < n \).

3. If \( m \geq k \) and \( k \leq m \leq M \), then (2.1) has the solution
   \[
   u(x) = \sum_{r=k}^M c_r \oplus_B^{r-k} \delta,
   \]
   which is only a singular distribution.

**Proof.**

1. For \( m = 0 \), we have \( \oplus_B^k u(x) = c_0 \delta \), and by Theorem 2.1 we obtain
   \[
   u(x) = \left( (-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*1} \right).
   \]
   Now, \( (-1)^{3k} S_{6k}(x) \) and \( R_{6k}(x) \) are the analytic function for \( 6k \geq n \) and also \( (-1)^{3k} S_{6k}(x) \) and \( R_{6k}(x) \) are analytic functions by (1.7). It follows that \( (-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*1} \) is an ordinary function for \( 6k \geq n \). By Lemma 1.5, \( (-1)^{3k} S_{6k}(x) \) and \( R_{6k}(x) \) are tempered distributions with \( 6k < n \), we obtain \( (-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*1} \) exits and is a tempered distribution.
(2) For the case $0 < m < k$, we have
\[ \oplus_B^k u(x) = c_1 \oplus_B \delta + c_2 \oplus_B^2 \delta + \cdots + c_m \oplus_B^m \delta. \]

We convolved both sides of the above equation by $(-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x)^{s-1})$ to obtain
\[ \oplus_B^k \left((-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x)^{s-1})\right) \ast u(x) \]
\[ = c_1 \oplus_B \left((-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x)^{s-1})\right) \]
\[ + c_2 \oplus_B^2 \left((-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x)^{s-1})\right) \]
\[ + \cdots + c_m \oplus_B^m \left((-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x)^{s-1})\right). \]

By Theorem 2.1, we obtain
\[ u(x) = c_1 \left((-1)^{3(k-1)} S_{6(k-1)}(x) \ast R_{6(k-1)}(x) \ast (C^{*(k-1)}(x)^{s-1})\right) \]
\[ + c_2 \left((-1)^{3(k-2)} S_{6(k-2)}(x) \ast R_{6(k-2)}(x) \ast (C^{*(k-2)}(x)^{s-1})\right) \]
\[ + \cdots + c_m \left((-1)^{3(k-m)} S_{6(k-m)}(x) \ast R_{6(k-m)}(x) \ast (C^{*(k-m)}(x)^{s-1})\right), \]

or
\[ u(x) = \sum_{r=1}^{m} c_r \left((-1)^{3(k-r)} S_{6(k-r)}(x) \ast R_{6(k-r)}(x) \ast (C^{*(k-r)}(x)^{s-1})\right). \]

Similarly, as in case (1), $u(x)$ is an ordinary function for $6k - 6r \geq n$ and is a tempered distribution for and $6k - 6r < n$.

(3) For the case $m \geq k$ and $k \leq m \leq M$, we have
\[ \oplus_B^k u(x) = c_k \oplus_B^k \delta + c_{k+1} \oplus_B^{k+1} \delta + \cdots + c_M \oplus_B^M \delta. \]

Convolved both sides of the above equation by
\[ (-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x)^{s-1}) \]

to obtain
\[ \oplus_B^k \left((-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x)^{s-1})\right) \ast u(x) \]
\[ = c_k \oplus_B^k \left((-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x)^{s-1})\right) \]
\[ + c_{k+1} \oplus_B^{k+1} \left((-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x)^{s-1})\right) \]
\[ + \cdots + c_M \oplus_B^M \left((-1)^{3k} S_{6k}(x) \ast R_{6k}(x) \ast (C^{*k}(x)^{s-1})\right). \]

By Theorem 2.1 again, we obtain
\[ u(x) = c_k \delta + c_{k+1} \oplus_B \delta + c_{k+2} \oplus_B^2 \delta + \cdots + c_M \oplus_B^{M-k} \delta = \sum_{r=k}^{M} c_r \oplus_B^{r-k} \delta. \]

Since $\oplus_B^{r-k} \delta$ is a singular distribution, hence $u(x)$ is only the singular distribution. This completes the proofs. \(\square\)
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