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PASSAGE OF PROPERTY (aw) FROM TWO OPERATORS TO THEIR TENSOR PRODUCT

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Abstract. A Banach space operator S satisfies property (aw) if $\sigma(S) \setminus \sigma_w(S) = E_a^0(S)$, where $E_a^0(S)$ is the set of all isolated point in the approximate point spectrum which are eigenvalues of finite multiplicity. Property (aw) does not transfer from operators A and B to their tensor product $A \otimes B$, so we give necessary and/or sufficient conditions ensuring the passage of property (aw) from A and B to $A \otimes B$. Perturbations by Riesz operators are considered.

1. Introduction

For a bounded linear operator $S \in \mathscr{L}(\mathbb{X})$, let $\sigma(S)$, $\sigma_p(S)$, $\sigma_a(S)$ denote, respectively, the spectrum, the point spectrum and the approximate point spectrum of S and if G is a subset of $\mathbb{C}$, then G^{iso}, G^{acc} denote, the isolated points of G and the accumulation points of G. Let $\alpha(S)$ and $\beta(S)$ denote the nullity and the deficiency of S, defined by $\alpha(S) = \dim \ker(S)$ and $\beta(S) = \operatorname{codim} \Re(S)$. If the range $\Re(S)$ of S is closed and $\alpha(S) < \infty$ (resp. $\beta(S) < \infty$), then S is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator. If $S \in \mathscr{L}(\mathbb{X})$ is either upper or lower semi-Fredholm, then S is called a semi-Fredholm operator, and $\operatorname{ind}(S)$, the index of S, is then defined by $\operatorname{ind}(S) = \alpha(S) - \beta(S)$. If both $\alpha(S)$ and $\beta(S)$ are finite, then S is a Fredholm operator. The ascent, denoted $\operatorname{asc}(S)$, and the descent, denoted $\operatorname{dsc}(S)$, of S are given by $\operatorname{asc}(S) = \inf\{n \in \mathbb{N} : \ker(S^n) = \ker(S^{n+1})\}$, $\operatorname{dsc}(S) = \inf\{n \in \mathbb{N} : \Re(S^n) = \Re(S^{n+1})\}$ (where the infimum is taken over the set of non-negative integers); if no such integer

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n exists, then $\operatorname{asc}(S)=\infty$, respectively $\operatorname{dsc}(S)=\infty$). Let

$$\begin{aligned}
\Phi_+(S) &= \{\lambda\in\mathbb{C} : S-\lambda \text{ is upper semi-Fredholm}\},\\
\Phi(S) &= \{\lambda\in\mathbb{C} : S-\lambda \text{ is Fredholm}\},\\
\sigma_{SF_+}(S) &= \{S-\lambda\in\sigma_a(S) : \lambda\notin\Phi_+(S)\},\\
\sigma_{aw}(S) &= \Big\{\lambda\in\sigma_a(S) : \lambda\in\sigma_{SF_+}(S) \text{ or } \operatorname{ind}(S-\lambda)>0\Big\},\\
\sigma_{ab}(S) &= \Big\{\lambda\in\sigma_a(S) : \lambda\in\sigma_{SF_+}(S) \text{ or } \operatorname{asc}(S-\lambda)=\infty\Big\},\\
E^0(S) &= \Big\{\lambda\in\sigma^{iso}(S) : 0<\alpha(S-\lambda)<\infty\Big\},\\
E^0_a(S) &= \Big\{\lambda\in\sigma^{iso}_a(S) : 0<\alpha(S-\lambda)<\infty\Big\},\\
\pi^0_a(S) &= \Big\{\lambda\in\sigma^{iso}_a(S) : \lambda\in\Phi_+(S), \operatorname{asc}(S-\lambda)<\infty\Big\},\\
H_0(S) &= \Big\{x\in\mathbb{X} : \lim_{n\longrightarrow\infty}\|S^nx\|^{1/n}=0\Big\}.
\end{aligned}$$

Let $\pi(S)$ be the set of all poles of the resolvent of S and $\pi^0(T)$ is the set of all poles of the resolvent of finite rank, that is, $\pi^0(S)=\{\lambda\in\pi(S):\alpha(S-\lambda)<\infty\}$. Let

$$\begin{aligned}
\sigma_w(S) &= \{\lambda\in\sigma(S) : S-\lambda\notin\Phi(S) \text{ or } \operatorname{ind}(S-\lambda)\neq 0\},\\
\sigma_b(S) &= \{\lambda\in\sigma(S) : S-\lambda\notin\Phi(S) \text{ or } \operatorname{asc}(S-\lambda)\neq\operatorname{dsc}(S-\lambda)\} \quad \text{and}\\
\sigma_{ab}(S) &= \{\lambda\in\sigma_a(S) : S-\lambda \text{ is not Fredholm or } \operatorname{asc}(T-\lambda)=\infty\},
\end{aligned}$$

denote, respectively, the Weyl spectrum, the Browder spectrum and the essential approximate Browder spectrum of T. Now, let $\Delta(S)=\sigma(S)\setminus\sigma_w(S)$ and $\Delta_a(S)=\sigma_a(S)\setminus\sigma_{aw}(S)$. Then S satisfies Browder's theorem (in symbol, $S\in\mathcal{B}$) if $\sigma_b(S)=\sigma_w(S)$, or equivalently, $\Delta(S)=\pi^0(S)$. We say that $S\in\mathscr{L}(\mathbb{X})$ satisfies a-Browder's theorem (in symbol, $S\in a\mathcal{B}$) if $\sigma_{ab}(S)=\sigma_{aw}(S)$, or equivalently, $\Delta_a(S)=\pi^0_a(S)$. S satisfies Weyl's theorem (in symbol, $S\in\mathcal{W}$) if $\Delta(S)=E^0(S)$ and S satisfies a-Weyl's theorem (in symbol, $S\in a\mathcal{W}$) if $\Delta_a(S)=E^0_a(S)$.

Operators satisfying property (aw) have been studied in a number of papers, see [4,5] for additional references. It is known that an operator S satisfying property (aw) satisfies Browder's theorem, but the reverse implication is generally false; property (aw) neither implies nor is implied by a-Weyl's theorem. Following [14], we say that $T\in\mathscr{L}(\mathbb{X})$ satisfies property (w) if $\Delta_a(T)=\sigma_a(T)\setminus\sigma_{SF^-_+}(T)=E_0(T)$. The property (w) has been studied in [2, 14]. In [2, Theorem 2.8], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. According to [4], an operator $T\in\mathscr{L}(\mathbb{X})$ is said to satisfy property (b) if $\Delta_a(T)=\pi_0(T)$. It is shown in [4, Theorem 2.13] that an operator satisfies property (w) satisfies property (b) but the converse is not true in general. An operator $S\in\mathscr{L}(\mathbb{X})$ is a-isoloid (resp. isoloid) if points $\lambda\in\sigma^{iso}_a(S)$ (resp. $\lambda\in\sigma^{iso}(S)$) are eigenvalues of the operator. If S is finitely a-isoloid (i.e., if $\lambda\in\sigma^{iso}_a(S)$ implies λ is a finite multiplicity eigenvalue of

S), $R \in \mathscr{L}(\mathbb{X})$ is a Riesz operator which commutes with S, then S satisfies Weyl's theorem implies $S + R$ satisfies Weyl's theorem [12, Theorem 2.7].

Given Banach space operators $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$, write

$$A \otimes B : \sum_i x_i \otimes y_i \mapsto \sum_i Ax_i \otimes By_i \in \mathscr{L}(\mathbb{X} \otimes \mathbb{Y}),$$

for the operator induced on the (algebraic completion of the) tensor product, endowed with a reasonable cross norm, of $\mathbb{X}$ and $\mathbb{Y}$. Property (aw) does not transfer from A and B to $A \otimes B$: a necessary and sufficient condition for property (aw) to transfer from A and B to $A \otimes B$ is that $A \otimes B$ satisfies the Weyl spectrum equality $\sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$. We say that S has the single valued extension property, or SVEP, at $\lambda \in \mathbb{C}$ if for every open neighborhood U of λ, the only analytic solution f to the equation $(S - \mu)f(\mu) = 0$ for all $\mu \in U$ is the constant function $f \equiv 0$; we say that S has SVEP if S has a SVEP at every $\lambda \in \mathbb{C}$. It is well known that finite ascent implies SVEP; also, an operator has SVEP at every isolated point of its spectrum (as well as at every isolated point of its approximate point spectrum). An operator $S \in \mathscr{L}(\mathbb{X})$ is polaroid if every $\lambda \in \sigma^{iso}(S)$ is a pole of the resolvent operator $(S - \lambda)^{-1}$. If S is polaroid and S^* (resp. S) has SVEP, then S (resp. S^*) satisfies property (aw). This property extends to tensor products $A \otimes B$: if A and B are polaroid, and if A^* and B^* (resp. A and B) have SVEP, then $A \otimes B$ (resp. $A^* \otimes B^*$) satisfies property (aw). If $Q_1 \in \mathscr{L}(\mathbb{X})$ and $Q_2 \in \mathscr{L}(\mathbb{Y})$ are quasinilpotent operators such that Q_1 commutes with A and Q_2 commutes with B, then $A \otimes B$ satisfies property (aw) if and only if $(A + Q_1) \otimes (B + Q_2)$ satisfies property (aw). For finitely a-isoloid A and B which satisfy property (aw), and Riesz operators R_1 and R_2 such that A commutes with R_1, B commutes with $\sigma(A + R_1) = \sigma(A)$ and $\sigma(B + R_2) = \sigma(B)$, $A \otimes B$ satisfies property (aw) implies $(A + R_1) \otimes (B + R_2)$ satisfies property (aw) if and only if Browder's theorem transfers from $A + R_1$ and $B + R_2$ to $(A + R_1) \otimes (B + R_2)$.

2. Property (aw) and Tensor product

The problem of transferring generalized Weyl theorem, property (gw) and property (b) from operators A and B to their tensor product $A \otimes B$ was considered in [15–17]. The main objective of this section is to study the transfer of property (aw) from a bounded linear operator A acting on a Banach space $\mathbb{X}$ and a bounded linear operator B acting on a Banach space $\mathbb{Y}$ to their tensor product $A \otimes B$.

Let

$$\begin{aligned} \sigma_s(S) &= \{\lambda \in \sigma(S) : S - \lambda \text{ is not surjective}\}, \\ \sigma_{sb}(S) &= \{\lambda \in \sigma_s(S) : S - \lambda \text{ is not lower semi-Fredholm or } \operatorname{dsc}(S - \lambda) = \infty\} \text{ and} \\ \sigma_{sw}(S) &= \{\lambda \in \sigma_s(S) : S - \lambda \text{ is not lower semi-Fredholm or } \operatorname{ind}(S - \lambda) < 0\}, \end{aligned}$$

denote, respectively, the surjectivity spectrum, the Browder essential surjectivity spectrum and the Weyl essential surjectivity spectrum of $S \in \mathscr{L}(\mathbb{X})$. Then S satisfies s-Browder's theorem if $\sigma_{sb}(S) = \sigma_{sw}(S)$. Apparently, S satisfies s-Browder's theorem if and only if S^* satisfies a-Browder's theorem. A necessary and sufficient condition for S to satisfy a-Browder's theorem is that S has SVEP at every $\lambda \in \Delta_a(S)$ [8, Lemma 2.8]; by duality, S satisfies s-Browder's theorem if and only if S^* has SVEP at every $\lambda \in \sigma_s(S) \setminus \sigma_{sw}(S)$. More generally, if either of S and S^* has SVEP, then S and S^* satisfy both a-Browder's theorem and s-Browder's theorem. Either of a-Browder's theorem and a-Browder's theorem implies Browder's theorem, but the converse is false. a-Browder's theorem fails to transfer from A and B to $A \otimes B$ [9, Example 1].

Lemma 2.1. [1, Theorem 3.23] *If $S \in \mathscr{L}(S)$ has SVEP at $\lambda \in \sigma(S)\setminus\sigma_{SF_+}(S)$. Then $\lambda \in \sigma_a^{iso}(S)$ and* $\operatorname{asc}(S-\lambda) < \infty$.

Lemma 2.2. [7] *Let $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$. Then*

(a) *$\sigma_x(A \otimes B) = \sigma_x(A)\sigma_x(B)$, where $\sigma_x = \sigma$ or σ_a;*
(b) *$\sigma_{SF_+}(A \otimes B) = \sigma_{SF_+}(A)\sigma_a(B) \cup \sigma_a(A)\sigma_{SF_+}(B)$.*

Lemma 2.3. [9] *Let $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$, then*

$$\sigma_a^{iso}(A \otimes B) \subseteq \sigma_a^{iso}(A)\sigma_a^{iso}(B) \cup \{0\}\,.$$

Lemma 2.4. [11] *Let $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$. Then*

(a) *$\sigma_p(A)\sigma_p(B) \subseteq \sigma_p(A \otimes B)$;*
(b) *$\sigma_w(A\otimes B) \subseteq \sigma(A)\sigma_w(B)\cup\sigma_w(A)\sigma(B) \subseteq \sigma(A)\sigma_b(B)\cup\sigma_b(A)\sigma(B) = \sigma_b(A\otimes B)$;*
(c) *$0 \notin \sigma(A \otimes B) \setminus \sigma_w(A \otimes B)$;*
(d) *If $A \otimes B \in \mathcal{B}$, then $\sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$.*

Example 2.1. Let $U \in \mathscr{L}(\ell^2)$ denote the forward unilateral shift, and let $A, B \in \mathscr{L}(\ell^2 \otimes \ell^2)$ be the operators

$$A = (1 - UU^*) \oplus \left(\frac{1}{2}U - 1\right), \; B = -(1 - UU^*) \left(\frac{1}{2}U^* - 1\right).$$

Then A and B^* have SVEP, so $A, B \in a\mathcal{B}$. Furthermore, $1 \in \sigma(A \otimes B) \setminus \sigma_w(A \otimes B)$. However, since

$$\sigma(A \otimes B) = \left\{\{0,1\} \cup \left\{\frac{1}{2}\mathbb{D} - 1\right\}\right\} \cdot \left\{\{0,-1\} \cup \left\{\frac{1}{2}\mathbb{D} + 1\right\}\right\},$$

where $\mathbb{D}$ is the closed unit disc in the complex plane $\mathbb{C}$, $1 \in \sigma^{acc}(A \otimes B) \Longrightarrow 1 \in \sigma_b(A \otimes B)$. Then $A \otimes B \notin \mathcal{B}$, and hence $A \otimes B$ does not obey property (aw).

Lemma 2.5. *Suppose that A, B and $A \otimes B$ satisfy property (aw). If $\mu \in \pi^0(A)$ and $\nu \in \pi^0(B)$, then $\lambda = \mu\nu \in \pi^0(A \otimes B)$.*

Proof. Suppose that $\mu \in \sigma(A) \setminus \sigma_w(A)$, $\nu \in \sigma(B) \setminus \sigma_w(B)$ and $\sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$. Then $\lambda = \mu\nu \in \sigma(A \otimes B) \setminus \sigma_w(A \otimes B) = \pi^0(A \otimes B)$. ☐

Theorem 2.1. *If $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$ are a-isoloid operators which satisfy property (aw), then the following conditions are equivalent.*

(i) *$A \otimes B$ satisfies property (aw).*
(ii) *The Weyl spectrum equality $\sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$ is satisfied.*
(iii) *$A \otimes B$ satisfies Browder's theorem.*

Proof. Since property (aw) implies Browder's theorem, the equivalence (ii)$\Leftrightarrow$(iii) and (i)$\Rightarrow$(iii) follows from [9, Theorem 3]. We prove (iii)$\Rightarrow$(i). The hypothesis A and B satisfy property (aw) implies

$$\sigma(A)\setminus\sigma_w(A)=E_a^0(A),\ \sigma(B)\setminus\sigma_w(B)=E_a^0(B).$$

Observe that (iii) implies Browder's theorem transfers from A and B to $A\otimes B$: hence $\sigma(A\otimes B)\setminus\sigma_w(A\otimes B)=\pi^0(A\otimes B)$. Since $\pi^0(A\otimes B)\subseteq E_a^0(A\otimes B)$, we have to prove $E_a^0(A\otimes B)\subseteq\pi^0(A\otimes B)$. Let $\lambda\in E_a^0(A\otimes B)$. Then $0\neq\lambda$ and there exist $\mu\in\sigma_a^{iso}(A)$ and $\nu\in\sigma_a^{iso}(B)$ such that $\lambda=\mu\nu$. By hypotheses, A and B are a-isoloid, hence μ is an eigenvalue of A and ν is an eigenvalue of B. Since $A\otimes B-(\mu I\otimes\nu I)=(A-\mu)\otimes B+\mu(I\otimes(B-\nu))$, if either of $\alpha(A-\mu)$ or $\alpha(B-\nu)$ is infinite then so is $\alpha(A\otimes B-(\mu I\otimes\nu I))$. Hence $\mu\in E_a^0(A)=\sigma(A)\setminus\sigma_w(A)$ and $\nu\in E_a^0(B)=\sigma(B)\setminus\sigma_w(B)$, consequently, $\lambda\in\sigma(A\otimes B)\setminus\sigma_w(A\otimes B)$; hence $E_a^0(A\otimes B)\subseteq\sigma(A\otimes B)\setminus\sigma_w(A\otimes B)$. Conversely, if $\lambda\in\sigma(A\otimes B)\setminus\sigma_w(A\otimes B)$, then by Lemma 2.4, we have $\lambda\neq 0$, and there exist $\mu\in\sigma(A)\setminus\sigma_w(A)=E_a^0(A)$ and $\nu\in\sigma(B)\setminus\sigma_w(B)=E_a^0(B)$ such that $\lambda=\mu\nu$. So, $\lambda\in E_a^0(A\otimes B)$. Therefore, $\sigma(A\otimes B)\setminus\sigma_w(A\otimes B)=E_a^0(A\otimes B)$. □

The following example shows that property (aw) does not transfer from $A\in\mathscr{L}(\mathbb{X})$ and $B\in\mathscr{L}(\mathbb{Y})$ to $A\otimes B$.

Example 2.2. Let $Q\in\mathscr{L}(\ell^2)$ be an injective quasi-nilpotent, and let

$$A=B=(I+Q)\oplus\alpha\oplus\beta\in\mathscr{L}(\ell^2)\oplus\mathbb{C}\oplus\mathbb{C},$$

where $\alpha\beta=1\neq\alpha$. Then

$$\sigma(A)=\sigma(B)=\{1,\alpha,\beta\},\ \sigma_w(A)=\sigma_w(B)=\{1\},\ \sigma(A\otimes B)=\{1,\alpha,\beta,\alpha^2,\beta^2\}.$$

The operators A,B have SVEP, hence Browder's theorem transfers from A and B to $A\otimes B$, which implies that

$$\sigma_w(A\otimes B)=\{1,\alpha,\beta\},\ 1\notin\sigma(A\otimes B)\setminus\sigma_w(A\otimes B)\text{ and }1=\alpha\beta\in E_a^0(A\otimes B).$$

Note that the operators A and B are not a-isoloid.

Theorem 2.2. *Suppose that $A\in\mathscr{L}(\mathbb{X})$ and $B\in\mathscr{L}(\mathbb{Y})$ are a-isoloid operators which satisfy property (aw). If $\sigma_w(A\otimes B)=\sigma(A)\sigma_w(B)\cup\sigma_w(A)\sigma(B)$, then $A\otimes B$ satisfies property (aw).*

Proof. The hypotheses imply that $A\otimes B\in\mathcal{B}$, that is, $\sigma(A\otimes B)\setminus\sigma_w(A\otimes B)=\pi^0(A\otimes B)$. Since $\pi^0(A\otimes B)\subseteq E_a^0(A\otimes B)$, we have to prove that $E_a^0(A\otimes B)\subseteq\pi^0(A\otimes B)$. Let $\lambda\in E_a^0(A\otimes B)$. Then $(0\neq)\ \lambda=\mu\nu$ for some $\mu\in\sigma_a^{iso}(A)$ and $\nu\in\sigma_a^{iso}(B)$. The operators A and B being a-isoloid, it follows (from $\lambda\in E_a^0(A\otimes B)$) that $\mu\in E_a^0(A)=\pi^0(A)$ and $\nu\in E_a^0(B)=\pi^0(B)$. So it follows from Lemma 2.5 that $\lambda\in\pi^0(A\otimes B)$. □

Definition 2.1. An operator $T\in\mathscr{L}(\mathbb{X})$ is said to be polaroid if iso$\sigma(T)$ is empty or every isolated point of $\sigma(T)$ is a pole of the resolvent.

Definition 2.2. An operator $T \in \mathscr{L}(\mathbb{X})$ is said to be a-polaroid if iso$\sigma_a(T)$ is empty or every isolated point of $\sigma_a(T)$ is a pole of the resolvent.

Clearly,

$$T \; a\text{-polaroid} \Rightarrow T \text{ polaroid}.$$

Observe that if T^* has SVEP then $\sigma(T) = \sigma_a(T)$, see [1, Corollary 2.45], so that

$$T^* \text{ has SVEP and } T \text{ polaroid} \Rightarrow T \; a\text{-polaroid}.$$

If T is polaroid then T^* is polaroid [3]. Moreover, if T has SVEP then $\sigma(T) = \sigma_a(T^*)$, see [1, Corollary 2.45], hence

$$T \text{ has SVEP and } T \text{ polaroid} \Rightarrow T^* \; a\text{-polaroid}.$$

Lemma 2.6. *If $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$ are a-polaroid, then $A \otimes B$ is a-polaroid.*

Proof. If $\sigma_a^{iso}(A) = \sigma_a^{iso}(B) = \emptyset$, then $\sigma_a^{iso}(A \otimes B) = \emptyset$. Observe also that if either of $\sigma_a^{iso}(A)$ or $\sigma_a^{iso}(B)$ is the empty set, say $\sigma_a^{iso}(A) = \emptyset$, then $\sigma_a^{iso}(A \otimes B) \subseteq \{0\}$. If $\sigma_a^{iso}(A \otimes B) = \{0\}$, then $0 \in \sigma_a^{iso}(B)$. But then $0 \in \pi(B)$, which implies that $0 \in \pi(A \otimes B)$. Let $\lambda \in \sigma_a^{iso}(A \otimes B)$ be such that $\lambda = \mu\nu$, $\mu \in \sigma_a^{iso}(A)$ and $\nu \in \sigma_a^{iso}(B)$. Then $\mu \in \pi(A)$ and $\nu \in \pi(B)$. Hence, we have $\lambda \in \pi(A \otimes B)$. $\square$

Theorem 2.3. *Suppose that the operators $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$ are polaroid.*

(i) *If A^* and B^* have SVEP, then $A \otimes B$ satisfies property (aw).*
(ii) *If A and B have SVEP, then $A^* \otimes B^*$ satisfies property (aw).*

Proof. (i) The hypothesis A^* and B^* have SVEP implies

$$\sigma(A) = \sigma_a(A),\; \sigma(B) = \sigma_a(B),\; \sigma_{aw}(A) = \sigma_w(A),,\; \sigma_{aw}(B) = \sigma_w(B)$$

and

$$A^*, B^*, \text{ and } A^* \otimes B^* \text{ satisfy s-Browder's theorem.}$$

Thus s-Browder's theorem and Browder's theorem transform from A^* and B^* to $A^* \otimes B^*$. Hence

$$\begin{aligned}\sigma_{aw}(A \otimes B) &= \sigma_{sw}(A^* \otimes B^*) = \sigma_s(A^*)\sigma_{sw}(B^*) \cup \sigma_{sw}(A^*)\sigma_s(B^*)\\ &= \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)\end{aligned}$$

and

$$\begin{aligned}\sigma_w(A \otimes B) &= \sigma_w(A^* \otimes B^*) = \sigma(A^*)\sigma_w(B^*) \cup \sigma_w(A^*)\sigma(B^*)\\ &= \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B).\end{aligned}$$

Consequently,

$$\sigma_{aw}(A \otimes B) = \sigma_w(A \otimes B).$$

Already,

$$\sigma_a(A \otimes B) = \sigma_a(A)\sigma_a(B) = \sigma(A)\sigma(B) = \sigma(A \otimes B).$$

Since A and B are a-polaroid, then $A \otimes B$ is a-polaroid by Lemma 2.6. Combining this with $A \otimes B$ satisfies Browder's theorem, it follows that $A \otimes B$ satisfies property

(aw). That is, $\sigma(A \otimes B) \setminus \sigma_w(A \otimes B) = E_a^0(A \otimes B)$.
(ii) In this case $\sigma(A) = \sigma_a(A^*)$, $\sigma(B) = \sigma_a(B^*)$, $\sigma_w(A^*) = \sigma_{aw}(A^*)$, $\sigma_w(B^*) = \sigma_{aw}(B^*)$, $\sigma(A^* \otimes B^*) = \sigma_a(A^* \otimes B^*)$, a-polaroid property transfer from A and B to $A^* \otimes B^*$ and both Browder's theorem and s-Browder's theorem transfer from A and B to $A \otimes B$. Hence

$$\begin{aligned}\sigma_{aw}(A^* \otimes B^*) &= \sigma_{sw}(A \otimes B) = \sigma_s(A)\sigma_{sw}(B) \cup \sigma_{sw}(A)\sigma_s(B)\\ &= \sigma_a(A^*)\sigma_{aw}(B^*) \cup \sigma_{aw}(A^*)\sigma_a(B^*) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)\\ &= \sigma_w(A \otimes B) = \sigma_w(A^* \otimes B^*).\end{aligned}$$

Thus, since $A^* \otimes B^*$ is a-polaroid and $A \otimes B)$ satisfies Browder's theorem imply $A^* \otimes B^*$ satisfy Browder's theorem,

$$\sigma(A^* \otimes B^*) \setminus \sigma_w(A^* \otimes B^*) = \pi^0(A^* \otimes B^*) = E_a^0(A^* \otimes B^*),$$

that is, $A^* \otimes B^*$ satisfies property (aw). □

A part of an operator is its restriction to an invariant subspace. $S \in \mathscr{L}(\mathbb{X})$ is said to be hereditarily polaroid, $S \in \mathcal{HP}$, if every part of S is polaroid. $\mathcal{HP}$ operators have SVEP [8, Lemma 2.8].

Corollary 2.1. *Suppose that the operators $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$ are $\mathcal{HP}$, then $A^* \otimes B^*$ satisfies property (aw).*

The class of $\mathcal{HP}$ operators is substantial: it includes in particular subscalar operators and paranormal operators (see [8] for further examples).

3. Perturbations

Let $[A, Q] = AQ - QA$ denote the commutator of the operators A and Q. If $Q_1 \in \mathscr{L}(\mathbb{X})$ and $Q_2 \in \mathscr{L}(\mathbb{Y})$ are quasinilpotent operators such that $[Q_1, A] = [Q_2, B] = 0$ for some operators $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$, then

$$(A + Q_1) \otimes (B + Q_2) = (A \otimes B) + Q,$$

where $Q = Q_1 \otimes B + A \otimes Q_2 + Q_1 \otimes Q_2 \in \mathscr{L}(\mathbb{X} \otimes \mathbb{Y})$ is a quasinilpotent operator. If in the above, Q_1 and Q_2 are nilpotent then $(A + Q_1) \otimes (B + Q_2)$ is the perturbation of $A \otimes B$ by a commuting nilpotent operator.

A bounded operator S on a Banach space $\mathbb{X}$ is called finite a-isoloid if every isolated point of $\sigma_a(S)$ is an eigenvalue of S of finite multiplicity.

Theorem 3.1. *Let $Q_1 \in \mathscr{L}(\mathbb{X})$ and $Q_2 \in \mathscr{L}(\mathbb{Y})$ be quasinilpotent operators such that $[Q_1, A] = [Q_2, B] = 0$ for some operators $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$. If $A \otimes B$ is finitely a-isoloid, then $A \otimes B$ satisfies property (aw) implies $(A + Q_1) \otimes (B + Q_2)$ satisfies property (aw).*

Proof. Start by recalling that $\sigma((A + Q_1) \otimes (B + Q_2)) = \sigma(A \otimes B)$, $\sigma_a((A + Q_1) \otimes (B + Q_2)) = \sigma_a(A \otimes B)$, $\sigma_{aw}((A + Q_1) \otimes (B + Q_2)) = \sigma_{aw}(A \otimes B)$, $\pi^0(A \otimes B) = \pi^0((A + Q_1) \otimes (B + Q_2))$ and that the perturbation of an operator by a commuting quasinilpotent has SVEP if and only if the operator has SVEP. If $A \otimes B$ satisfies property (aw), then

$$\begin{aligned} E_a^0(A \otimes B) &= \sigma(A \otimes B) \setminus \sigma_w(A \otimes B) \\ &= \sigma((A + Q_1) \otimes (B + Q_2)) \setminus \sigma_w((A + Q_1) \otimes (B + Q_2)). \end{aligned}$$

We prove that $E_a^0(A\otimes B) = E_a^0((A+Q_1)\otimes(B+Q_2))$. Observe that if $\lambda \in \sigma_a^{iso}(A\otimes B)$, then $A^* \otimes B^*$ has SVEP at λ, equivalently, $(A^* + Q_1^*) \otimes (B^* + Q_2^*)$ has SVEP at λ. Let $\lambda \in E_a^0(A \otimes B)$, then $\lambda \in \sigma((A + Q_1) \otimes (B + Q_2)) \setminus \sigma_w((A + Q_1) \otimes (B + Q_2))$. Since $(A + Q_1)^* \otimes (B + Q_2)^*$ has SVEP at λ, it follows that $\lambda \notin \sigma_w((A + Q_1) \otimes (B + Q_2))$ and $\lambda \in \sigma_a^{iso}((A + Q_1) \otimes (B + Q_2))$. Thus $\lambda \in E_a^0((A + Q_1) \otimes (B + Q_2))$. Hence $E_a^0(A\otimes B) \subseteq E_a^0((A+Q_1)\otimes(B+Q_2))$. Conversely, if $\lambda \in E_a^0((A+Q_1)\otimes(B+Q_2))$, then $\lambda \in \sigma_a^{iso}(A\otimes B)$, and this, since $A\otimes B$ is finitely a-isoloid, implies that $\lambda \in E_a^0(A\otimes B)$. Therefore, $E_a^0((A + Q_1) \otimes (B + Q_2)) \subseteq E_a^0(A \otimes B)$. So, the proof of the theorem is achieved. □

Corollary 3.1. *If $Q_1 \in \mathscr{L}(\mathbb{X})$ and $Q_2 \in \mathscr{L}(\mathbb{Y})$ are nilpotent operators such that $[Q_1, A] = [Q_2, B] = 0$ for some operators $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$, then $A \otimes B$ satisfies property (aw) implies $(A + Q_1) \otimes (B + Q_2)$ satisfies property (aw).*

The situation for perturbations by commuting Riesz operators is a bit more delicate. The equality $\sigma_a(T) = \sigma_a(T + R)$ does not always hold for operators $T, R \in \mathscr{L}(\mathbb{X})$ such that R is Riesz and $[T, R] = 0$; the tensor product $T \otimes R$ is not a Riesz operator (the Fredholm spectrum $\sigma_e(T \otimes R) = \sigma(T)\sigma_e(R) \cup \sigma_e(T)\sigma(R) = \sigma_e(T)\sigma(R) = \{0\}$ for a particular choice of T only). However, σ_w (also, σ_b) is stable under perturbation by commuting Riesz operators [18], and so T satisfies Browder's theorem if and only if $T + R$ satisfies Browder's theorem. Thus, if $\sigma(T) = \sigma(T + R)$ for a certain choice of operators $T, R \in \mathscr{L}(\mathbb{X})$ (such that R is Riesz and $[T, R] = 0$), then

$$\pi^0(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T + R) \setminus \sigma_w(T + R) = \pi^0(T + R),$$

where $\pi^0(T)$ is the set of $\lambda \in \sigma^{iso}(T)$ which are finite rank poles of the resolvent of T. If we now suppose additionally that T satisfies property (aw), then

$$E_a^0(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T + R) \setminus \sigma_w(T + R)$$

and a necessary and sufficient condition for $T + R$ to satisfy property (aw) is that $E_a^0(T + R) = E_a^0(T)$. One such condition, namely T is finitely a-isoloid.

Proposition 3.1. *Let $T, R \in \mathscr{L}(\mathbb{X})$, where R is Riesz, $[T, R] = 0$ and T is finitely a-isoloid. Then T satisfies property (aw) implies $T + R$ satisfies property (aw).*

Proof. Since Browder's theorem holds for $T + R$ by Lemma 2.2 of [12], it suffices to show that $\pi^0(T+R) = E_a^0(T+R)$. If $T-\lambda$ is invertible, then $T+R-\lambda$ is a Fredholm, and hence $\lambda \in E_a^0(T + R)$. Suppose $\lambda \in \sigma(T)$, then by Proposition 2.4 of [13] it follows that λ is an isolated point of $\sigma(T)$, and since by assumption T is finite-isoloid, we have $\lambda \in E_a^0(T)$. But property (aw) holds for T implies that $E_a^0(T) \cap \sigma_w(T) = \emptyset$.

Therefore, $T-\lambda$ is Fredholm and hence so is $T+R-\lambda$. Thus, $\lambda \in \pi^0(T+R)$. The other inclusion is trivial, therefore $T+R$ satisfies property (aw). □

Theorem 3.2. *Let $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$ be finitely a-isoloid operators which satisfy property (aw). If $R_1 \in \mathscr{L}(\mathbb{X})$ and $R_2 \in \mathscr{L}(\mathbb{Y})$ are Riesz operators such that $[A, R_1] = [B, R_2] = 0$, $\sigma(A+R_1) = \sigma(A)$ and $\sigma(B+R_2) = \sigma(B)$, then $A \otimes B$ satisfies property (aw) implies $(A+R_1) \otimes (B+R_2)$ satisfies property (aw) if and only if Browder's theorem transforms from $A+R_1$ and $B+R_2$ to their tensor product.*

Proof. The hypotheses imply (by Proposition 3.1) that both $A+R_1$ and $B+R_2$ satisfy property (aw). Suppose that $A \otimes B$ satisfies property (aw). Then $\sigma(A\otimes B) \setminus \sigma_{aw}(A \otimes B) = \pi^0(A \otimes B)$. Evidently $A \otimes B$ satisfies Browder's theorem, and so the hypothesis A and B satisfy property (aw) implies that Browder's theorem transfers from A and B to $A \otimes B$. Furthermore, since $\sigma(A+R_1) = \sigma(A)$, $\sigma(B+R_2) = \sigma(B)$ and σ_w is stable under perturbations by commuting Riesz operators,

$$\begin{aligned}\sigma_w(A\otimes B) &= \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)\\ &= \sigma(A+R_1)\sigma_w(B+R_2) \cup \sigma_w(A+R_1)\sigma(B+R_2).\end{aligned}$$

Suppose now that Browder's theorem transfers from $A+R_1$ and $B+R_2$ to $(A+R_1)\otimes(B+R_2)$. Then

$$\sigma_w(A\otimes B) = \sigma_w((A+R_1)\otimes(B+R_2))$$

and

$$E_a^0(A\otimes B) = \sigma((A+R_1)\otimes(B+R_2)) \setminus \sigma_w((A+R_1)\otimes(B+R_2)).$$

Let $\lambda \in E_a^0(A\otimes B)$. Then $\lambda \neq 0$, and hence there exist $\mu \in \sigma(A+R_1) \setminus \sigma_w(A+R_1)$ and $\nu \in \sigma(B+R_2) \setminus \sigma_w(B+R_2)$ such that $\lambda = \mu\nu$. As observed above, both $A+R_1$ and $B+R_2$ satisfy property (aw); hence $\mu \in E_a^0(A+R_1)$ and $\nu \in E_a^0(B+R_2)$. This, since $\lambda \in \sigma(A\otimes B) = \sigma((A+R_1)\otimes(B+R_2))$, implies $\lambda \in E_a^0((A+R_1)\otimes(B+R_2))$. Conversely, if $\lambda \in E_a^0((A+R_1)\otimes(B+R_2))$, then $\lambda \neq 0$ and there exist $\mu \in E_a^0(A+R_1) \subseteq \sigma_a^{iso}(A)$ and $\nu \in E_a^0(B+R_2) \subseteq \sigma_a^{iso}(B)$ such that $\lambda = \mu\nu$. Recall that $E_a^0((A+R_1)\otimes(B+R_2)) \subseteq E_a^0(A+R_1)E_a^0(B+R_2)$. Since A and B are finite a-isoloid, $\mu \in E_a^0(A)$ and $\nu \in E_a^0(B)$. Hence, since $\sigma((A+R_1)\otimes(B+R_2)) = \sigma(A\otimes B)$, $\lambda = \mu\nu \in E_a^0(A\otimes B)$. To complete the proof, we observe that if the implication of the statement of the theorem holds, then (necessarily) $(A+R_1)\otimes(B+R_2)$ satisfies Browder's theorem. This, since $A+R_1$ and $B+R_2$ satisfy Browder's theorem, implies Browder's theorem transfers from $A+R_1$ and $B+R_2$ to $(A+R_1)\otimes(B+R_2)$. □

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