SHARP INEQUALITIES INVOLVING THE RICCI CURVATURE FOR RIEMANNIAN SUBMERSIONS

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ABSTRACT. In this paper, we obtain sharp inequalities on Riemannian manifolds admitting a Riemannian submersion and give some characterizations using these inequalities. We improve Chen-Ricci inequality for Riemannian submersion and present some examples which satisfy this inequality.

1. INTRODUCTION

Riemannian invariants play the most fundamental role in Riemannian geometry. These invariants determine the intrinsic and extrinsic characteristics of Riemannian manifolds which affect the behaviour of the Riemannian manifold in general form. Thus, in 1999, B.-Y. Chen studied the intrinsic and extrinsic invariants who established an inequality involving Ricci curvature and squared mean curvature of a submanifold in a real space form $\mathbb{R}^m(c)$ (see [4]). A generalization of this inequality was proved by B.-Y. Chen in 2005 for arbitrary submanifolds in an arbitrary Riemannian manifold (see [6]). Later, this inequality has been intensively studied for different ambient spaces by several authors who are obtained some results (see [9, 10, 12, 17–19, 23–25]). So, this inequality is well-known as Chen-Ricci inequality.

On the other hand, the notion of submersion is used in Physics as well as Differential Geometry because of its applications in Kaluza-Klein theory, Yang-Mills theory and general relativity. Hence, submersions are studied for different kinds of spaces by several authors and new submersions are obtained such as Riemannian submersion, almost Hermitian submersion, semi-Riemannian submersion and etc. (see [1, 2, 5, 7, 13–16, 21]).

In the present paper, our goal is to give some inequalities involving Ricci curvatures and study Chen-Ricci inequality using Riemannian invariants which are the intrinsic and extrinsic characteristics of Riemannian manifolds admitting a Riemannian submersion in Section 3, we compute the Ricci curvatures Ric' of (B, g') and $\widehat{\mathrm{Ric}}$ of any fibre of Riemannian submersion π. Considering these computations, some characterizations are obtained. Next, the main notion of our paper, namely Chen-Ricci inequality for Riemannian submersions, is given by Section 4. Here, we obtain some results involving the intrinsic and extrinsic characteristics such as Ricci curvature, scalar curvature and the squared mean curvature $\|H\|^2$. The last Section is devoted to provide examples of Riemannian submersion which are satisfied Chen-Ricci inequality.

2. Preliminaries

Let (M, g) and (B, g') be m and n dimensional C^∞-Riemannian manifolds, respectively. $\pi : M \to B$ is a surjective map of M onto B is called a *Riemannian submersion* if π has a maximal rank and the differential π_* preserves the lengths of the horizontal vectors. For any $b \in B$, the closed r-dimensional $(r = m - n)$ submanifold $\pi^{-1}(b)$ of M is obtained and it is called fibre of Riemannian submersion π. For any $p \in M$, denoting $\mathcal{V}_p = \ker(\pi_*(p))$ and it follows that $\mathcal{V}$ is an integrable distribution that is called the *vertical distribution*. The sections of $\mathcal{V}$ are called the vertical vector fields.

Let $\mathcal{H}$ be a complementary distribution of $\mathcal{V}$ determined by the Riemannian metric g. Then, one has:

$$T_p M = \mathcal{V}_p \oplus \mathcal{H}_p.$$

Here, $\mathcal{H}_p$ is called the *horizontal space* at p. If a vector X on M is always orthogonal to fibres, then it is called the *horizontal vector* on M.

Let $\chi^h(M)$ and $\chi^v(M)$ are the space of horizontal and vertical vector fields, respectively. A Riemannian submersion is determined the invariant tensors T and A which are defined as follows

$$A_E F = h\nabla_{hE} vF + v\nabla_{hE} hF,$$
$$T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF,$$

where h and v are the projection morphisms of $E, F \in \chi(M)$ to $\chi^h(M)$ and $\chi^v(M)$, respectively.

Let ∇ be the Levi-Civita connection of M with respect to the Riemannian metric g and denote

$$(2.1) \qquad\qquad T^{\mathcal{H}} : \chi^v(M) \times \chi^v(M) \mapsto \chi^h(M),$$
$$(U, V) \mapsto T^{\mathcal{H}}(U, V) = h\nabla_U V,$$
$$(2.2) \qquad\qquad T^{\mathcal{V}} : \chi^v(M) \times \chi^h(M) \mapsto \chi^v(M),$$
$$(U, X) \mapsto T^{\mathcal{V}}(U, X) = v\nabla_U X,$$

and

$$(2.3) \qquad A^{\mathcal{H}} : \chi^h(M) \times \chi^v(M) \mapsto \chi^h(M),$$
$$(X, U) \mapsto A^{\mathcal{H}}(X, U) = h\nabla_X U,$$
$$(2.4) \qquad A^{\mathcal{V}} : \chi^h(M) \times \chi^h(M) \mapsto \chi^v(M),$$
$$(X, Y) \mapsto A^{\mathcal{V}}(X, Y) = v\nabla_X Y,$$

where $U, V \in \chi^v(M)$ and $X, Y \in \chi^h(M)$. Here, we note that $T^{\mathcal{H}}$ is a symmetric operator on $\chi^v(M) \times \chi^v(M)$ and $A^{\mathcal{V}}$ is an anti-symmetric operator on $\chi^h(M) \times \chi^h(M)$. From (2.1), (2.2), (2.3) and (2.4), we have

$$\nabla_U V = T^{\mathcal{H}}(U, V) + \widehat{\nabla}_U V,$$
$$\nabla_V X = h\nabla_V X + T^{\mathcal{V}}(U, X),$$
$$\nabla_X U = A^{\mathcal{H}}(X, U) + v\nabla_X U,$$
$$\nabla_X Y = h\nabla_X Y + A^{\mathcal{V}}(X, Y),$$

for any $U, V \in \chi^v(M)$ and $X, Y \in \chi^h(M)$ (see [11]).

Denote R, R', $\widehat{R}$ and R^* the Riemannian curvature tensors of Riemannian manifolds M, B, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, respectively. Then, the Gauss- Codazzi type equations are given by the following

$$
\begin{aligned}
R(U, V, F, W) =& \widehat{R}(U, V, F, W) + g\left(T^{\mathcal{H}}(U, W), T^{\mathcal{H}}(V, F)\right) \\
& - g\left(T^{\mathcal{H}}(V, W), T^{\mathcal{H}}(U, F)\right),
\end{aligned}
$$
$$
\begin{aligned}
R(X, Y, Z, H) =& R^*(X, Y, Z, H) - 2g\left(A^{\mathcal{V}}(X, Y), A^{\mathcal{V}}(Z, H)\right) \\
& + g\left(A^{\mathcal{V}}(Y, Z), A^{\mathcal{V}}(X, H)\right) - g\left(A^{\mathcal{V}}(X, Z), A^{\mathcal{V}}(Y, H)\right),
\end{aligned}
$$
$$
\begin{aligned}
R(X, V, Y, W) =& g\left((\nabla_X T)(V, W), Y\right) + g\left((\nabla_V A)(X, Y), W\right) \\
& - g\left(T^{\mathcal{V}}(V, X), T^{\mathcal{V}}(W, Y)\right) + g\left(A^{\mathcal{H}}(X, V), A^{\mathcal{H}}(Y, W)\right),
\end{aligned}
$$

where

$$\pi_*(R^*(X, Y, Z)) = R'(\pi_*(X), \pi_*(Y), \pi_*(Z)),$$

for any $X, Y, Z, H \in \chi^h(M)$ and $U, V, F, W \in \chi^v(M)$ (for details, see [11]).

Using the above equalities, we get the following equations involving the sectional curvatures:

(i) If $\alpha = \mathrm{Span}\{U, V\}$ and $U, V \in \chi^v(M)$, then

$$(2.5) \qquad K(\alpha) = \widehat{K}(\alpha) - \left\|T^{\mathcal{H}}(U, V)\right\|^2 + g\left(T^{\mathcal{H}}(U, U), T^{\mathcal{H}}(V, V)\right).$$

(ii) If $\alpha = \mathrm{Span}\{X, Y\}$ and $\alpha' = \mathrm{Span}\{\pi_* X, \pi_* Y\}$ and for any vector fields $X, Y \in \chi^h(M)$, then

$$(2.6) \qquad K(\alpha) = K'(\alpha') + 3\left\|A^{\mathcal{V}}(X, Y)\right\|^2.$$

(iii) If $\alpha = \text{Span}\{X, V\}$ for each vector fields $U \in \chi^v(M)$ and $X \in \chi^h(M)$, then

$$(2.7) \qquad K(\alpha) = -g((\nabla_X T)(V, V), X) + \left\|T^{\mathcal{V}}(V, X)\right\|^2 - \left\|A^{\mathcal{H}}(X, V)\right\|^2.$$

Let $\pi : M \to B$ be a smooth map between Riemannian manifolds and let ∇, $\nabla^{\pi^{-1}TB}$ denote respectively, the Levi-Civita connection on M and the pull-back connection. Then, we say that π is harmonic if its tension field $\tau(p)$ vanishes identically, that is

$$\tau(p) = \text{trace}_g(\nabla.\pi_*\cdot) = \sum_{i=1}^{m}(\nabla\pi_*)(E_i, E_i) = 0,$$

where $\{E_i\}_{i=1\ldots m}$ is an orthonormal basis of M and $\nabla\pi_*$ denote the second fundamental form of π, which is defined by

$$(\nabla\pi_*)(E, F) = \nabla_E^{\pi^{-1}TB}(\pi_*F) - \pi_*(\nabla_E F),$$

for any $E, F \in \chi(M)$.

Moreover, the mean curvature vector field $H(p)$ of any fibre of Riemannian submersion π at any $p \in M$ is given by

$$\mathcal{N} = rH,$$

where

$$\mathcal{N} = \sum_{j=1}^{r}T^{\mathcal{H}}(U_j, U_j)$$

and $\{U_1, U_2, \ldots U_r\}$ is an orthonormal basis of the vertical distribution $\mathcal{V}$.

We here note that the horizontal vector field N vanishes if and only if any fibre of Riemannian submersion π is minimal submanifold on M and this implies that the tension field of π vanishes, identically. Hence, the Riemannian submersion π is harmonic (for details, see [8]).

Furthermore, it is said to be π *has totally geodesic fibres* if both $T^{\mathcal{H}}$ and $T^{\mathcal{V}}$ vanish on $\chi^h(M)$ and $\chi^v(M)$, respectively and π has *totally umbilical fibres* if

$$T^{\mathcal{H}}(U, V) = g(U, V) H,$$

where $U, V \in \chi^v(M)$ and H is the mean curvature vector field of any fibre.

The horizontal distribution $\mathcal{H}$ is *integrable* if both $A^{\mathcal{H}}$ and $A^{\mathcal{V}}$ vanish on $\chi^h(M)$ and $\chi^v(M)$, respectively.

Let $\{U_1, \ldots, U_r\}$ be an orthonormal basis of $\chi^v(M)$. Then, it follows that

$$g(\nabla_E\mathcal{N}, X) = \sum_{j=1}^{r}g((\nabla_E T)(U_j, U_j), X),$$

for any $E \in \chi(M)$ and $X \in \chi^h(M)$ (see [11]).

The horizontal divergence of any vector field X on $\chi^h(M)$ is given by $\check{\delta}(X)$ and defined by

$$\check{\delta}(X) = \sum_{i=1}^{n}g(\nabla_{X_i}X, X_i),$$

where $\{X_1, \ldots, X_n\}$ is an orthonormal basis of $\chi^h(M)$. Then, one has

$$(2.8) \qquad \check{\delta}(\mathcal{N}) = \sum_{i=1}^{n} \sum_{j=1}^{r} g\left((\nabla_{X_i} T)(U_j, U_j), X_i\right).$$

For more details, we refer to [3, pp. 243].

3. Ricci Curvature for Riemannian Submersions

In the present section, we study some inequalities involving Ricci curvatures on the vertical and horizontal distributions for Riemannian submersions. Also, we consider the equality cases of these inequalities and give some characterizations for Riemannian submersions involving the fundamental tensors.

We begin to this section with the following lemma.

Lemma 3.1. *Let (M, g) and (B, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$ and $\{U_1, \ldots, U_r, X_1, \ldots X_n\}$ be an orthonormal basis of T_pM at any point $p \in M$, such that $\mathcal{V} = \mathrm{Span}\{U_1, \ldots, U_r\}$ and $\mathcal{H} = \mathrm{Span}\{X_1, \ldots X_n\}$. Then, one has*

$$(3.1) \quad \mathrm{Ric}(U_i) = \widehat{\mathrm{Ric}}(U_i) + \sum_{j=1}^{r} \left(g\left(T^{\mathcal{H}}(U_i, U_i), T^{\mathcal{H}}(U_j, U_j)\right) \right.$$

$$\left. -g\left(T^{\mathcal{H}}(U_i, U_j), T^{\mathcal{H}}(U_i, U_j)\right) \right) + \sum_{j=1}^{n} \left(g\left(T^{\mathcal{V}}(U_i, X_j), T^{\mathcal{V}}(U_i, X_j)\right) \right.$$

$$\left. -g\left(A^{\mathcal{H}}(X_j, U_i), A^{\mathcal{H}}(x_j, U_i)\right) - g\left((\nabla_{X_j} T)(U_i, U_i), X_j\right) \right),$$

and

$$(3.2) \qquad \mathrm{Ric}(X_i) = \mathrm{Ric}^*(X_i) + \sum_{j=1}^{r} \left(g\left(T^{\mathcal{V}}(U_j, X_i), T^{\mathcal{V}}(U_j, X_i)\right) \right.$$

$$\left. -g\left(A^{\mathcal{H}}(X_i, U_j), A^{\mathcal{H}}(X_i, U_j)\right) - g\left((\nabla_{X_i} T)(U_j, U_j), X_i\right) \right)$$

$$+ 3\sum_{j=1}^{n} g\left(A^{\mathcal{V}}(X_i, X_j), A^{\mathcal{V}}(X_i, X_j)\right),$$

where

$$\widehat{\mathrm{Ric}}(U_i) = \sum_{j=1}^{r} \widehat{R}(U_i, U_j, U_j, U_i)$$

and

$$\mathrm{Ric}^*(X_i) = \sum_{j=1}^{n} R^*(X_i, X_j, X_j, X_i).$$

Proof. For $U_i \in \chi^v(M)$ and $X_i \in \chi^h(M)$, the Ricci curvatures are given as follows

$$(3.3) \qquad \mathrm{Ric}(U_i) = \sum_{j=1}^{r} R(U_i, U_j, U_j, U_i) + \sum_{j=1}^{n} R(U_i, X_j, X_j, U_i)$$

and

$$(3.4) \qquad \text{Ric}(X_i) = \sum_{j=1}^{r} R(X_i, U_j, U_j, X_i) + \sum_{j=1}^{n} R(X_i, X_j, X_j, X_i).$$

If we put Gauss-Codazzi type equations for Riemannian submersions in above equalities (3.3) and (3.4), we get the required equalities. $\qquad \square$

Notation. Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion between Riemannian manifolds. Then, we have the following linear maps

$$T_1^{\mathcal{H}} : \chi^v(M) \to \chi^h(M)^* \;\; ; \;\; T_1^{\mathcal{V}} : \chi^v(M) \to \chi^v(M)^*$$
$$T_1^{\mathcal{H}}(U) = T^{\mathcal{H}}(U, V), \qquad T_1^{\mathcal{V}}(U) = T^{\mathcal{V}}(U, X),$$
$$T_2^{\mathcal{V}} : \chi^h(M) \to \chi^v(M)^*$$
$$T_2^{\mathcal{V}}(X) = T^{\mathcal{V}}(U, X),$$

and

$$A_1^{\mathcal{V}} : \chi^h(M) \to \chi^v(M)^* \;\; ; \;\; A_1^{\mathcal{H}} : \chi^h(M) \to \chi^h(M)^*$$
$$A_1^{\mathcal{V}}(X) = A^{\mathcal{V}}(X, Y), \qquad A_1^{\mathcal{H}}(X) = A^{\mathcal{H}}(X, U),$$
$$A_2^{\mathcal{H}} : \chi^v(M) \to \chi^h(M)^*$$
$$A_2^{\mathcal{H}}(U) = A^{\mathcal{H}}(X, U),$$

where $\chi^h(M)^*$ and $\chi^v(M)^*$ are the dual vector space of the horizontal and vertical vector spaces $\chi^h(M)$ and $\chi^v(M)$, respectively. Moreover, the squared norms of above linear maps are given as follows

$$\|T_1^{\mathcal{H}}(U)\|^2 = \sum_{j=1}^{r} g\left(T^{\mathcal{H}}(U, U_j), T^{\mathcal{H}}(U, U_j)\right),$$

$$\|T_1^{\mathcal{V}}(U)\|^2 = \sum_{j=1}^{n} g\left(T^{\mathcal{V}}(U, X_j), T^{\mathcal{V}}(U, X_j)\right),$$

$$\|T_2^{\mathcal{V}}(X)\|^2 = \sum_{j=1}^{r} g\left(T^{\mathcal{V}}(U_j, X), T^{\mathcal{V}}(U_j, X)\right),$$

$$\|A_1^{\mathcal{V}}(X)\|^2 = \sum_{j=1}^{n} g\left(A^{\mathcal{V}}(X, X_j), A^{\mathcal{V}}(X, X_j)\right),$$

$$\|A_1^{\mathcal{H}}(X)\|^2 = \sum_{j=1}^{r} g\left(A^{\mathcal{H}}(X, U_j), A^{\mathcal{H}}(X, U_j)\right),$$

$$\|A_2^{\mathcal{H}}(U)\|^2 = \sum_{j=1}^{n} g\left(A^{\mathcal{H}}(X_j, U), A^{\mathcal{H}}(X_j, U)\right),$$

where $\{U_1, U_2, \ldots U_r\}$ and $\{X_1, X_2, \ldots X_n\}$ are the orthonormal basis of the vertical distribution $\chi^v(M)$ and the horizontal distribution $\chi^h(M)$, respectively.

Considering above notions, we may give the following.

Theorem 3.1. *Let (M, g) and (B, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. Then, one has*

$$(3.5) \qquad \mathrm{Ric}(U) \leq \widehat{\mathrm{Ric}}(U) + r.g\left(T^{\mathcal{H}}(U, U), N\right) + \|T_1^{\mathcal{V}}(U)\|^2 - \check{\delta}\left(T^{\mathcal{H}}(U, U)\right).$$

The equality case of (3.5) holds for a unit vertical vector field $U \in \chi^v(M)$ if and only if

$$T^{\mathcal{H}}(U, V) = 0,$$
$$A^{\mathcal{H}}(X, U) = 0,$$

for any $V \in \chi^v(M)$ and $X \in \chi^h(M)$, respectively. Here, we note that the equality case of (3.5) holds for any $U \in \chi^v(M)$ if and only if both $T^{\mathcal{H}}$ and $A^{\mathcal{H}}$ vanish identically.

$$(3.6) \qquad \mathrm{Ric}(U) \geq \widehat{\mathrm{Ric}}(U) + r.g\left(T^{\mathcal{H}}(U, U), H(p)\right) - \|T_1^{\mathcal{H}}(U)\|^2$$
$$- \|A_2^{\mathcal{H}}(U)\|^2 - \check{\delta}\left(T^{\mathcal{H}}(U, U)\right).$$

The equality case of (3.6) holds for a unit vertical vector field $U \in \chi^v(M)$ if and only if

$$T^{\mathcal{V}}(U, X) = 0, \quad \text{for any } X \in \chi^h(M).$$

Notice that the equality case of (3.6) holds for any $U \in \chi^v(M)$ if and only if $T^{\mathcal{V}}$ vanishes identically.

Proof. At any point $p \in M$, we have the equality (3.1) for unit vertical vector field $U \in \chi^v(M)$. Using above Notation in equality (3.1),

$$(3.7) \qquad \mathrm{Ric}(U_i) = \widehat{\mathrm{Ric}}(U_i) + r.g\left(T^{\mathcal{H}}(U_i, U_i), H(p)\right) - \|T_1^{\mathcal{H}}(U_i)\|^2$$
$$+ \|T_1^{\mathcal{V}}(U_i)\|^2 - \|A_2^{\mathcal{H}}(U_i)\|^2 - \check{\delta}\left(T^{\mathcal{H}}(U, U)\right).$$

Putting $U = U_i$ for $(1 \leq i \leq r)$ in (3.7), the required statement is obtained. $\square$

Theorem 3.2. *Let (M, g) and (B, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. Then, one has*

$$(3.8) \qquad \mathrm{Ric}(X) \leq \mathrm{Ric}^*(X) + g\left(\nabla_X N, X\right) + \|T_2^{\mathcal{V}}(X)\|^2 + 3\|A_1^{\mathcal{V}}(X)\|^2.$$

The equality case of (3.8) holds for a unit horizontal vector field $X \in \chi^h(M)$ if and only if

$$(3.9) \qquad A^{\mathcal{H}}(X, V) = 0, \quad \text{for any } V \in \chi^v(M).$$

Here, we note that the equality case of (3.8) holds for all unit horizontal vector field $X \in \chi^h(M)$ if and only if $A^{\mathcal{H}}$ vanishes identically.

$$(3.10) \qquad \mathrm{Ric}(X) \geq \mathrm{Ric}^*(X) + g\left(\nabla_X N, X\right) - \|A_1^{\mathcal{H}}(X)\|^2.$$

The equality case of (3.10) *holds for a unit horizontal vector field* $X \in \chi^h(M)$ *if and only if*

$$T^{\mathcal{V}}(V, X) = 0, \quad \text{for any } V \in \chi^v(M),$$
$$A^{\mathcal{V}}(X, Y) = 0, \quad \text{for any } Y \in \chi^h(M).$$

Notice that the equality case of (3.10) *holds for all unit horizontal vector fields* $X \in \chi^h(M)$ *if and only if both* $T^{\mathcal{V}}$ *and* $A^{\mathcal{V}}$ *vanish identically.*

Proof. At any point $p \in M$, we have the equality (3.2) for unit horizontal vector field $X \in \chi^h(M)$. Using above Notation in equality (3.2),

$$(3.11) \qquad \mathrm{Ric}(X_i) = \mathrm{Ric}^*(X_i) + g\left(\nabla_X \mathcal{N}, X\right) + \|T_2^{\mathcal{V}}(X)\|^2 - \|A_1^{\mathcal{H}}(X)\|^2$$
$$- 3\|A_1^{\mathcal{V}}(X)\|^2.$$

Putting $X = X_i$ for $(1 \le i \le n)$ in (3.11), the required statement is obtained. $\qquad \square$

Remark 3.1. Let $\pi : M \to B$ be a Riemannian submersion between Riemannian manifolds. Using M. Falcitelli, S. Ianus and A. M. Pastore's book (see [11]), we recall that the fundamental tensor A vanishes identically if and only if the horizontal distribution $\mathcal{H}$ is integrable. In particular, the vanishing of fundamental tensor T imply that the Riemannian submersion π has any fibre which is totally geodesic submanifold of M.

Remark 3.2. Considering the equalities (2.5), (2.6) and (2.7), if M has non-positive sectional curvatures, then the horizontal distibution $\mathcal{H}$ is integrable and the Riemannian manifold B has non-positive sectional curvature. Also, in [20], C. Pro and F. Wilhelm proved that there is no Riemannian submersion $\pi : M \to B$ to a space B with the non-positive Ricci curvature for any compact Riemannian manifold with the positive Ricci curvature.

Corollary 3.1. *Let* (M, g) *and* (B, g') *be Riemannian manifolds and* $\pi : M \to B$ *be a Riemannian submersion with totally geodesic fibres. Considering above Remark, one can see that the Riemannian submersion* π *preserves the positive Ricci curvature.*

In particular, if M *is an Einstein manifold and the equality case of* (3.8) *holds for any unit horizontal vector field* $X \in \chi^h(M)$, *then both manifolds* M *and* B *are flat.*

4. Chen-Ricci Inequality

In this section, we give our main notion of the present paper which is about Chen-Ricci inequality for Riemannian submersions. We study relations between the intrinsic and extrinsic invariants using fundamental tensors and obtain some characterizations for Riemannian submersions.

We begin to this section with some notions as follows.

Notation. Let $\pi : M \to B$ be a Riemannian submersion between Riemannian manifolds and $\{X_i, U_j\}_{1 \leq i \leq n; 1 \leq j \leq r}$ be a local orthonormal frame on M. Then, we recall the squared norms of invariant tensors as follows:

$$\|T^{\mathcal{H}}\|^2 = \sum_{i,j=1}^{r} g\left(T^{\mathcal{H}}(U_i, U_j), T^{\mathcal{H}}(U_i, U_j)\right),$$

$$\|T^{\mathcal{V}}\|^2 = \sum_{i=1}^{r} \sum_{j=1}^{n} g\left(T^{\mathcal{V}}(U_i, X_j), T^{\mathcal{V}}(U_i X_j)\right),$$

$$\|A^{\mathcal{H}}\|^2 = \sum_{i=1}^{r} \sum_{j=1}^{n} g\left(A^{\mathcal{H}}(X_j, U_i), A^{\mathcal{H}}(X_j, U_i)\right),$$

$$\|A^{\mathcal{V}}\|^2 = \sum_{i,j=1}^{n} g\left(A^{\mathcal{V}}(X_i, X_j), A^{\mathcal{V}}(X_i, X_j)\right).$$

Now, we need the following lemma to prove our main inequality.

Lemma 4.1. *Let (M, g) and (B, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. A local orthonormal frame $\{X_i, U_j\}_{1 \leq i \leq n; 1 \leq j \leq r}$ on M, such that the horizontal and vertical distributions are spanned by $\{X_i\}_{1 \leq i \leq n}$ and $\{U_j\}_{1 \leq j \leq r}$, respectively. Then, one has:*

$$(4.1) \qquad 2\tau(p) = 2\widehat{\tau}(p) + 2\tau^*(p) + r^2 \|H(p)\|^2 - \left\|T^{\mathcal{H}}\right\|^2 + 3\left\|A^{\mathcal{V}}\right\|^2$$
$$- \widecheck{\delta}(\mathcal{N}) + \left\|T^{\mathcal{V}}\right\|^2 - \left\|A^{\mathcal{H}}\right\|^2,$$

where

$$\widehat{\tau}(p) = \sum_{1 \leq i < j \leq r} \widehat{K}(U_i, U_j)$$

and

$$\tau^*(p) = \sum_{1 \leq i < j \leq n} K^*(X_i, X_j)$$

are the scalar curvatures of the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, respectively.

Proof. At any point $p \in M$, the scalar curvature $\tau(p)$ is given by

$$(4.2) \qquad \tau(p) = \frac{1}{2} \sum_{i,j=1}^{r} R(U_i, U_j, U_j, U_i) + \frac{1}{2} \sum_{i,j=1}^{n} R(X_i, X_j, X_j, X_i)$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} R(X_i, U_j, U_j, X_i).$$

If we consider equality (2.8) and use above notation in (4.2), the equality (4.1) is obtained. $\qquad \square$

Lemma 4.2. *Let (M, g) and (B, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. A local orthonormal frame $\{X_i, U_j\}_{1 \leq i \leq n; 1 \leq j \leq r}$ on*

M, such that the horizontal and vertical distributions are spanned by $\{X_i\}_{1\le i\le n}$ and $\{U_j\}_{1\le j\le r}$, respectively and at any point $p \in M$, we get

$$(4.3) \qquad \|T^{\mathcal{H}}\|^2 = \frac{1}{2}r^2\|H(p)\|^2 + \frac{1}{2}\sum_{s=1}^{n}(T_{11}^s - T_{22}^s - \cdots - T_{rr}^s)^2$$
$$+ 2\sum_{s=1}^{n}\sum_{j=2}^{r}(T_{1j}^s)^2 - 2\sum_{s=1}^{n}\sum_{2\le i<j\le n}\left(T_{ii}^s T_{jj}^s - (T_{ij}^s)^2\right),$$

where $T_{ij}^s = g\left(T^{\mathcal{H}}(U_i, U_j), X_s\right)$.

Theorem 4.1 (Chen-Ricci inequality). *Let (M, g) and (B, g') be a Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. A local orthonormal frame $\{X_i, U_j\}_{1\le i\le n; 1\le j\le r}$ on M, such that the horizontal and vertical distributions are spanned by $\{X_i\}_{1\le i\le n}$ and $\{U_j\}_{1\le j\le r}$, respectively.*
a) *For any unit vertical vector field $U \in \chi^v(M)$, we have*

$$(4.4) \qquad \mathrm{Ric}_{\mathcal{V}}(U) - \widehat{\mathrm{Ric}}(U) - \tau^*(p) \le \frac{1}{4}r^2\|H(p)\|^2 + \frac{1}{2}\|T^{\mathcal{V}}\|^2 + \frac{3}{2}\|A^{\mathcal{V}}\|^2 - \frac{1}{2}\check{\delta}(\mathcal{N}),$$

where

$$\mathrm{Ric}_{\mathcal{V}}(U) = \sum_{j=1}^{r} R(U, U_j, U_j, U).$$

b) *The equality case of (4.4) holds for a unit vertical vector field $U \in \chi^v(M)$ if and only if $A^{\mathcal{H}}$ vanishes identically and*

$$(4.5) \qquad T(U, V) = 0, \quad \text{for all } V \in \chi^h(M) \text{ orthogonol to } U,$$
$$T^{\mathcal{H}}(U, U) = \frac{r}{2}H(p)$$

are satisfied.
c) *For all unit vertical vector fields $U \in \chi^v(M)$, the equality case of (4.4) satisfies if and only if $A^{\mathcal{H}}$ vanishes identically and we have either:*
  (i) *if $r = 2$, π has totally umbilical fibres, or,*
  (ii) *if $r \ne 2$, π has totally geodesic fibres.*

Proof. If we put (4.3) in (4.1), we have

$$2\tau(p) = 2\hat{\tau}(p) + 2\tau^*(p) + \frac{1}{2}r^2\|H(p)\|^2 - \frac{1}{2}\sum_{s=1}^{n}(T_{11}^s - T_{22}^s - \cdots - T_{rr}^s)^2$$
$$- 2\sum_{s=1}^{n}\sum_{j=2}^{r}(T_{1j}^s)^2 + 2\sum_{s=1}^{n}\sum_{2\le i<j\le n}\left(T_{ii}^s T_{jj}^s - (T_{ij}^s)^2\right) + 3\left\|A^{\mathcal{V}}\right\|^2$$
$$- \check{\delta}(\mathcal{N}) + \left\|T^{\mathcal{V}}\right\|^2 - \left\|A^{\mathcal{H}}\right\|^2.$$

Since

$$\sum_{s=1}^{n}\sum_{2\le i,j\le n}\left(T_{ii}^s T_{jj}^s - (T_{ij}^s)^2\right) = \tau(p) - \mathrm{Ric}_{\mathcal{V}}(U_1) + \widehat{\mathrm{Ric}}(U_1) - \hat{\tau}(p),$$

we get

$$(4.6) \quad \tau^*(p) + \widehat{\mathrm{Ric}}(U_1) - \mathrm{Ric}_{\mathcal{V}}(U_1) \leq \frac{1}{4}r^2\|H(p)\|^2 + \frac{3}{2}\|A^{\mathcal{V}}\|^2 + \frac{1}{2}\|T^{\mathcal{V}}\|^2 + \frac{1}{2}\check{\delta}(\mathcal{N}),$$

Putting $U = U_1$ in (4.6), we obtain (4.4).

The equality case of (4.4) is valid for a unit vertical vector field $U \in \chi^v(M)$ if and only if $A^{\mathcal{H}}$ vanishes identically,

$$(4.7) \qquad T_{12}^s = \cdots = T_{1r}^s = 0 \quad \text{and} \quad T_{11}^s = T_{22}^s + \cdots + T_{rr}^s, \ s \in \{1, \ldots, n\},$$

which is equivalent to (4.5).

Now, we shall prove the next statement. Suppose that the equality case of (4.4) is valid for all unit vertical vector field $U \in \chi^v(M)$, since $T^{\mathcal{H}}$ is a symmetric operator on $\chi^{\mathcal{V}}(M) \times \chi^{\mathcal{V}}(M)$ and in view of (4.7), then we have

$$(4.8) \qquad\qquad\qquad T_{ij}^s = 0, \quad 2T_{ii}^s = T_{11}^s + T_{22}^s + \cdots T_{rr}^s,$$

for any $i \neq j \in \{1, \ldots, r\}$ and $s \in \{1, \ldots, n\}$. From (4.8), we get

$$(4.9) \qquad\qquad\qquad (r-2)(T_{11}^s + T_{22}^s + \cdots + T_{rr}^s) = 0.$$

Hence, either $T_{11}^s + \cdots + T_{rr}^s = 0$ or $r = 2$. If $T_{11}^s + T_{22}^s + \cdots + T_{rr}^s = 0$, then, from (4.8), we obtain $T_{ii}^s = 0$ for all $i \in \{1, \ldots, r\}$ and $s \in \{1, \ldots, n\}$. Also, using (4.9), one has $T_{ij}^s = 0$ for all $i, j \in \{1, \ldots, r\}$ and $s \in \{1, \ldots, n\}$ which implies that π has totally geodesic fibres at $p \in M$. If $r = 2$, then from (4.8) we have $2T_{11}^s = 2T_{22}^s = (T_{11}^s + T_{22}^s)$ for all $s \in \{1, \ldots, n\}$, which shows that π has totally umbilical fibres at $p \in M$. The proof of the converse is straightforward. $\qquad\qquad\square$

In particular case, we obtain the following.

Corollary 4.1. *Let $(\mathbb{R}^m, g)$ be m-dimensional Euclidean space and (B, g') be n-dimensional Riemanian manifold admitting a Riemannian submersion $\pi : \mathbb{R}^m \to B$. Then, one has as follows*
(i) For any unit vertical vector field $U \in \chi^v(\mathbb{R}^m)$, one has

$$(4.10) \qquad\qquad \mathrm{Ric}_{\mathcal{V}}(U) - \widehat{\mathrm{Ric}}(U) \leq \frac{1}{4}r^2\|H(p)\|^2 + \frac{3}{2}\|A^{\mathcal{V}}\|^2 - \frac{1}{2}\check{\delta}(\mathcal{N}).$$

(ii) The equality case of (4.10) holds for a unit vertical vector field $U \in \chi^v(\mathbb{R}^m)$ if and only if the equality (4.5) is satisfied.
(iii) The equality case of (4.10) holds for all unit vertical vector field $U \in \chi^v(\mathbb{R}^m)$ if and only if $A^{\mathcal{H}}$ vanishes, identically and either $r = 2$, π has totally umbilical fibres or π has totally geodesic fibres.

We recall here the following definition from [22].

Definition 4.1. Let $\pi : (M, J, g) \to (B, g')$ be an anti-invariant Riemannian submersion from almost Hermitian manifold to Riemannian manifold. Then, π is called a Lagrangian Riemannian submersion if dimension of the vertical distribution $\mathcal{V}$ is equal to the dimension of the horizontal distribution $\mathcal{H}$, i.e., $\dim(\ker \pi_*) = \dim(\ker \pi_*)^\perp$.

In this case, an almost complex structure J reverses the vertical and horizontal distributions, i.e., $J\mathcal{V} = \mathcal{H}$.

In particular case, above Chen-Ricci inequality for Lagrangian Riemannian submersion π is satisfied, as follows.

Corollary 4.2. *Let $\pi : (M, J, g) \to (B, g')$ be a Lagrangian Riemannian submersion from a Kähler manifold to Riemannian manifold. Then,*
(a) For any unit vertical vector field $U \in \chi^v(M)$, we have

$$\text{(4.11)} \qquad \text{Ric}_\mathcal{V}(U) - \widehat{\text{Ric}}(U) - \tau^*(p) \leq \frac{1}{4}n^2\|H(p)\|^2 + \frac{1}{2}\|T^\mathcal{V}\|^2 - \frac{1}{2}\check{\delta}(\mathcal{N}).$$

(b) The equality case of (4.11) holds for a unit vertical vector field $U \in \chi^v(M)$ if and only if $T^\mathcal{H}(U, V) = 0$, for any $V \in \chi^v(M)$ orthogonal to U and $T^\mathcal{H}(U, U) = \frac{n}{2}H(p)$, are satisfied.
(c) For any unit vector field $U \in \chi^v(M)$, the equality case of (4.11) holds if and only if we have either
  (i) if $n = 2$, one has

$$\widehat{\tau}(p) - n(n - 1)\|H(p)\|^2 = \tau^*(p),$$

  (ii) or, if $n \neq 2$,

$$\widehat{\tau}(p) - n^2\|H(p)\|^2 = \tau^*(p),$$

where $\widehat{\tau}(p)$ and $\tau^*(p)$ denote the scalar curvatures of the vertical and horizontal distributions of π, respectively.

Moreover, we here note that the fundamental tensor field A vanishes, identically in the theory of Lagrangian Riemannian submersion. Hence, one can see that the inequality (4.11) is satisfied for such a submersion which is given above relation (4.4) for Riemannian submersion.

5. SOME EXAMPLES OF RIEMANNIAN SUBMERSIONS SATISFY CHEN-RICCI INEQUALITY

In the last section, we give two examples which satisfy inequality (4.4).

Example 5.1. Let M be a submanifold of $\mathbb{R}^5$ with coordinates $\{x_1, x_2, x_3, x_4, x_5\}$, such that

$$\cot x_3 = \frac{x_1}{x_2}, \quad x_2 \neq 0, \quad x_3 \in \left(0, \frac{\pi}{2}\right).$$

Let us consider the mapping $\pi : M \to \mathbb{R}^3$ is given by

$$\pi(x_1, x_2, x_3, x_4, x_5) = (x_1 \cos x_3 + x_2 \cos x_3, x_4, x_5).$$

Then, the Jacobian matrix J of π is equal to

$$J = \begin{pmatrix} \sin x_3 & \cos x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since rank $J = \mathbb{R}^3$, the mapping π is a submersion. On the other hand, the horizontal space and the vertical space of M are as follows

$$\mathcal{H} = \text{Span} \left\{ X_1 = \sin x_3 \frac{\partial}{\partial x_1} + \cos x_3 \frac{\partial}{\partial x_2}, \ X_2 = \frac{\partial}{\partial x_4}, \ X_3 = \frac{\partial}{\partial x_5} \right\}$$

and

(5.1) $$\mathcal{V} = \text{Span} \left\{ U_1 = -\cos x_3 \frac{\partial}{\partial x_1} + \sin x_3 \frac{\partial}{\partial x_2}, \ U_2 = \frac{\partial}{\partial x_3} \right\},$$

respectively. Hence, it is clear that $\pi : (M, g) \to \mathbb{R}^3$ is a Riemannian submersion. By straightforward computations, we have

$$T^{\mathcal{V}}(U_2, X_1) = -U_1 \quad \text{and} \quad T^{\mathcal{H}}(U_1, U_2) = X_1.$$

Other components of operators $T^{\mathcal{H}}$, $T^{\mathcal{V}}$, $A^{\mathcal{H}}$ and $A^{\mathcal{V}}$ vanish identically. Furthermore,

$$\tau^*(p) = 0, \quad \widehat{\text{Ric}}(U_1) = 1, \quad \text{and} \ \text{Ric}_{\mathcal{V}}(U_1) = 0.$$

Thus, it is clear that the Riemannian submersion π in Example (5.1) which is satisfied inequality (4.4).

The next example is satisfied inequality (4.4) as follows.

Example 5.2. Let $\mathcal{C}$ be the catenoid given by the following parametrization

$$X(v, u) = (\cosh v \cos u, \cosh v \sin u, v)$$

and $\pi : \mathcal{C} \to B$ be a submersion such that the manifold B is the profile curve and the projection π is a mapping which carries $(\cosh v \cos u, \cosh v \sin u, v)$ to $(\cosh v, v)$. Then, the horizontal and vertical spaces of $\mathcal{C}$, respectively as follows

$$\mathcal{H} = \text{Span}\{X_v = (\sinh v \cos u, \sinh v \sin u, 1)\},$$
$$\mathcal{V} = \text{Span}\{X_u = (-\cosh v \sin u, \cosh v \cos u, 0)\}.$$

By straightforward computation, we obtain

$$\langle X_v, X_v \rangle = \cosh^2 v, \quad \langle X_u, X_v \rangle = 0, \quad \langle X_u, X_u \rangle = \cosh^2 v,$$

where $\langle , \rangle$ is the inner product of the induced metric g of $\mathbb{R}^3$. On the other hand,

$$\langle X_{vv}, X_v \rangle = -\langle X_{uu}, X_v \rangle = \langle X_{uv}, X_u \rangle = \cosh v \sinh v,$$
$$\langle X_{vv}, X_u \rangle = \langle X_{uv}, X_v \rangle = \langle X_{uu}, X_u \rangle = 0.$$

If we choose an orthonormal basis of $T_p\mathcal{C}$ as

$$\left\{ e_1 = \frac{1}{\cosh v} X_u, \quad e_2 = \frac{1}{\cosh v} X_v \right\},$$

we get

$$T^{\mathcal{H}}(e_1, e_1) = -\sinh v e_2, \quad T^{\mathcal{V}}(e_1, e_2) = \sinh v e_2, \quad \text{and} \ A^{\mathcal{V}}(e_2, e_2) = A^{\mathcal{H}}(e_2, e_1) = 0.$$

Then, the Gauss curvature of catenoid $\mathcal{C}$

$$R(e_1, e_2, e_2, e_1) = -\frac{1}{\cosh^4 v}$$

is obtained.

Acknowledgements. The authors would like to thank the reviewers for their valuable comments and constructive suggestions.

References

[1] P. Alegre, B.-Y. Chen, M. I. Munteanu, *Riemannian submersions, δ-invariants and optimal inequality*, Ann. Glob. Anal. Geom. **42** (2012), 317–331.

[2] M. Atçeken, *Anti-invariant Riemannian submersions from a locally Riemannian product manifold to any Riemannian manifold*, Gulf Journal of Mathematics **1**(2013), 25–35.

[3] A. L. Besse, *Einstein Manifolds*, Berlin-Heidelberg-New York, Spinger-Verlag, 1987.

[4] B.-Y. Chen, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*, Glasg. Math. J. **41**(1999), 33–41.

[5] B.-Y. Chen, *Riemannian submersions, minimal immersions and cohomology class*, Proc. Japan Acad. Ser. A. **81** (2005), 162–167.

[6] B.-Y. Chen, *A General optimal inequality for arbitrary Riemannian submanifolds*, Journal of Inequalities in Pure and Applied Mathematics **6**(3) (2005), Article ID: 77.

[7] B.-Y. Chen, *Pseudo-Riemannian Geometry, δ-Invariants and Applications*, World Scientific Publishing, Hackensack, NJ, 2011.

[8] B.-Y. Chen, S. W. Wei, *p-harmonic morphisms, cohomology classes and submersions*, Tamkang J. Math. **40** (2009), 377–382.

[9] S. Deng, *An improved Chen-Ricci inequality*, Int. Electron. J. of Geom. **2** (2009) 39–45.

[10] Ş. Eken, M. Gülbahar, E. Kılıç: *Some Inequalities for Riemannian submersions*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Math. (N.S.), (in press).

[11] M. Falciteli, S Ianus, A. M. Pastore, *Riemannian Submersions and Related Topics*, World Scientific Publishing Co. Pte. Ltd, 2004.

[12] S. Hong, K. Matsumoto, M. M. Tripathi, *Certain basic inequalities for submanifolds of locally conformal Kähler space forms*, SUT J. Math. **41**(1) (2005), 75–94.

[13] Y. Gündüzalp, *Slant submersions from almost product Riemannian manifolds*, Turkish. J. Math. **37** (2013), 863–873.

[14] S. Ianus, R. Mazzocco, G. E. Vilcu, *Riemannian submersions from quaternionic manifolds*, Acta Appl. Math. **104** (2008), 83–89.

[15] S. Ianus, A. M. Ionescu, R. Mazzocco, G. E. Vilcu, *Riemannian submersions from almost contact metric manifolds,* Abh. Math. Semin. Univ. Hambg. **81** (2011), 101–114.

[16] S. Ianus, A. M. Ionescu, R. Mocanu, G. E. Vilcu, *Riemannian Submersions from almost contact metric manifolds*, Abh. Math. Semin. Univ. Hambg. **81** (2011), 101–114.

[17] J. S. Kim, M. M. Tripathi, J. Choi, *Ricci curvature of submanifolds in locally conformal almost cosyplectic manifolds*, Indian J. Pure App. Math. **35** (2004), 259–271.

[18] A. Mihai, *Inequalities on the Ricci curvature*, J. of Math. Ineq. **9**(2015), 811–822.

[19] C. Özgür, C. Murathan, *Chen inequalities for submanifolds of a cosymplectic space form with a semi-symmetric metric connection*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Math. (N.S.) **2** (2012), 395–408.

[20] C. Pro, F. Wilhelm, *Riemannian submersions need not preserve positive Ricci curvature*, Proc. Amer. Math. Soc. **142** (2014), 2529–2535.

[21] B. Şahin, *Riemannian submersion from almost Hermitian manifolds*, Taiwan. J. Math. **17** (2013), 629–659.

[22] H. M. Taştan, *On Lagrangian submersions*, Hacet. J. Math. Stat. **43** (2014), 993–1000.

[23] M. M. Tripathi, *Chen-Ricci inequality for submanifolds of contact metric manifolds*, J. Adv. Math. Stud. **1** (2008), 111–134.

[24] G. E. Vilcu, *B.-Y. Chen inequalities for slant submanifolds in quaternionic space forms*, Turkish. J. Math. **34** (2010), 115–128.

[25] D. W. Yoon, *Inequality for Ricci curvature of slant submanifolds in cosymplectic space forms*, Turkish. J. Math. **30** (2006), 43–56.

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