Kragujevac Journal of Mathematics
Volume 42(2) (2018), Pages 229–237.

ON GRADIENT η-EINSTEIN SOLITONS

A. M. BLAGA[1]

Abstract. If the potential vector field of an η-Einstein soliton is of gradient type, using Bochner formula, we derive from the soliton equation a nonlinear second order PDE. Under certain conditions, the existence of an η-Einstein soliton forces the manifold to be of constant scalar curvature.

1. Introduction

In the same way like the Ricci solitons generate self-similar solutions to Ricci flow

$$\frac{\partial}{\partial t}g = -2S, \tag{1.1}$$

the notion of *Einstein solitons*, which generate self-similar solutions to Einstein flow

$$\frac{\partial}{\partial t}g = -2\left(S - \frac{\mathrm{scal}}{2}g\right), \tag{1.2}$$

was introduced by G. Catino and L. Mazzieri [4]. The interest in studying this equation from different points of view arises from concrete physical problems. On the other hand, gradient vector fields play a central role in Morse-Smale theory [16], aspects of gradient η-Ricci solitons being discussed in [3].

In what follows, after characterizing the manifold of constant scalar curvature via the existence of η-Einstein solitons, we focus on the case when the potential vector field ξ is of gradient type (i.e., $\xi = \mathrm{grad}(f)$, for f a nonconstant smooth function on M) and give the Laplacian equation satisfied by f. Under certain assumptions, the existence of an η-Einstein soliton implies that the manifold is quasi-Einstein. Remark that quasi-Einstein manifolds arose during the study of exact solutions of Einstein field equations.

Key words and phrases. Gradient η-Einstein soliton, gradient vector field, Laplace equation.
2010 *Mathematics Subject Classification.* Primary: 53C21, 53C25. Secondary: 53B50, 53C15.
Received: September 15, 2016.
Accepted: February 27, 2017.

2. η-Einstein soliton equation

In the study of the η-Einstein soliton equation we will consider certain assumptions, one essential condition being $\nabla\xi = I_{\chi(M)} - \eta \otimes \xi$ which naturally arises in different geometries: Kenmotsu [10], Lorenzian-Kenmotsu [1], para-Kenmotsu [12] etc. Immediate properties of this structure, which will be used later, are given in the next proposition.

Proposition 2.1. [3] *Let (M, g) be an m-dimensional Riemannian manifold and η be the g-dual 1-form of the nonzero vector field ξ. If ξ satisfies $\nabla\xi = I_{\chi(M)} - \eta \otimes \xi$, where ∇ is the Levi-Civita connection associated to g, then:*

(a) $\mathcal{L}_\xi g = 2(g - \eta \otimes \eta)$;
(b) $R(X, Y)\xi = \eta(X)Y - \eta(Y)X$, *for any* X, $Y \in \chi(M)$;
(c) $S(\xi, \xi) = (1 - m)|\xi|^2$.

Consider the equation

$$\mathcal{L}_\xi g + 2S + (2\lambda - \mathrm{scal})g + 2\mu\eta \otimes \eta = 0, \tag{2.1}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field, scal is the scalar curvature of the Riemannian metric g, and λ and μ are real constants. For $\mu \neq 0$, the data (g, ξ, λ, μ) will be called *η-Einstein soliton*. Remark that if the scalar curvature scal of the manifold is constant, then the η-Einstein soliton (g, ξ, λ, μ) reduces to an η-Ricci soliton $\left(g, \xi, \lambda - \frac{\mathrm{scal}}{2}, \mu\right)$ and, moreover, if $\mu = 0$, to a Ricci soliton $\left(g, \xi, \lambda - \frac{\mathrm{scal}}{2}\right)$. Therefore, the two concepts of η-Einstein soliton and η-Ricci soliton are distinct on manifolds of nonconstant scalar curvature.

Writing now $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain

$$2S(X, Y) = -g(\nabla_X\xi, Y) - g(X, \nabla_Y\xi) - (2\lambda - \mathrm{scal})g(X, Y) - 2\mu\eta(X)\eta(Y), \tag{2.2}$$

for any $X, Y \in \chi(M)$.

An important geometrical object in studying Ricci solitons is a symmetric $(0, 2)$-tensor field which is parallel with respect to the Levi-Civita connection [2], [7]. The existence of such tensors on smooth manifolds carrying different structures such as contact [14], K-contact [15], P-Sasakian [11], α-Sasakian [8] etc. was investigated by many authors. The starting point was the Eisenhart problem of finding symmetric and (skew symmetric) parallel tensors on various spaces. He proved in [9] that if a positive definite Riemannian manifold admits a second order parallel symmetric covariant tensor field other than a constant multiple of the metric, then it is reducible. In our case, we show that if we ask to be satisfied the condition $\nabla\xi = I_{\chi(M)} - \eta \otimes \xi$, then any symmetric ∇-parallel $(0, 2)$-tensor field must be a constant multiple of the metric. If we take $\alpha := \mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta$, the above mentioned result leads us to characterize the existence of the soliton (g, ξ, λ, μ) in terms of a scalar curvature property. Following the ideas of Călin and Crasmareanu [5] we shall study the equation (2.1), applying similar techniques.

Let α be such a symmetric $(0,2)$-tensor field which is parallel with respect to the Levi-Civita connection ($\nabla\alpha = 0$). From the Ricci identity $\nabla^2\alpha(X,Y;Z,W) - \nabla^2\alpha(X,Y;W,Z) = 0$, one obtains for any X, Y, Z, $W \in \chi(M)$ [13]

$$\alpha(R(X,Y)Z,W) + \alpha(Z,R(X,Y)W) = 0. \tag{2.3}$$

In particular, for $Z = W := \xi$ from the symmetry of α follows $\alpha(R(X,Y)\xi,\xi) = 0$, for any $X,Y \in \chi(M)$.

If $\nabla\xi = I_{\chi(M)} - \eta \otimes \xi$, from Proposition 2.1 we have $R(X,Y)\xi = \eta(X)Y - \eta(Y)X$ and replacing this expression in α we get

$$\alpha(Y,\xi) - \eta(Y)\alpha(\xi,\xi) = 0, \tag{2.4}$$

for any $Y \in \chi(M)$, equivalent to

$$\alpha(Y,\xi) - \alpha(\xi,\xi)g(Y,\xi) = 0, \tag{2.5}$$

for any $Y \in \chi(M)$. Differentiating the equation (2.5) covariantly with respect to the vector field $X \in \chi(M)$ we obtain

$$\alpha(\nabla_X Y,\xi) + \alpha(Y,\nabla_X\xi) = \alpha(\xi,\xi)[g(\nabla_X Y,\xi) + g(Y,\nabla_X\xi)],$$

and substituting the expression of $\nabla_X\xi = X - \eta(X)\xi$ we get

$$\alpha(Y,X) = \alpha(\xi,\xi)g(Y,X), \tag{2.6}$$

for any $X,Y \in \chi(M)$. As α is ∇-parallel, follows $\alpha(\xi,\xi)$ is constant and we conclude the following.

Proposition 2.2. *Under the hypotheses above, any parallel symmetric* $(0,2)$*-tensor field is a constant multiple of the metric.*

Applying these results, we conclude the following theorem.

Theorem 2.1. *Let η be the g-dual* 1*-form of the unitary vector field ξ on the Riemannian manifold (M,g) such that ξ satisfies $\nabla\xi = I_{\chi(M)} - \eta\otimes\xi$, where ∇ is the Levi-Civita connection associated to g. Assume that the symmetric* $(0,2)$*-tensor field $\alpha := \mathcal{L}_\xi g + 2S + 2\mu\eta\otimes\eta$ is parallel with respect to ∇. Then $\left(g,\xi,\lambda := -\frac{1}{2}[\alpha(\xi,\xi) - \mathrm{scal}],\mu\right)$ satisfies equation* (2.1) *if and only if M is of constant scalar curvature.*

Proof. Compute $\alpha(\xi,\xi)$ and from (2.1) we get

$$\alpha(\xi,\xi) = (\mathcal{L}_\xi g)(\xi,\xi) + 2S(\xi,\xi) + 2\mu\eta(\xi)\eta(\xi) = -2\lambda + \mathrm{scal},$$

and taking into account that $\nabla\alpha = 0$, we deduce that $\mathrm{scal} = c$ (a real constant), so $\lambda = -\frac{1}{2}[\alpha(\xi,\xi) - c]$.

Conversely, if $\mathrm{scal} = c$ (a real constant), from (2.6) and $\nabla\alpha = 0$ we get $\alpha(X,Y) = \alpha(\xi,\xi)g(X,Y) = -(2\lambda - c)g(X,Y)$, for any X, $Y \in \chi(M)$. Therefore, $\mathcal{L}_\xi g + 2S + 2\mu\eta\otimes\eta = -(2\lambda - c)g$, i.e. (g,ξ,λ,μ) satisfies equation (2.1). $\square$

The above condition we asked to be satisfied by the potential vector field ξ, namely $\nabla_X\xi = X-\eta(X)\xi$, naturally arises if $(M, \varphi, \xi, \eta, g)$ is for example, Kenmotsu manifold [10].

Example 2.1. Let $M = \{(x, y, z) \in \mathbb{R}^3, z > 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Set

$$\varphi := -\frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \quad \xi := -z\frac{\partial}{\partial z}, \quad \eta := -\frac{1}{z}dz,$$

$$g := \frac{1}{z^2}(dx \otimes dx + dy \otimes dy + dz \otimes dz).$$

Then (φ, ξ, η, g) is a Kenmotsu structure on M.

Consider the linearly independent system of vector fields:

$$E_1 := z\frac{\partial}{\partial x}, \quad E_2 := z\frac{\partial}{\partial y}, \quad E_3 := -z\frac{\partial}{\partial z}.$$

Follows

$$\varphi E_1 = -E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0,$$
$$\eta(E_1) = 0, \quad \eta(E_2) = 0, \quad \eta(E_3) = 1,$$
$$[E_1, E_2] = 0, \quad [E_2, E_3] = E_2, \quad [E_3, E_1] = -E_1,$$

and the Levi-Civita connection ∇ is deduced from Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) =& X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &- g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

precisely

$$\nabla_{E_1}E_1 = -E_3, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_1}E_3 = E_1,$$
$$\nabla_{E_2}E_1 = 0, \quad \nabla_{E_2}E_2 = -E_3, \quad \nabla_{E_2}E_3 = E_2,$$
$$\nabla_{E_3}E_1 = 0, \quad \nabla_{E_3}E_2 = 0, \quad \nabla_{E_3}E_3 = 0.$$

Then the Riemann and the Ricci curvature tensor fields are given by:

$$R(E_1, E_2)E_2 = -E_1, \quad R(E_1, E_3)E_3 = -E_1, \quad R(E_2, E_1)E_1 = -E_2,$$
$$R(E_2, E_3)E_3 = -E_2, \quad R(E_3, E_1)E_1 = -E_3, \quad R(E_3, E_2)E_2 = -E_3,$$
$$S(E_1, E_1) = S(E_2, E_2) = S(E_3, E_3) = -2,$$

and the scalar curvature $\mathrm{scal} = -6$.

From (2.1) computed in (E_i, E_i) follows

$$2[g(E_i, E_i) - \eta(E_i)\eta(E_i)] + 2S(E_i, E_i) + (2\lambda - \mathrm{scal})g(E_i, E_i) + 2\mu\eta(E_i)\eta(E_i) = 0,$$

for all $i \in \{1, 2, 3\}$, and we have

$$2(1 - \delta_{i3}) - 4 + 2\lambda + 6 + 2\mu\delta_{i3} = 0 \quad \Longleftrightarrow \quad 2\lambda + 4 + 2(\mu - 1)\delta_{i3} = 0,$$

for all $i \in \{1, 2, 3\}$. Therefore, $\mu = 1$ and $\lambda = -2$ define an η-Einstein soliton on $(M, \varphi, \xi, \eta, g)$.

The condition $\nabla_X \xi = X - \eta(X)\xi$ implies $\mathcal{L}_\xi g = 2(g - \eta \otimes \eta)$ and the equation (2.2) becomes

$$S(X, Y) = -\left(\lambda + 1 - \frac{\text{scal}}{2}\right) g(X, Y) - (\mu - 1)\eta(X)\eta(Y). \tag{2.7}$$

Recall that the manifold is called *quasi-Einstein* if the Ricci curvature tensor field S is a linear combination (with real scalars λ and μ respectively, with $\mu \neq 0$) of g and the tensor product of a nonzero 1-form η satisfying $\eta(X) = g(X, \xi)$, for ξ a unit vector field [6] and respectively, *Einstein* if S is collinear with g. Sufficient conditions for (M, g) to be quasi-Einstein or Einstein are given in the next two propositions.

Proposition 2.3. *Let η be the g-dual 1-form of the nonzero vector field ξ on the Riemannian manifold (M, g) such that ξ satisfies $\nabla \xi = I_{\chi(M)} - \eta \otimes \xi$, where ∇ is the Levi-Civita connection associated to g. If (g, ξ, λ, μ) satisfy the equation (2.1), then (M, g) is quasi-Einstein.*

Proof. It follows from (2.7). □

If we ask for certain curvature conditions, namely, $R(\xi, X) \cdot S = 0$ and $S(\xi, X) \cdot R = 0$, we deduce that M is either Einstein or get its scalar curvature depending on the constants (λ, μ) that define the η-Einstein soliton on M respectively, where by $\cdot$ we denote the derivation of the tensor algebra at each point of the tangent space:

- $(R(\xi, X) \cdot S)(Y, Z) := ((\xi \wedge_R X) \cdot S)(Y, Z) := S((\xi \wedge_R X)Y, Z) + S(Y, (\xi \wedge_R X)Z)$, for $(X \wedge_R Y)Z := R(X, Y)Z$;
- $S((\xi, X) \cdot R)(Y, Z)W := ((\xi \wedge_S X) \cdot R)(Y, Z)W := (\xi \wedge_S X)R(Y, Z)W + R((\xi \wedge_S X)Y, Z)W + R(Y, (\xi \wedge_S X)Z)W + R(Y, Z)(\xi \wedge_S X)W$, for $(X \wedge_S Y)Z := S(Y, Z)X - S(X, Z)Y$.

Proposition 2.4. *Let η be the g-dual 1-form of the nonzero and nonunitary vector field ξ on the Riemannian manifold (M, g) such that ξ satisfies $\nabla \xi = I_{\chi(M)} - \eta \otimes \xi$, where ∇ is the Levi-Civita connection associated to g. If (g, ξ, λ, μ) satisfy the equation (2.1) and $R(\xi, X) \cdot S = 0$, then (M, g) is Einstein manifold.*

Proof. The condition that must be satisfied by S is:

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0, \tag{2.8}$$

for any X, Y, $Z \in \chi(M)$.

Replacing the expression of S from (2.7) and from the symmetries of R we get

$$(\mu - 1)[\eta(Y)g(X, Z) + \eta(Z)g(X, Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0, \tag{2.9}$$

for any $X, Y, Z \in \chi(M)$.

For $X = Y = Z := \xi$ we have

$$(\mu - 1)[\eta(\xi)]^2[\eta(\xi) - 1] = 0, \tag{2.10}$$

which implies $\mu = 1$. □

Remark 2.1. Under the hypotheses of Proposition 2.4, there is no Ricci soliton with the potential vector field ξ.

Proposition 2.5. *Let η be the g-dual 1-form of the nonzero and nonunitary vector field ξ on the Riemannian manifold (M, g) such that ξ satisfies $\nabla\xi = I_{\chi(M)} - \eta \otimes \xi$, where ∇ is the Levi-Civita connection associated to g. If (g, ξ, λ, μ) satisfy the equation* (2.1) *and $S(\xi, X) \cdot R = 0$, then (λ, μ) satisfy $2(\lambda + 1) + |\xi|^2(\mu - 1) = \mathrm{scal}$.*

Proof. The condition that must be satisfied by S is:

$$\begin{aligned}
&S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W \\
&-S(\xi, Y)R(X, Z)W + S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W \\
&+S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0,
\end{aligned} \tag{2.11}$$

for any $X, Y, Z, W \in \chi(M)$.

Taking the inner product with ξ, the relation (2.11) becomes

$$\begin{aligned}
&S(X, R(Y, Z)W)|\xi|^2 - S(\xi, R(Y, Z)W)\eta(X) + S(X, Y)\eta(R(\xi, Z)W) \\
&-S(\xi, Y)\eta(R(X, Z)W) + S(X, Z)\eta(R(Y, \xi)W) - S(\xi, Z)\eta(R(Y, X)W) \\
&+S(X, W)\eta(R(Y, Z)\xi) - S(\xi, W)\eta(R(Y, Z)X) = 0,
\end{aligned} \tag{2.12}$$

for any $X, Y, Z, W \in \chi(M)$.

For $W := \xi$ and from the symmetries of R we get

$$S(X, R(Y, Z)\xi)|\xi|^2 - S(\xi, R(Y, Z)\xi)\eta(X) + S(\xi, \xi)g(R(Y, Z)\xi, X) = 0, \tag{2.13}$$

for any $X, Y, Z \in \chi(M)$.

Replacing the expression of S from (2.7), we get

$$|\xi|^2[2\lambda + 2 - \mathrm{scal} + (\mu - 1)|\xi|^2][\eta(Y)g(X, Z) - \eta(Z)g(X, Y)] = 0, \tag{2.14}$$

for any $X, Y, Z \in \chi(M)$.

For $Z := \xi$ we have

$$|\xi|^2[2\lambda + 2 - \mathrm{scal} + (\mu - 1)|\xi|^2][\eta(X)\eta(Y) - |\xi|^2 g(X, Y)] = 0, \tag{2.15}$$

for any $X, Y \in \chi(M)$ and we obtain

$$2\lambda + 2 - \mathrm{scal} + (\mu - 1)|\xi|^2 = 0, \tag{2.16}$$

which is stated. □

Corollary 2.1. *Let η be the g-dual 1-form of the nonzero and nonunitary vector field ξ on the Riemannian manifold (M, g) such that ξ satisfies $\nabla\xi = I_{\chi(M)} - \eta \otimes \xi$, where ∇ is the Levi-Civita connection associated to g. If $(g, \xi, \lambda, 0)$ satisfy the equation* (2.1) *and $S(\xi, X) \cdot R = 0$, then $\lambda = \frac{|\xi|^2 + \mathrm{scal}}{2} - 1$.*

3. Gradient η-Einstein solitons

We are interested in gradient η-Einstein solitons, as solutions of the equation

$$\mathcal{L}_\xi g + 2S + (2\lambda - \mathrm{scal})g + 2\mu\eta \otimes \eta = 0, \tag{3.1}$$

where g is a Riemannian metric, S is the Ricci curvature, scal is the scalar curvature, η is a 1-form whose g-dual vector field ξ is of gradient type, $\xi := \mathrm{grad}(f)$, for f a smooth function on M, and λ and μ are real constants ($\mu \neq 0$). The data (g, ξ, λ, μ) which satisfy the equation (3.1) is said to be a *gradient η-Einstein soliton* on M.

Taking the trace of the relation (3.1), applying then ∇_ξ and observing that if $\xi = \sum_{i=1}^m \xi^i E_i$, for $\{E_i\}_{1\leq i\leq m}$ a local orthonormal frame field with $\nabla_{E_i} E_j = 0$ in a point,

$$\begin{aligned}\mathrm{trace}(\eta\otimes\eta) &= \sum_{i=1}^m [df(E_i)]^2 = \sum_{1\leq i,j,k\leq m} \xi^j\xi^k g(E_i,E_j)g(E_i,E_k) = \sum_{i=1}^m (\xi^i)^2 = \\ &= \sum_{1\leq i,j\leq m} \xi^i\xi^j g(E_i,E_j) = |\xi|^2,\end{aligned}$$

we get

$$\xi(\mathrm{div}(\xi)) + \left(1 - \frac{m}{2}\right)\nabla_\xi(\mathrm{scal}) + \mu\xi(|\xi|^2) = 0, \tag{3.2}$$

and because $\mathcal{L}_\xi g = 2\,\mathrm{Hess}(f)$, taking the divergence of the same relation and computing it in ξ we obtain

$$(\mathrm{div}(\mathcal{L}_\xi g))(\xi) + 2(\mathrm{div}(S))(\xi) - \mathrm{trace}(d(\mathrm{scal})\otimes d(\mathrm{scal})) + 2\mu(\mathrm{div}(df\otimes df))(\xi) = 0. \tag{3.3}$$

From (3.1) we deduce

$$S(\xi,\xi) = -\frac{1}{2}\xi(|\xi|^2) - \lambda|\xi|^2 + \frac{\mathrm{scal}}{2}|\xi|^2 - \mu|\xi|^4. \tag{3.4}$$

Multiplying (3.3) by $1-\frac{m}{2}$, substracting (3.2), using the fact that $\nabla(\mathrm{scal}) = 2\,\mathrm{div}(S)$, we get

$$(\mathrm{div}(\mathcal{L}_\xi g))(\xi) = \Delta(|\xi|^2) - 2|\nabla\xi|^2, \tag{3.5}$$

$$\Delta(|\xi|^2) - 2|\nabla\xi|^2 = 2S(\xi,\xi) + 2\xi(\mathrm{div}(\xi)), \tag{3.6}$$

$$(\mathrm{div}(df\otimes df))(\xi) = \frac{1}{2}\mathrm{trace}(df\otimes d(|\xi|^2)) + |\xi|^2\,\mathrm{div}(\xi), \tag{3.7}$$

and with (3.4), we get

$$\begin{aligned}&\left(\frac{1-m}{2}\right)\Delta(|\xi|^2) \\ &= (1-m)|\nabla\xi|^2 + \frac{1}{2}\xi(|\xi|^2) + \lambda|\xi|^2 - \frac{\mathrm{scal}}{2}|\xi|^2 + \left(1-\frac{m}{2}\right)\cdot\mathrm{trace}(d(\mathrm{scal})\otimes d(\mathrm{scal})) \\ &\quad + \mu\left\{|\xi|^4 - \left(1-\frac{m}{2}\right)\cdot\mathrm{trace}(df\otimes d(|\xi|^2)) - 2\left(1-\frac{m}{2}\right)|\xi|^2\,\mathrm{div}(\xi) + \xi(|\xi|^2)\right\}.\end{aligned} \tag{3.8}$$

Theorem 3.1. *Let* (M, g) *be an* m*-dimensional Riemannian manifold* $(m > 2)$ *and* η *be the* g*-dual* 1*-form of the gradient vector field* $\xi := \mathrm{grad}(f)$*. Assume that* (3.1) *defines an* η*-Einstein soliton on* M *and* ξ *satisfies* $\nabla\xi = I_{\chi(M)} - \eta \otimes \xi$*, where* ∇ *is the Levi-Civita connection associated to* g*. If* $\mu = 1$*, then* M *is of constant scalar curvature; if* $\mu \neq 1$*, then* M *is of constant scalar curvature if and only if* ξ *is of constant length, and in this case, the Laplacian equation becomes*

$$\Delta(f) = \frac{m-1}{\mu}. \tag{3.9}$$

Proof. We have

$$\xi(|\xi|^2) = 2(|\xi|^2 - |\xi|^4), \tag{3.10}$$

and

$$\xi(|\xi|^4) = 4(|\xi|^4 - |\xi|^6), \tag{3.11}$$

and from (2.7) we get

$$S(\xi, \xi) = -\left(\lambda + 1 - \frac{\mathrm{scal}}{2}\right)|\xi|^2 - (\mu - 1)|\xi|^4. \tag{3.12}$$

Also from Proposition 2.1:

$$S(\xi, \xi) = |\xi|^2 - m|\xi|^2, \tag{3.13}$$

therefore

$$|\xi|^2 = \left(m - 1 - \lambda + \frac{\mathrm{scal}}{2}\right)|\xi|^2 - (\mu - 1)|\xi|^4. \tag{3.14}$$

We obtain

$$|\xi|^2(\mu - 1) = m - 2 - \lambda + \frac{\mathrm{scal}}{2}. \tag{3.15}$$

If $\mu = 1$, then from (3.15) we obtain $\mathrm{scal} = 2(\lambda + 2 - m)$, i.e. M is of constant scalar curvature.

Let $\mu \neq 1$. If the scalar curvature is constant, then $|\xi|$ is constant. Conversely, if ξ is of constant length, from (3.10) follows $|\xi| = 1$ and from (3.15) we obtain $\mathrm{scal} = 2(\lambda + \mu + 1 - m)$, i.e. M is of constant scalar curvature.

We also have

$$|\nabla\xi|^2 := \sum_{i=1}^{m} g(\nabla_{E_i}\xi, \nabla_{E_i}\xi) = \sum_{i=1}^{m}\{1 + (|\xi|^2 - 2)[\eta(E_i)]^2\} = m + |\xi|^2(|\xi|^2 - 2), \tag{3.16}$$

for $\{E_i\}_{1 \leq i \leq m}$ a local orthonormal frame field with $\nabla_{E_i}E_j = 0$ in a point.

Now using the relations above, (3.8) becomes (3.9). □

Remark that in this case, the soliton (3.1) is completely determined by f, m and scal.

Example 3.1. The soliton considered in Example 2.1 is a gradient η-Einstein soliton, as the potential vector field ξ is of gradient type, $\xi = \mathrm{grad}(f)$, where $f(x, y, z) := -\ln z$.

Acknowledgements. The author thanks to the referees for their valuable suggestions that definitely improved the paper.

References

[1] N. S. Basavarajappa, C. S. Bagewadi and D. G. Prakasha, *Some results on Lorentzian β-Kenmotsu manifolds*, An. Univ. Craiova Ser. Mat. Inform. **35** (2008), 7–14.

[2] C. L. Bejan and M. Crasmareanu, *Ricci solitons in manifolds with quasi-constant curvature*, Publ. Math. Debrecen **78**(1) (2011), 235–243.

[3] A. M. Blaga, *On warped product gradient η-Ricci solitons*, arXiv:1705.04092 (2017).

[4] G. Catino and L. Mazzieri, *Gradient Einstein solitons*, Nonlinear Anal. **132** (2016), 66–94.

[5] C. Călin and M. Crasmareanu, *Eta-Ricci solitons on Hopf hypersurfaces in complex space forms*, Rev. Roumaine Math. Pures Appl. **57**(1) (2012), 55–63.

[6] M. C. Chaki and R. K. Maity, *On quasi Einstein manifolds*, Publ. Math. Debrecen **57** (2000), 297–306.

[7] M. Crasmareanu, *Parallel tensors and Ricci solitons in $N(k)$-quasi Einstein manifolds*, Indian J. Pure Appl. Math. **43**(4) (2012), 359–369.

[8] L. Das, *Second order parallel tensor on α-Sasakian manifold*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **23**(1) (2007), 65–69.

[9] L. P. Eisenhart, *Symmetric tensors of the second order whose first covariant derivatives are zero*, Trans. Amer. Math. Soc. **25**(2) (1923), 297–306.

[10] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J. **24** (1972), 93–103.

[11] Z. Li, *Second order parallel tensors on P-Sasakian manifolds with a coefficient k*, Soochow J. Math. **23**(1) (1997), 97–102.

[12] Z. Olszak, *The Schouten-van Kampen affine connection adapted to an almost (para) contact metric structure*, Publ. Inst. Math. (Beograd) (N.S.) **94**(108) (2013), 31–42.

[13] R. Sharma, *Second order parallel tensor in real and complex space forms*, Int. J. Math. Math. Sci. **12**(4) (1989), 787–790.

[14] R. Sharma, *Second order parallel tensor on contact manifolds*, Algebras Groups Geom. **7** (1990), 787–790.

[15] R. Sharma, *Second order parallel tensor on contact manifolds II*, C. R. Math. Acad. Sci. Soc. R. Can. **13**(6) (1991), 259–264.

[16] S. Smale, *On gradient dynamical systems*, Ann. Math. **74**(1) (1961), 199–206.

[1]Department of Mathematics,
West University of Timişoara,
Bld. V. Pârvan nr. 4, 300223, Timişoara, România
E-mail address: adarablaga@yahoo.com