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LAPLACIAN ENERGY OF GENERALIZED COMPLEMENTS OF A GRAPH

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Abstract. Let $P = \{V_1, V_2, V_3, \dots, V_k\}$ be a partition of vertex set $V(G)$ of order $k \geq 2$. For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j in graph G and add the edges between V_i and V_j which are not in G. The graph G_k^P thus obtained is called the $k-$*complement* of graph G with respect to a partition P. For each set V_r in P, remove the edges of graph G inside V_r and add the edges of $\overline{G}$ (the complement of G) joining the vertices of V_r. The graph $G_{k(i)}^P$ thus obtained is called the $k(i)-$*complement of graph* G with respect to a partition P. In this paper, we study Laplacian energy of generalized complements of some families of graph. An effort is made to throw some light on showing variation in Laplacian energy due to changes in a partition of the graph.

1. Introduction

Let G be a graph on n vertices and m edges. The *complement of a graph* G, denoted as $\overline{G}$ has the same vertex set as that of G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. If G is isomorphic to $\overline{G}$ then G is said to be *self-complementary graph.* For all notations and terminologies we refer [2, 15, 21]. E. Sampathkumar et al. have introduced two types of generalized complements [19] of a graph. For completeness we produce these here.

Let $P = \{V_1, V_2, V_3, \dots, V_k\}$ be a partition of vertex set $V(G)$ of order $k \geq 2$. For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j in graph G and add the edges between V_i and V_j which are not in G. The graph G_k^P thus obtained is called the $k-$*complement* of graph G with respect to a partition P. For each set V_r in P, remove the edges of graph G inside V_r and add the edges of $\overline{G}$ joining the vertices

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of V_r. The graph $G^P_{k(i)}$ thus obtained is called the $k(i)$*-complement of graph* G with respect to a partition P.

The energy of the graph is first defined by Ivan Gutman [10] in 1978 as the sum of absolute eigenvalues of graph G. For more information on energy of a graph we refer [1,3,5,7–9,12,13,17,18,20]. Let $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees, and $A(G)$ is the adjacency matrix. Then $L(G) = D(G) - A(G)$ is the Laplacian matrix of graph G. The characteristic polynomial of the Laplacian matrix is denoted by $\phi(L(G), \mu) = \det(\mu I_n - L(G))$. Let $\{\mu_1, \mu_2, \cdots, \mu_n\}$ be the Laplacian eigenvalues of graph G, i.e., the roots of $\phi(L(G), \mu)$. The Laplacian energy[14], denoted by $LE(G)$, is defined as

$$LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|.$$

The Laplacian energy $LE(G)$ is a very recently defined graph invariant. The basic properties for Laplacian energy have been established in [4, 6, 11, 14, 22, 23], and it has found remarkable chemical applications. In this paper we study the Laplacian energy of generalized complements of some classes of graphs.

2. Preliminaries

Proposition 2.1. [19] *The* k*-complement and* $k(i)$ *complement of* G *are related as follows:*

(i) $\overline{G^P_k} \cong G^P_{k(i)}$;
(ii) $\overline{G^P_{k(i)}} \cong G^P_k$.

Theorem 2.1. [6] *Let* G *be a graph with* n *vertices and* $\overline{G}$ *be its complement. If the Laplacian spectrum of* G *is* $\{\mu_1, \mu_2, \ldots, \mu_n\}$*, then the Laplacian spectrum of* $\overline{G}$ *is* $\{n - \mu_{n-1}, n - \mu_{n-2}, \ldots, n - \mu_1, 0\}$.

Definition 2.1. [6] Let f_i, $i = 1, 2, \ldots, k$, $1 \leq k \leq \left\lfloor \frac{p}{2} \right\rfloor$, be independent edges of the complete graph K_p, $p \geq 3$. The graph $Kb_p(k)$ is obtained by deleting f_i, $i = 1, 2, \ldots, k$, from K_p. In addition $Kb_p(0) \cong K_p$.

Proposition 2.2. [6] *For* $p \geq 3$ *and* $0 \leq k \leq \left\lfloor \frac{p}{2} \right\rfloor$,

$$LE(Kb_p(k)) = (2p - 2) + \left(2 - \frac{4}{p}\right) k - \frac{4k^2}{p}.$$

Lemma 2.1. [16] *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a 2×2 *block symmetric matrix. Then the eigenvalues of* A *are the eigenvalues of the matrices* $A_0 + A_1$ *and* $A_0 - A_1$.

3. Laplacian energy of generalized complements of classes of graphs

Now we find Laplacian energy of generalized complements of some standard graphs like complete, complete bipartite, path, cycle, double star, friendship and cocktail party graph. For some graphs we take partition of order k and for some partition of order two.

Theorem 3.1. *Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of the complete graph K_n.*

(i) *If $< V_i >= K_i$, for $i = 1, 2, \ldots, n$, then*

$$LE\left((K_n)_k^P\right) = \frac{2k\sum\limits_{i=1}^{k} {}^{n_i}C_2}{n} + \sum_{i=1}^{k}(n_i - 1)\left|n_i - \frac{2k\sum\limits_{i=1}^{k} {}^{n_i}C_2}{n}\right|$$

and

$$LE\left((K_n)_{k(i)}^P\right) = n(k-1) + (k+2)\left(\frac{2\left({}^{n}C_2 - \sum\limits_{i=1}^{k} {}^{n_i}C_2\right)}{n}\right) + \sum_{i=1}^{k}(n_i - 1)\left|(n - n_i) - \frac{2\left({}^{n}C_2 - \sum\limits_{i=1}^{k} {}^{n_i}C_2\right)}{n}\right|.$$

(ii) *If $|V_i| = 2$ and one of $|V_i| = 1$ when n is odd, then*

$$LE\left((K_n)_k^P\right) = \frac{4k(n-k)}{n} \quad \text{and} \quad LE\left((K_n)_{k(i)}^P\right) = \frac{2(3n-2k)(k-1)}{n}.$$

Proof. For a partition $P = \{V_1, V_2, \ldots, V_k\}$, let $G = (K_n)_k^P$ be a graph.

(i) If $< V_i >= K_i$, for $i = 1, 2, \ldots, n$, then G is the union of k disconnected complete subgraphs of order n_i such that $\sum\limits_{i=1}^{n} n_i = n$. If $n_i \geq 2$, the Laplacian spectrum of K_{n_i} consists of $\{0, n_i(n_{i-1} \text{ times})\}$, $i = 1, 2, \ldots, n$.

Then the Laplacian spectrum of $(K_n)_k^P$ is $\begin{Bmatrix} 0 & n_1 & n_2 & \ldots & n_k \\ k & n_1 - 1 & n_2 - 1 & \ldots & n_k - 1 \end{Bmatrix}$ and the average degree of $(K_n)_k^P = \frac{2\sum\limits_{i=1}^{k} {}^{n_i}C_2}{n}$,

$$LE\left((K_n)_k^P\right) = \frac{2k\sum\limits_{i=1}^{k} {}^{n_i}C_2}{n} + \sum_{i=1}^{k}(n_i - 1)\left|n_i - \frac{2k\sum\limits_{i=1}^{k} {}^{n_i}C_2}{n}\right|,$$

where $|V_i| = n_i$, $i = 1, 2, \ldots, k$.

By noting that $(K_n)^P_{k(i)} = \overline{(K_n)^P_k}$ and from Theorem 2.1, we obtain Laplacian spectrum of $(K_n)^P_{k(i)}$ is $\begin{Bmatrix} 0 & n & n-n_1 & n-n_2 & \dots & n-n_k \\ 1 & k-1 & n_1-1 & n_2-1 & \dots & n_k-1 \end{Bmatrix}$ and the average degree of $(K_n)^P_{k(i)}$ is $\frac{2\left({}^nC_2-\sum_{i=1}^{k}{}^{n_i}C_2\right)}{n}$,

$$LE\left((K_n)^P_{k(i)}\right) = n(k-1) + (k+2)\left(\frac{2\left({}^nC_2 - \sum_{i=1}^{k} {}^{n_i}C_2\right)}{n}\right)$$

$$+ \sum_{i=1}^{k}(n_i-1)\left|(n-n_i) - \frac{2\left({}^nC_2 - \sum_{i=1}^{k} {}^{n_i}C_2\right)}{n}\right|,$$

where $|V_i| = n_i$, $i = 1, 2, \dots, k$.

(ii) If $|V_i| = 2$, then $k = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even}, \\ \frac{n+1}{2}, & \text{if } n \text{ is odd}. \end{cases}$
It follows by substituting the value of k in Theorem 3.1 of statement (i). Also, the Laplacian spectrum of G is given by $\{0(k \text{ times}), 2(n-k \text{ times})\}$. Average degree of G is $\frac{2m}{n} = \frac{2(n-k)}{n}$. Thus,

$$\begin{aligned} LE(G) &= \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| \\ &= k\left|\frac{2(n-k)}{n}\right| + (n-k)\left|2 - \frac{2(n-k)}{n}\right| \\ &= \frac{2k(n-k)}{n} + \frac{2k(n-k)}{n} \\ &= \frac{4k(n-k)}{n}, \quad k \geq 1. \end{aligned}$$

Hence, Laplacian spectrum of $\overline{G}$ consists of $\{0, (n-2)(n-k \text{ times}), n(k-1 \text{ times})\}$. Average degree of G is $\frac{2m}{n} = \frac{n(n-1)-2(n-k)}{n}$.

$$\begin{aligned} LE(\overline{G}) &= \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| \\ &= \left|\frac{n(n-1)-2(n-k)}{n}\right| \\ &\quad + (n-k)\left|(n-2) - \frac{n(n-1)-2(n-k)}{n}\right| \end{aligned}$$

$$+ (k-1)\left|n - \frac{n(n-1)-2(n-k)}{n}\right|$$
$$=\frac{2(3n-2k)(k-1)}{n}, \quad k \geq 1. \qquad \square$$

Remark 3.1. Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of the complete graph K_n with $|V_i| = 2$ and one of $|V_i| = 1$, if n is odd. Then,

$$LE\left((K_n)_k^P\right) + LE\left((K_n)_{k(i)}^P\right) = \begin{cases} 3n-4, & \text{if } n \text{ is even,} \\ 3(n-1), & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 3.2. *Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of path P_n.*

(i) *If any one of the pendant vertex is in V_1 or V_k, and remaining $k-1$ sets are K_2's, then*

$$LE\left((P_n)_{k(i)}^P\right) = \frac{2}{n}\{k(k-1) + (n-k)(n-k+1)\},$$

whereas

$$LE\left((P_n)_k^P\right) = LE\left[Kb_n\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right] = 2n-2+\left(2-\frac{4}{n}\right)\left\lfloor\frac{n}{2}\right\rfloor - 4\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^2,$$

for odd $n \geq 3$.

(ii) *If any one of the non pendant vertex is in V_i, $3 \leq i \leq n-2$ and remaining $k-1$ sets are K_2's, then*

$$LE\left((P_n)_{k(i)}^P\right) = \frac{2}{n}[(n-k)^2 + k(k-2) + n],$$

whereas

$$LE\left((P_n)_k^P\right) = \frac{2}{n}[n(n-k) + 2k(k-1)], \quad k = \left\lceil\frac{n}{2}\right\rceil \text{ for odd } n \geq 5.$$

(iii) *If $< V_i >= K_2$, then*

$$LE\left((P_n)_{k(i)}^P\right) = \frac{4}{n}(k-1)(n-k+1) \quad \text{and} \quad LE\left((P_n)_k^P\right) = 2(n-1),$$

for even $n \geq 2$.

Proof. (i) If any one of the pendant vertex is in V_1 or V_k, and remaining $k-1$ sets are K_2's, then $(P_n)_{k(i)}^P$ is the union of $(k-1)$ K_2's and one isolated vertex. Laplacian spectrum of $(P_n)_{k(i)}^P$ is $\{0(k \text{ times}), 2(n-k \text{ times})\}$. The average degree of $(P_n)_{k(i)}^P$ is $\frac{2}{n}(k-1)$, i.e.,

$$LE\left((P_n)_{k(i)}^P\right) = k\left(\frac{2}{n}(k-1)\right) + (n-k)\left|2 - \frac{2}{n}(k-1)\right|$$
$$=\frac{2}{n}[(n-k)^2 + k(k-2) + n].$$

Note that $(P_n)_k^P$ is the graph obtained from K_n by deleting all the independent edges, $(P_n)_k^P = Kb_n\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$. Hence, from Proposition 2.2,

$$LE\left((P_n)_k^P\right) = LE\left[Kb_n\left(\left\lfloor \frac{n}{2} \right\rfloor\right)\right] = 2n - 2 + \left(2 - \frac{4}{n}\right)\left\lfloor \frac{n}{2} \right\rfloor - 4\left(\left\lfloor \frac{n}{2} \right\rfloor\right)^2.$$

(ii) If any one of the non pendant vertex is in V_i, $3 \leq i \leq n-2$ and remaining $k-1$ sets are K_2's, then $(P_n)_{k(i)}^P$ is the union of $K_{1,2}$, $(n-k-2)$ K_2's and two isolated vertices. Hence, Laplacian spectrum of $(P_n)_{k(i)}^P$ is $\{0(k \text{ times}), 1, 2(n-k-2 \text{ times}), 3\}$. The average degree of $(P_n)_{k(i)}^P$ is $\frac{2(k-1)}{n}$, i.e.,

$$\begin{aligned}LE\left((P_n)_{k(i)}^P\right) =& k\left(\frac{2}{n}(k-1)\right) + (n-k-2)\left|2 - \frac{2}{n}(k-1)\right| \\ &+ \left|1 - \frac{2}{n}(k-1)\right| + \left|3 - \frac{2}{n}(k-1)\right| \\ =& \frac{2}{n}[(n-k)^2 + k(k-2) + n].\end{aligned}$$

Also from Theorem 2.1, Laplacian spectrum of $(P_n)_k^P$ is $\{0, n-3, n-2(n-k-2 \text{ times}), n-1, n(k-1 \text{ times})\}$. Average degree of $(P_n)_k^P$ is $\frac{n(n-1)-2(k-1)}{n}$. Hence,

$$\begin{aligned}LE\left((P_n)_k^P\right) =& \frac{n(n-1)-2(k-1)}{n} + \left|(n-3) - \frac{n(n-1)-2(k-1)}{n}\right| \\ &+ (n-k-2)\left|(n-2) - \frac{n(n-1)-2(k-1)}{n}\right| \\ &+ \left|(n-1) - \frac{n(n-1)-2(k-1)}{n}\right| \\ &+ (k-1)\left|n - \frac{n(n-1)-2(k-1)}{n}\right| \\ =& \frac{2}{n}[(n-k)^2 + k(k-2) + n].\end{aligned}$$

(iii) If $< V_i >= K_2$, then $(P_n)_{k(i)}^P$ is the union of $(k-1)$ K_2's and two isolated vertices. Its Laplacian spectrum is $\{0(n-k+1 \text{ times}), 2(k-1 \text{ times})\}$ and average degree is $\frac{2}{n}(k-1)$. Hence,

$$\begin{aligned}LE\left((P_n)_{k(i)}^P\right) &= (n-k+1)\frac{2}{n}(k-1) + (k-1)\left|2 - \frac{2}{n}(k-1)\right| \\ &= \frac{4}{n}(k-1)(n-k+1).\end{aligned}$$

Also from Theorem 2.1, Laplacian spectrum of $(P_n)_k^P$ is $\{0, n-2(n-k-1 \text{ times}), n(k \text{ times})\}$. Average degree of $(P_n)_k^P$ is $\frac{n(n-1)-2(k-1)}{n}$. Thus,

$$LE\left((P_n)_k^P\right) = \frac{1}{n}[n(n+k-1)+2(k-1)^2+(n-k-1)(n-2(k-1))].$$

Note that $k = \frac{n}{2}$ for path of even order, hence, we obtain, $LE\left((P_n)_k^P\right) = 2(n-1)$. □

Remark 3.2. As $k = \frac{n}{2}$ for path of even order, $LE\left((P_n)_{k(i)}^P\right) = \frac{n^2-4}{n}$.

Theorem 3.3. *Let $P = \{V_1, V_2, \dots, V_k\}$ be a partition of cycle C_n.*

(i) *If any of the V_i is K_1 and remaining V_j's are all K_2's, where $i \neq j$. Then*

$$LE\left((C_n)_{k(i)}^P\right) = \frac{2}{n}\left[(n-k)^2+k^2\right],$$

whereas

$$LE\left((C_n)_k^P\right) = \frac{4k}{n}\left[n-2\right], \quad \text{for odd } n \geq 3.$$

(ii) *If each V_i consists of K_2, then*

$$LE\left((C_n)_{k(i)}^P\right) = \frac{2}{n}[k^2+(n-k)^2],$$

whereas

$$LE\left((C_n)_k^P\right) = (2n-2)+\left(2-\frac{4}{n}\right)\frac{2}{n}-\frac{4}{n}\left(\frac{n}{2}\right)^2, \quad \text{for even } n \geq 4.$$

Proof. (i) If any of the V_i is K_1 and remaining V_j's are all K_2's, where $i \neq j$. Then $(C_n)_{k(i)}^P$ is the union of $K_{1,2}$ and $(n-k-1)$ K_2's. Hence, Laplacian spectrum of $(C_n)_{k(i)}^P$ is $\{0(k-1 \text{ times}), 1, 2(n-k-1 \text{ times}), 3\}$. The average degree of $(C_n)_{k(i)}^P$ is $\frac{2k}{n}$ and

$$\begin{aligned} LE\left((C_n)_{k(i)}^n\right) =&(k-1)\left|-\frac{2k}{n}\right|+\left|1-\frac{2k}{n}\right| \\ &+(n-k-1)\left|2-\frac{2k}{n}\right|+\left|3-\frac{2k}{n}\right| \\ =&\frac{2}{n}\left[(n-k)^2+k^2\right]. \end{aligned}$$

Also according to Theorem 2.1, Laplacian spectrum of $(C_n)_k^P$ is $\{0, n-3, n-2 \times(n-k-1 \text{ times}), n-1, n(k-2 \text{ times})\}$. The average degree of $(C_n)_k^P$ is $\frac{n(n-1)-2k}{n}$. Thus

$$\begin{aligned} LE\left((C_n)_k^P\right) =&\left|\frac{n(n-1)-2k}{n}\right|+\left|(n-3)-\left(\frac{n(n-1)-2k}{n}\right)\right| \\ &+\left|(n-1)-\left(\frac{n(n-1)-2k}{n}\right)\right| \end{aligned}$$

$$+ (n-k-1)\left|(n-2) - \left(\frac{n(n-1)-2k}{n}\right)\right|$$
$$+ (k-2)\left|n - \left(\frac{n(n-1)-2k}{n}\right)\right|$$
$$= \frac{4k}{n}[n-2].$$

(ii) If each V_i consists of K_2, then $(C_n)^P_{k(i)}$ has k components of K_2. Hence, Laplacian specrtum of $(C_n)^P_{k(i)}$ is $\{0(k \text{ times}), 2(n-k \text{ times})\}$. The average degree of $(C_n)^P_{k(i)}$ is $\frac{2k}{n}$ and

$$LE((C_n)^P_{k(i)}) = k\left|-\frac{2k}{n}\right| + (n-k)\left|2 - \frac{2k}{n}\right| = \frac{2}{n}\left[k^2 + (n-k)^2\right].$$

Also from Theorem 2.1, Laplacian spectrum of $(C_n)^P_k$ is $\{0, n-2(k \text{ times}), n(n-k-1 \text{ times})\}$. As $(C_n)^P_{k(i)}$ has k edges, $(C_n)^P_k$ has $({}^nC_2 - k)$ edges. Also, note that $(C_n)^P_k$ has $K_n - k$ independent edges. For cycle of even order, $k = \frac{n}{2}$, we have

$$LE\left((C_n)^P_k\right) = LE\left(Kb_n\left(\frac{n}{2}\right)\right) = (2n-2) + \left(2 - \frac{4}{n}\right)\frac{2}{n} - \frac{4}{n}\left(\frac{n}{2}\right)^2. \qquad \square$$

Theorem 3.4. *Let $K_{m,n} = \{U_m, U_n\}$ be complete bipartite graph with partition $P = \{V_1, V_2\}$. Then,*

(i) *If $< V_1 >= K_{s_1,s_2}$ and $< V_2 >= K_{m-s_1,n-s_2}$, where s_1, s_2 denote number of vertices of V_1 such that s_1 vertices belong to U_m and s_2 vertices belong to U_n and $s_1 < m$, $s_2 < n$, then*

$$LE\left((K_{m,n})^P_2\right) = 2q + 2(n-s_1+s_2)(m-s_2+s_1) \times \frac{|(n-s_1+s_2)-(m-s_2+s_1)|}{(n-s_1+s_2)+(m-s_2+s_1)},$$

where

$$q = \begin{cases} n-s_1+s_2, & \text{if } n-s_1+s_2 \le m-s_2+s_1, \\ m-s_2+s_1, & \text{if } m-s_2+s_1 < n-s_1+s_2 \end{cases}$$

and

$$LE\left((K_{m,n})^P_{2(i)}\right) = 2(m+n-2).$$

(ii) *If $|V_1| = m-1$ such that all the vertices of V_1 are from first partite set of $K_{m,n}$ and $|V_2| = n+1$, then*

$$LE\left((K_{m,n})^P_2\right) = 2\left(\frac{n^2-2n+2}{n}\right)$$

and

$$LE\left((K_{m,n})^P_{2(i)}\right) = 2(m+n-2).$$

Proof. (i) If $< V_1 >= K_{s_1,s_2}$ and $< V_2 >= K_{m-s_1,n-s_2}$, then

$$(K_{m,n})_2^P \cong K_{n-s_1+s_2,m-s_2+s_1}.$$

Hence,

$$LE\left((K_{m,n})_2^P\right) = 2q + 2(n-s_1+s_2)(m-s_2+s_1) \times \frac{|(n-s_1+s_2)-(m-s_2+s_1)|}{(n-s_1+s_2)+(m-s_2+s_1)},$$

where

$$q = \begin{cases} n-s_1+s_2, & \text{if } n-s_1+s_2 \leq m-s_2+s_1, \\ m-s_2+s_1, & \text{if } m-s_2+s_1 < n-s_1+s_2. \end{cases}$$

Also

$$LE\left((K_{m,n})_{2(i)}^P\right) \cong K_m \cup K_n.$$

Hence,

$$LE\left((K_{m,n})_{2(i)}^P\right) = LE(K_m) + LE(K_n) = 2(m+n-2).$$

(ii) If $|V_1| = m-1$ such that all the vertices of V_1 are from first partite set of $K_{m,n}$ and $|V_2| = n+1$, then $(K_{m,n})_2^P \cong K_{1,m+n-1}$. Hence $LE\left((K_{m,n})_2^P\right) = 2\left(\frac{n^2-2n+2}{n}\right)$. Also $(K_{m,n})_{2(i)}^P \cong K_1 \cup K_{m+n-1}$. Thus, $LE\left((K_{m,n})_{2(i)}^P\right) = 2(m+n-2)$, which is stated. □

Theorem 3.5. *Let $S(m,n)$ be double star graph with partition $P = \{V_1, V_2\}$, such that the vertices of V_1 and V_2 are of distance two. Then*

$$LE\left(S(m,n)_{2(i)}^P\right) = \begin{cases} \dfrac{2n(m(2+m-n)+2(n-1))}{m+n}, & \text{if } m > n, \\ \dfrac{2m(n(2+n-m)+2(m-1))}{m+n}, & \text{if } n > m, \\ \dfrac{4(m^2+n^2-2)}{m+n}, & \text{if } m = n, \end{cases}$$

and

$$LE\left(S(m,n))_2^P\right) = \frac{12(n-1)(m-1)}{m+n}.$$

Proof. Let V_1 and V_2 be the partition of vertices of $S(m,n)$ such that the vertices of V_1 and V_2 are of distance two, i.e., V_1=$\{v_1, v_2, v_3, \ldots, v_{m-1}, u_1\}$ and V_2=$\{v_m, u_2, u_3, \ldots,$

$u_{n-1}, u_n\}$. We have

$$L(S(m,n)^P_{2(i)}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{m-1} \\ v_m \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \begin{array}{c} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & \dots & v_{m-1} & v_m & u_1 & u_2 & \dots & u_n \end{array} \\ \left[\begin{array}{cccccccccc} m & -1 & -1 & \cdots & -1 & -1 & -1 & 0 & \cdots & 0 \\ -1 & m & -1 & \cdots & -1 & -1 & -1 & 0 & \cdots & 0 \\ -1 & -1 & m & \cdots & -1 & -1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & m & -1 & -1 & 0 & \dots & 0 \\ -1 & -1 & -1 & \cdots & -1 & n+m-1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & n+m-1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 & n & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & n \end{array}\right] \end{array}.$$

Consider $\det\left(\lambda I - L\left(S(m,n)^P_{2(i)}\right)\right)$.

Step 1: Replace R_i by $R_i - R_{i+1}$, for $i = v_1, v_2, v_3, \ldots, v_{m-2}, v_m$ and replace R_i by $R_i - R_{i-1}$, for $i = u_n, u_{n-1}, \ldots, u_4, u_3$. Then $\det\left(\lambda I - L\left(S(m,n)^P_{2(i)}\right)\right)$ is of the form

$$(\lambda - (m+1))^{m-2}(\lambda - (n+1))^{n-2}(\lambda - (n+m))\det(D).$$

Step 2: In $\det(D)$, replace C_i by $C_i - C_{i-1}$, for $i = v_2, v_3, v_4, \ldots, v_{m-1}$ and replace C_i by $C_i - C_{i+1}$, for $i = u_{n-1}, u_{n-2}, \ldots, u_2, u_1$. Then it reduces to a new determinant

$$\det(E) = \begin{vmatrix} \lambda - 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ m-1 & 1 & \lambda - m & n-1 \\ 0 & 1 & \lambda - 1 & \lambda - 2 \end{vmatrix},$$

i.e., $\det(E) = \lambda(\lambda-2)(\lambda-(n+m))$. The Laplacian spectrum of $\left(S(m,n)^P_{k(i)}\right)$ is $\{0, 2, n+m(2 \text{ times}), m+1(m-2 \text{ times}), n+1(n-2 \text{ times})\}$ and the average degree of $\left(S(m,n)^P_{2(i)}\right)$ is $\frac{n(n+1)+m(m+1)-2}{m+n}$. Hence,

$$\begin{aligned} LE\left((S(m,n)^P_{2(i)}\right) =& \frac{n(n+1)+m(m+1)-2}{m+n} \\ &+ \left|2 - \frac{n(n+1)+m(m+1)-2}{m+n}\right| \\ &+ 2\left|(n+m) - \frac{n(n+1)+m(m+1)-2}{m+n}\right| \\ &+ (m-2)\left|(m+1) - \frac{n(n+1)+m(m+1)-2}{m+n}\right| \\ &+ (n-2)\left|(n+1) - \frac{n(n+1)+m(m+1)-2}{m+n}\right|. \end{aligned}$$

If $m > n$,

$$LE\left(S(m,n)^P_{2(i)}\right) = \frac{2n(m(2+m-n)+2(n-1))}{m+n}.$$

If $n > m$,

$$LE\left(S(m,n)^P_{2(i)}\right) = \frac{2m(n(2+n-m)+2(m-1))}{m+n}.$$

If $m = n$,

$$LE\left(S(m,n)^P_{2(i)}\right) = \frac{4(m^2+n^2-2)}{m+n}.$$

Also from Theorem 2.1, Laplacian spectrum of $\left(S(m,n)^P_2\right)$ is $\{0(3 \text{ times}), m-1(n-2 \text{ times}), n-1(m-2 \text{ times}), n+m-2\}$. Average degree of $\left(S(m,n)^P_2\right)$ is $\frac{2(m-1)(n-1)}{4n+2}$. Thus,

$$\begin{aligned} LE\left(S(m,n)^P_2\right) =& 3\left(\frac{2(m-1)(n-1)}{4n+2}\right) \\ &+ (n-2)\left|(m-1) - \frac{2(m-1)(n-1)}{4n+2}\right| \\ &+ (m-2)\left|(n-1) - \frac{2(m-1)(n-1)}{4n+2}\right| \\ &+ \left|(n+m-2) - \frac{2(m-1)(n-1)}{4n+2}\right| \\ =& \frac{4(m^2+n^2-2)}{m+n}. \end{aligned}$$

□

Theorem 3.6. *Let F_n be friendship graph with partition $P = \{V_1, V_2\}$, such that a partition V_1 contains central vertex and remaining vertices are in a partition V_2. Then*

$$LE\left((F_n)^P_{2(i)}\right) = \frac{2n(4n+1)}{2n+1} \quad \text{and} \quad LE\left((F_n)^P_2\right) = \frac{4n(n+1)}{2n+1}.$$

Proof. Let V_1 and V_2 be the partition of vertices of F_n and a partition V_1 contains only the central vertex and remaining vertices are in a partition V_2. We have

$$L((F_n)^P_{2(i)}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-1} \\ v_n \\ v_{n+1} \\ \vdots \\ v_{2n} \\ v_{2n+1} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} v_1 & v_2 & v_3 & v_4 & \cdots & v_{n-1} & v_n & v_{n+1} & \cdots & v_{2n} & v_{2n+1} \end{array} \\ \left[\begin{array}{ccccccccccc} 2n & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & 2n-1 & 0 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & 0 & 2n-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & 2n-1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & 2n-1 & 0 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & 0 & 2n-1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & 2n-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & 2n-1 & 0 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & 0 & 2n-1 \end{array}\right] \end{array}.$$

Consider $\det\left(\lambda I - L(F_n)^P_{2(i)}\right)$.

Step 1: Replace R_i by $R_i - R_{i-1}$, where $i = v_{2n+1}, v_{2n}, v_{2n-1}, \ldots, v_3, v_2$. Then we conclude that $\det\left(\lambda I - L(F_n)^P_{2(i)}\right)$ is of the form

$$(\lambda - (2n-1))^n \det(D).$$

Step 2: In $\det(D)$, replace C_i by $C_i + C_{i+1}$, where $i = v_{2n}, v_{2n-1}, v_{2n-2}, \ldots, v_1$. We get a new determinant, let it be $\det(E)$.

Step 3: In $\det(E)$, replace R_i by $R_i + R_{i-1}$, where $i = v_4, v_6, v_8, \ldots, v_{2n-2}, v_{2n}$, then the entries below the principal diagonal are zeros. Hence $\det(E) = \lambda(\lambda - (2n+1))^n$.

Thus,

$$\det\left(\lambda I - L(F_n)^P_{2(i)}\right) = \lambda(\lambda - (2n+1))^n(\lambda - (2n-1))^n.$$

Therefore, Laplacian spectrum of $(F_n)^P_{2(i)}$ is $\{0, 2n-1(n \text{ times}), 2n+1(n \text{ times})\}$ and the average degree of $(F_n)^P_{2(i)}$ is $\frac{4n^2}{2n+1}$. Hence,

$$LE\left((F_n)^P_{2(i)}\right) = \frac{4n^2}{2n+1} + n\left|(2n-1) - \frac{4n^2}{2n+1}\right| + n\left|(2n-1) - \frac{4n^2}{2n+1}\right| = \frac{2n(4n+1)}{2n+1}.$$

Also from Theorem 2.1, Laplacian spectrum of $(F_n)^P_2$ is $\{0(n+1) \text{ times}, 2(n \text{ times})\}$. Average degree of $(F_n)^P_2$ is $\frac{2n}{2n+1}$. Thus,

$$LE\left((F_n)^P_2\right) = \frac{2n}{2n+1} + n\left|2 - \frac{2n}{2n+1}\right| = \frac{4n(n+1)}{2n+1}.$$ □

Theorem 3.7. *Let F_n be friendship graph with partition $P = \{V_1, V_2\}$, such that a partition V_1 contains one triangle and remaining vertices are in a partition V_2. Then*

$$LE\left((F_n)^P_{2(i)}\right) = \frac{24(n^2 - 2n + 1)}{2n+1}$$

and

$$LE\left((F_n)^P_2\right) = \begin{cases} \dfrac{4(3n^2 - n + 1)}{2n+1}, & \text{if } n = 3, \\ \dfrac{4(4n^2 - 6n + 5)}{2n+1}, & \text{if } n > 3. \end{cases}$$

Proof. If V_1 and V_2 be the partition of F_n, such that a partition V_1 contains one triangle and remaining vertices are in a partition V_2. Then

$$L((F_n)^P_{2(i)}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ \vdots \\ v_{n-1} \\ v_n \\ v_{n+1} \\ \vdots \\ v_{2n} \\ v_{2n+1} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & \cdots & v_{n-1} & v_n & v_{n+1} & \cdots & v_{2n+1} \end{array} \\ \begin{bmatrix} 2n-2 & 0 & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 2n-3 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & 0 & 0 & 0 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & 0 & 0 & -1 & 2n-3 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & -1 & -1 & \cdots & 2n-3 & 0 & -1 & \cdots & -1 \\ -1 & 0 & 0 & -1 & -1 & \cdots & 0 & 2n-3 & -1 & \cdots & -1 \\ -1 & 0 & 0 & -1 & -1 & \cdots & -1 & -1 & 2n-3 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & 0 \\ -1 & 0 & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & 2n-3 \end{bmatrix} \end{array}.$$

Consider $\det\left(\lambda I - L(F_n)^P_{2(i)}\right)$.

Step 1: Replace R_i by $R_i - R_{i-1}$, where $i = v_{2n+1}, v_{2n}, v_{2n-1}, \ldots, v_5$ and replace R_{v_4} by $R_{v_4} - R_{v_1}$. Then $\det\left(\lambda I - L(F_n)^P_{2(i)}\right)$ is of the form

$$\lambda^2(\lambda - (2n-3))^{n-1}\det(D).$$

Step 2: In $\det(D)$, replace C_i by $C_i + C_{i+1}$, where $i = v_{2n}, v_{2n-1}, v_{2n-2}, \ldots, v_4$. We get a new determinant, let it be $\det(E)$.

Step 3: In $\det(E)$, replace R_i by $R_i + R_{i-1}$, where $i = v_{2n}, v_{2n-2}, v_{2n-4}, \ldots, v_6$ and simplifying we get,

$$\begin{aligned} \det(E) &= (\lambda - (2n-1))^{n-2} \begin{vmatrix} \lambda - (2n-2) & 0 & 0 & 2n-2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2n-\lambda-1 & 0 & 0 & \lambda - 2n + 1 \end{vmatrix} \\ &= (\lambda - (2n-1))^{n-2}\lambda(\lambda - (2n-1)). \end{aligned}$$

Thus,

$$\det\left(\lambda I - L\left((F_n)^P_{2(i)}\right)\right) = \lambda^3(\lambda - (2n-1))^{n-1}(\lambda - (2n-3))^{n-1}.$$

Therefore, the Laplacian spectrum of $(F_n)^P_{2(i)}$ is $\{0(3 \text{ times}), 2n-1(n-1 \text{ times}), 2n-3(n-1 \text{ times})\}$ and the average degree of $(F_n)^P_{2(i)}$ is $\frac{4(n^2-2n+1)}{2n+1}$. Hence,

$$\begin{aligned} LE\left((F_n)^P_{2(i)}\right) =& 3\left(\frac{4(n^2-2n+1)}{2n+1}\right) + (n-1)\left|(2n-1) - \frac{4(n^2-2n+1)}{2n+1}\right| \\ &+ (n-1)\left|(2n-3) - \frac{4(n^2-2n+1)}{2n+1}\right| \\ =& \frac{24(n^2-2n+1)}{2n+1}. \end{aligned}$$

Also from Theorem 2.1, Laplacian spectrum of $(F_n)^P_2$ is $\{0, 2(n-1 \text{ times}), 4(n-1 \text{ times}), 2n+1(2 \text{ times})\}$. Average degree of $(F_n)^P_2$ is $\frac{10n-4}{2n+1}$.

If $n = 3$, then

$$\begin{aligned} LE\left((F_n)_2^P\right) =& \frac{10n-4}{2n+1} + (n-1)\left|2 - \frac{10n-4}{2n+1}\right| + (n-1)\left|4 - \frac{10n-4}{2n+1}\right| \\ &+ 2\left|(2n+1) - \frac{10n-4}{2n+1}\right| \\ =& \frac{4(3n^2-n+1)}{2n+1}. \end{aligned}$$

If $n > 3$,

$$LE\left((F_n)_2^P\right) = \frac{4(4n^2-6n+5)}{2n+1}. \qquad \square$$

Theorem 3.8. *Let $K_{n\times 2}$ be cocktail party graph with vertex set $V = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and partition $P = \{V_1, V_2\}$, such that V_1 contains v_i vertices, and remaining u_i vertices are in V_2, where $i = 1, 2, 3, \ldots, n$. Then*

$$LE\left((K_{n\times 2})_{2(i)}^P\right) = LE\left((K_{n\times 2})_2^P\right) = 4(n-1).$$

Proof. Let $K_{n\times 2}$ be a cocktail party graph with vertex set $V = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and $P = \{V_1, V_2\}$ be a partition of vertices of $K_{n\times 2}$ such that $V_1 = \{v_1, \ldots, v_n\}$ and V_2=$\{u_1, u_2, u_3, \ldots, u_n\}$. We have

$$L(K_{n\times 2})_{2(i)}^P = \left[\begin{array}{c|c} (n-1)I_n & (I-J)_n \\ \hline (I-J)_n & (n-1)I_n \end{array}\right].$$

It is of the form $\left[\begin{array}{c|c} A_0 & A_1 \\ \hline A_1 & A_0 \end{array}\right]$. Hence, from Lemma 2.1, we get, Laplacian spectrum of $(K_{n\times 2})_{2(i)}^P$ is $\{0, 2n-2, n(n-1 \text{ times}), n-2(n-1 \text{ times})\}$ and the average degree of $(K_{n\times 2})_{2(i)}^P$ is $\frac{n^2-1}{n}$. Hence,

$$\begin{aligned} LE\left((K_{n\times 2})_{2(i)}^P\right) =& \frac{n^2-1}{n} + \left|(2n-2) - \frac{n^2-1}{n}\right| + (n-1)\left|n - \frac{n^2-1}{n}\right| \\ &+ (n-1)\left|(n-2) - \frac{n^2-1}{n}\right| \\ =& 4(n-1). \end{aligned}$$

Also from Theorem 2.1, Laplacian spectrum of $(K_{n\times 2})_2^P$ is $\{0, 2, n(n-1 \text{ times}), n+2(n-1 \text{ times})\}$. Average degree of $(K_{n\times 2})_2^P$ is n. Thus,

$$LE\left((K_{n\times 2})_2^P\right) = n + |2-n| + (n-1)|n-n| + (n-1)|(n+2)-n| = 4(n-1). \quad \square$$

Theorem 3.9. *For cocktail graph $K_{2n\times 2}$ with a partition $P = \{V_1, V_2, \ldots, V_{2n}\}$, such that V_i consists of K_2 in respective partition, where $i = 1, 2, \ldots, 2n$. Then*

$$LE\left((K_{2n\times 2})_{k(i)}^P\right) = 10n-6 \quad \textit{and} \quad LE\left((K_{2n\times 2})_k^P\right) = 4n.$$

Proof. Let $P = \{V_1, V_2, \dots, V_{2n}\}$ be a partition of cocktail party graph $K_{2n\times 2}$, such that V_i consists of K_2 in respective partition, where $i = 1, 2, \dots, 2n$. Then, we have

$$L((K_{n\times 2})^P_{k(i)}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{2n-1} \\ v_{2n} \\ u_1 \\ u_2 \\ \vdots \\ u_{2n-1} \\ u_{2n} \end{array} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & \cdots & v_{2n-1} & v_{2n} & u_1 & u_2 & \cdots & u_{2n} \\ 4n-3 & 0 & -1 & \cdots & -1 & -1 & 0 & -1 & \cdots & -1 \\ 0 & 4n-3 & -1 & \cdots & -1 & -1 & -1 & 0 & \cdots & -1 \\ -1 & -1 & 4n-3 & \cdots & -1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 & -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 4n-3 & 0 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \cdots & 0 & 4n-3 & -1 & -1 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & -1 & 4n-3 & 0 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 & -1 & 0 & 4n-3 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 0 & -1 & -1 & -1 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & -1 & -1 & \cdots & 4n-3 \end{array}.$$

It is of the form $\left[\begin{array}{c|c} A_0 & A_1 \\ \hline A_1 & A_0 \end{array}\right]$. Hence, from Lemma 2.1, we get Laplacian spectrum of $(K_{n\times 2})^P_{k(i)}$ is $\{0, 4n(n-1 \text{ times}), 4n-2(2n \text{ times}),\ 4n-4(n \text{ times})\}$ and the average degree of $(K_{n\times 2})^P_{k(i)}$ is $4n-3$. Hence,

$$\begin{aligned} LE\left((K_{2n\times 2})^P_{k(i)}\right) =& (4n-3) + (n-1)\,|4n-(4n-3)| + 2n\,|(4n-2)-(4n-3)| \\ & + n\,|(4n-4)-(4n-3)| \\ =& 10n-6. \end{aligned}$$

Also from Theorem 2.1, Laplacian spectrum of $(K_{2n\times 2})^P_k$ is $\{0(n \text{ times}), 2(2n \text{ times}), 4(n \text{ times})\}$. Average degree of $(K_{2n\times 2})^P_k$ is 2. Thus,

$$LE\left((K_{2n\times 2})^P_k\right) = 2n + 2n\,|2-2| + n\,|4-2| = 4n. \qquad \square$$

Theorem 3.10. *For cocktail graph $K_{(2n+1)\times 2}$ with a partition $P = \{V_1, V_2, \dots, V_{2n+2}\}$, such that V_i consists of K_2 in respective partition, where $i = 1, 2, \dots, 2n$ and V_j consists of K_1 in respective partition, where $j = 1, 2$. Then*

$$LE\left((K_{(2n+1)\times 2})^P_{k(i)}\right) = \frac{4n(5n+2)}{2n+1} \quad \text{and} \quad LE\left((K_{(2n+1)\times 2})^P_k\right) = \frac{|8n^2-26n-2|}{2n+1}.$$

Proof. For the given partition

$$L((K_{(2n+1)\times 2})^P_{k(i)}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{2n} \\ v_{2n+1} \\ u_1 \\ u_2 \\ \vdots \\ u_{2n} \\ u_{2n+1} \end{array} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & \cdots & v_{2n+1} & u_1 & u_2 & \cdots & u_{2n} & u_{2n} \\ 4n-1 & 0 & -1 & \cdots & -1 & 0 & -1 & \cdots & -1 & -1 \\ 0 & 4n-1 & -1 & \cdots & -1 & -1 & 0 & \cdots & -1 & -1 \\ -1 & -1 & 4n-1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & 0 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \dots & 0 & -1 \\ -1 & -1 & -1 & \cdots & 4n & -1 & -1 & \cdots & -1 & 0 \\ 0 & -1 & -1 & \cdots & -1 & 4n-1 & 0 & \cdots & -1 & -1 \\ -1 & 0 & -1 & \cdots & -1 & 0 & 4n-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & 4n-1 & -1 \\ -1 & -1 & -1 & \cdots & 0 & -1 & -1 & \cdots & -1 & 4n \end{array}.$$

It is of the form $\left[\begin{array}{c|c} A_0 & A_1 \\ \hline A_1 & A_0 \end{array}\right]$. Hence, from Lemma 2.1, we get Laplacian Spectrum of $(K_{(2n+1)\times 2})^P_{k(i)}$ is $\{0, 4n(2n+1 \text{ times}), 4n-2(n \text{ times}), 4n+2(n \text{ times})\}$ and the average degree of $(K_{(2n+1)\times 2})^P_{k(i)}$ is $\frac{4(4n^2+n)}{4n+2}$. Hence,

$$\begin{aligned} LE\left((K_{(2n+1)\times 2})^P_{k(i)}\right) =& \frac{4(4n^2+n)}{4n+2} + (2n+1)\left|4n - \frac{4(4n^2+n)}{4n+2}\right| \\ &+ n\left|(4n-2) - \frac{4(4n^2+n)}{4n+2}\right| + n\left|(4n+2) - \frac{4(4n^2+n)}{4n+2}\right| \\ =& \frac{4n(5n+2)}{2n+1}. \end{aligned}$$

Also from Theorem 2.1, Laplacian spectrum of $(K_{(2n+1)\times 2})^P_k$ is $\{0(n+1 \text{ times}), 2(2n+1 \text{ times}), 4(n \text{ times})\}$. Average degree of $(K_{(2n+1)\times 2})^P_k$ is $\frac{4n+1}{2n+1}$. Thus,

$$\begin{aligned} LE\left((K_{(2n+1)\times 2})^P_k\right) =& n+1\left(\frac{4n+1}{2n+1}\right) + (2n+1)\left|2 - \frac{4n+1}{2n+1}\right| + n\left|4 - \frac{4n+1}{2n+1}\right| \\ =& \frac{|8n^2-26n-2|}{2n+1}. \end{aligned}$$

□

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