WHEN IS A BI-JORDAN HOMOMORPHISM BI-HOMOMORPHISM?

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Abstract. For Banach algebras $A$ and $B$, we show that if $U = A \times B$ is commutative (weakly commutative), then each bi-Jordan homomorphism from $U$ into a semisimple commutative Banach algebra $D$ is a bi-homomorphism. We also prove the same result for 3-bi-Jordan homomorphism with the additional hypothesis that the Banach algebra $U$ is unital.

1. Introduction

Let $A$ and $B$ be complex Banach algebras and $\varphi : A \to B$ be a linear map. Then $\varphi$ is called an $n$-homomorphism if for all $a_1, a_2, \ldots, a_n \in A$,

$$\varphi(a_1a_2\ldots a_n) = \varphi(a_1)\varphi(a_2)\ldots\varphi(a_n).$$

The concept of an $n$-homomorphism was studied for complex algebras by Hejazian et al. in [5]. A 2-homomorphism is then just a homomorphism, in the usual sense. One may refer to [1] for certain properties of 3-homomorphisms.

The notion of $n$-Jordan homomorphisms was dealt with firstly by Herstein in [6]. A linear map $\varphi$ between Banach algebras $A$ and $B$ is called an $n$-Jordan homomorphism if

$$\varphi(a^n) = \varphi(a)^n, \quad a \in A.$$

A 2-Jordan homomorphism is called simply a Jordan homomorphism.

It is obvious that each $n$-homomorphism is an $n$-Jordan homomorphism, but in general the converse is false. The converse statement may be true under certain conditions. For example, it is shown in [2] that every $n$-Jordan homomorphism between two commutative Banach algebras is an $n$-homomorphism for $n \in \{2, 3, 4\},$
and this result extended to the case \( n = 5 \) in [3]. Lee in [7] generalized this result and proved it for all \( n \in \mathbb{N} \). See also [4] for another proof of Lee’s theorem.

Zelazko in [9] has given a characterization of Jordan homomorphism, that we mention in the following (see also [8]). We refer to [10] for another approach to the same result.

**Theorem 1.1.** Suppose that \( A \) is a Banach algebra, which need not be commutative, and suppose that \( B \) is a semisimple commutative Banach algebra. Then each Jordan homomorphism \( \varphi : A \rightarrow B \) is a homomorphism.

Also it is shown in [11] that Theorem 1.1 is valid for 3-Jordan homomorphism with the extra condition that the Banach algebra \( A \) is unital. Some significant results concerning Jordan homomorphisms and their generalizations on Banach algebras obtained by the author in [12].

Throughout the paper, let \( U = A \times B \). Then \( U \) is a Banach algebra for the multiplication
\[
(a, b)(x, y) = (ax, by), \quad (a, b), (x, y) \in U,
\]
and with norm
\[
\|(a, b)\| = \|a\| + \|b\|.
\]

Let \( D \) be a complex Banach algebra. A bilinear map is a function \( \varphi : U \rightarrow D \) such that for any \( a \in A \) the map \( b \mapsto \varphi(a, b) \) is linear map from \( B \) to \( D \), and for any \( b \in B \) the map \( a \mapsto \varphi(a, b) \) is linear map from \( A \) to \( D \).

A bilinear map \( \varphi \) is called an \( n \)-bi-homomorphism if for all \((a_i, b_i) \in U\),
\[
\varphi(a_1a_2...a_n, b_1b_2...b_n) = \varphi(a_1, b_1)\varphi(a_2, b_2)...\varphi(a_n, b_n),
\]
and it is called an \( n \)-bi-Jordan homomorphism if
\[
\varphi(a^n, b^n) = \varphi(a, b)^n, \quad (a, b) \in U.
\]

The concept of an \( n \)-bi-Jordan homomorphism introduced by the author in [13]. A (2-bi-Jordan) 2-bi-homomorphism is called simply a (bi-Jordan) bi-homomorphism.

It is obvious that each \( n \)-bi-homomorphism is \( n \)-bi-Jordan homomorphism, but in general the converse is not true.

Recently, the author proved [13] that every bi-Jordan homomorphism from unital commutative Banach algebra \( U \) into a semisimple commutative Banach algebra \( D \) is a bi-homomorphism.

In this paper, we extended this result for nonunital Banach algebra \( U \). We also prove the same result for 3-bi-Jordan homomorphism with the additional hypothesis that the Banach algebra \( U \) is unital.

**2. Characterization of Bi-Jordan Homomorphisms**

The following Theorem is the generalization of Theorem 4 of [13].
Theorem 2.1. Every bi-Jordan homomorphism \( \varphi \) from commutative Banach algebra \( U \) into a semisimple commutative Banach algebra \( D \) is a bi-homomorphism.

Proof. We first assume that \( D = \mathbb{C} \) and let \( \varphi : U \rightarrow \mathbb{C} \) be a bi-Jordan homomorphism. Then for all \( (a, b) \in U \),
\[
(2.1) \quad \varphi(a^2, b^2) = \varphi(a, b)^2.
\]
Replacing \( a \) by \( a + x \) and \( b \) by \( b + y \) in (2.1), gives
\[
(2.2) \quad \varphi(a^2 + x^2 + 2ax, b^2 + y^2 + 2by) = \varphi(a + x, b + y)^2.
\]

By Lemma 1 of [13], for all \( (a, b), (x, y) \in U \) we have
\[
(2.3) \quad \varphi(a^2, by) = \varphi(a, b)\varphi(a, y) \quad \text{and} \quad \varphi(ax, b^2) = \varphi(a, b)\varphi(x, b).
\]

It follows from (2.2) and (2.3) that
\[
(2.4) \quad 2\varphi(ax, by) = \varphi(a, b)\varphi(x, y) + \varphi(a, y)\varphi(x, b),
\]
for all \( (a, b), (x, y) \in U \). Take \( I = \varphi(a, b)\varphi(x, y), \quad J = \varphi(a, y)\varphi(x, b) \) and \( t = I - J \).

Then we get
\[
(2.5) \quad t^2 = I^2 + J^2 - 2IJ, \quad 4\varphi(ax, by)^2 = I^2 + J^2 + 2IJ.
\]

By (2.4) and (2.5), we deduce
\[
4\varphi(ax, by)^2 + t^2 = 2(I^2 + J^2) = 2[\varphi(a, b)^2\varphi(x, y)^2 + \varphi(a, y)^2\varphi(x, b)^2] = 2[\varphi(a^2, b^2)\varphi(x^2, y^2) + \varphi(a^2, y^2)\varphi(x^2, b^2)] = 4\varphi(a^2x^2, b^2y^2) = 4\varphi(ax, by)^2.
\]

Hence, \( t = 0 \), which proves that \( I = J \). Thus, by (2.4) we have
\[
\varphi(ax, by) = \varphi(a, b)\varphi(x, y),
\]
for all \( (a, b), (x, y) \in U \), so \( \varphi \) is a bi-homomorphism.

Now suppose that \( D \) is semisimple and commutative. Let \( \mathcal{M}(D) \) be the maximal ideal space of \( D \). We associate with each \( f \in \mathcal{M}(D) \) a function \( \varphi_f : U \rightarrow \mathbb{C} \) defined by
\[
\varphi_f(a, b) := f(\varphi(a, b)), \quad (a, b) \in U.
\]

Pick \( f \in \mathcal{M}(D) \) arbitrary. It is easy to see that \( \varphi_f \) is a bi-Jordan homomorphism, so by the above argument it is a bi-homomorphism. Thus, by the definition of \( \varphi_f \) we have
\[
f(\varphi(ax, by)) = f(\varphi(a, b))f(\varphi(x, y)) = f(\varphi(a, b)\varphi(x, y)).
\]

Since \( f \in \mathcal{M}(D) \) was arbitrary and \( D \) is assumed to be semisimple,
\[
\varphi(ax, by) = \varphi(a, b)\varphi(x, y),
\]
for all \( (a, b), (x, y) \in U \). This complete the proof. \( \square \)
A bilinear map \( \varphi : \mathcal{U} \to \mathcal{D} \) is called co-bi-homomorphism if
\[
\varphi(ax, by) = -\varphi(a, b)\varphi(x, y),
\]
for all \((a, b), (x, y) \in \mathcal{U}\), and it is called co-bi-Jordan homomorphism if
\[
\varphi(a^2, b^2) = -\varphi(a, b)^2, \quad (a, b) \in \mathcal{U}.
\]

By a same method as Theorem 2.1, we have the following result for co-bi-Jordan homomorphisms.

**Theorem 2.2.** Every co-bi-Jordan homomorphism from commutative Banach algebra \( \mathcal{U} \) into a semisimple commutative Banach algebra \( \mathcal{D} \) is a co-bi-homomorphism.

We say that the Banach algebra \( \mathcal{A} \) is weakly commutative if
\[
(ax)^2 = a^2x^2 \quad \text{and} \quad ax^2a = x^2a^2,
\]
for all \(a, x \in \mathcal{A}\). Clearly, every commutative Banach algebra is weakly commutative, but in general, the converse is false. For example, let
\[
\mathcal{A} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : \quad a, b \in \mathbb{R} \right\}.
\]
Then it is obvious to check that with the usual matrix product for all \(x, y \in \mathcal{A}\),
\[
(xy)^2 = x^2y^2 \quad \text{and} \quad xy^2x = y^2x^2.
\]
Thus, \( \mathcal{A} \) is weakly commutative, but it is neither unital nor commutative.

Note that a unital Banach algebra is weakly commutative if and only if it is commutative.

**Lemma 2.1.** Let \( \mathcal{U} \) be a weakly commutative Banach algebra, and \( \varphi : \mathcal{U} \to \mathbb{C} \) be a bi-Jordan homomorphism. Then
\[
\varphi(ax, by) = \varphi(ax, yb) = \varphi(xa, by),
\]
for all \((a, b), (x, y) \in \mathcal{U}\).

**Proof.** By Lemma 1 of [13],
\[
(2.6) \quad \varphi(a^2, by + yb) = 2\varphi(a, b)\varphi(a, y), \quad (a, b), (a, y) \in \mathcal{U}.
\]
Replacing \(a\) by \(ax\) in (2.6) we get
\[
(2.7) \quad \varphi((ax)^2, by + yb) = 2\varphi(ax, b)\varphi(ax, y).
\]
Replacing \(b\) by \(by\) and \(y\) by \(yb\) in (2.7), gives
\[
(2.8) \quad \varphi((ax)^2, by^2b + yb^2y) = 2\varphi(ax, by)\varphi(ax, yb).
\]
Since $U$ is weakly commutative, by (2.8) we have
\[
2\varphi(ax, by)\varphi(ax, yb) = \varphi((ax)^2, by^2b + yb^2y) \\
= \varphi((ax)^2, yb^2 + b^2y^2) \\
= \varphi((ax)^2, b^2y^2) + \varphi((ax)^2, yb^2) \\
= \varphi(ax, by)^2 + \varphi(ax, yb)^2.
\]
Thus,
\[
(\varphi(ax, by) - \varphi(ax, yb))^2 = 0,
\]
which proves that
\[
\varphi(ax, by) = \varphi(ax, yb),
\]
for all $(a, b), (x, y) \in U$. In a similar way, we can prove that $\varphi(ax, by) = \varphi(xa, by)$. This complete the proof. □

The next result is the generalization of Theorem 2.1.

**Theorem 2.3.** Suppose that $\varphi$ is a bi-Jordan homomorphism from weakly commutative Banach algebra $U$ into a semisimple commutative Banach algebra $D$. Then $\varphi$ is a bi-homomorphism.

**Proof.** We first assume that $D = \mathbb{C}$ and let $\varphi : U \to \mathbb{C}$ be a bi-Jordan homomorphism. Then for all $(a, b) \in U$,
\[
(2.9) \quad \varphi(a^2, b^2) = \varphi(a, b)^2.
\]
Replacing $a$ by $a + x$ and $b$ by $b + y$ in (2.9), gives
\[
(2.10) \quad \varphi(ax + xa, by + yb) = 2\varphi(a, b)\varphi(x, y) + 2\varphi(a, y)\varphi(x, b),
\]
for all $(a, b), (x, y) \in U$. It follows from (2.10) and Lemma 2.1 that
\[
4\varphi(ax, by) = \varphi(ax + xa, by + yb) \\
= 2\varphi(a, b)\varphi(x, y) + 2\varphi(a, y)\varphi(x, b).
\]
Hence,
\[
2\varphi(ax, by) = \varphi(a, b)\varphi(x, y) + \varphi(a, y)\varphi(x, b),
\]
for all $(a, b), (x, y) \in U$. Thus, the relation (2.4) in Theorem 2.1 holds. Now the rest of proof is similar to the proof of Theorem 2.1. □

As a consequence of Theorem 2.3 we have the next result.

**Corollary 2.1.** Suppose that $U$ is weakly commutative and $\varphi : U \to \mathbb{C}$ satisfies
\[
(2.11) \quad |\varphi(ax, by) - \varphi(a, b)\varphi(x, y)| \leq \delta(\|(a, b)\| + \|(x, y)\|),
\]
for all $(a, b), (x, y) \in U$ and for some $\delta \geq 0$. Then $\varphi$ is a bi-homomorphism.
Proof. Replacing \((x, y)\) by \((a, b)\) in (2.11), gives
\[(2.12) \quad \vert \varphi(a^2, b^2) - \varphi(a, b)^2 \vert \leq 2\delta(\|a\| + \|b\|),\]
for all \((a, b) \in \mathcal{U}\). Take \(a = 2^n x\) and \(b = 2^n y\) in (2.12), then
\[\vert \varphi(x^2, y^2) - \varphi(x, y)^2 \vert \leq \frac{2^{n+1}\delta(\|x\| + \|y\|)}{2^{4n}} \to 0,\]
as \(n \to \infty\). Hence,
\[\varphi(x^2, y^2) = \varphi(x, y)^2, \quad (x, y) \in \mathcal{U}.\]
Therefore, \(\varphi\) is a bi-Jordan and so it is a bi-homomorphism by Theorem 2.3. \(\square\)

Example 2.1. Let
\[
\mathcal{U} = \left\{ \left( \begin{array}{cc} a & b \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} x & y \\ 0 & 0 \end{array} \right) : a, b, x, y \in \mathbb{R} \right\}.
\]
Then \(\mathcal{U}\) is a weakly commutative Banach algebra, but it is not commutative. Hence by Theorem 2.3, each bi-Jordan homomorphism from \(\mathcal{U}\) into a semisimple commutative Banach algebra \(\mathcal{D}\) is a bi-homomorphism and via versa.

The commutativity of Banach algebra \(\mathcal{D}\) in Theorem 2.3 is essential. For example, let
\[
\mathcal{A} = \left\{ \left( \begin{array}{cc} a & b \\ 0 & 0 \end{array} \right) : a, b \in \mathbb{R} \right\},
\]
as above and let \(\mathcal{A}^2\) be the unitization of \(\mathcal{A}\) with the identity matrix as a unit. Set \(\mathcal{U} = \mathcal{A} \times \mathcal{A}^2\) and define \(\varphi : \mathcal{U} \to \mathcal{A}\) by \(\varphi(x, y) = xy\). Then for all \((x, y) \in \mathcal{U}\),
\[\varphi(x^2, y^2) = \varphi(x, y)^2.\]
Hence \(\varphi\) is bi-Jordan homomorphism, but it is not bi-homomorphism. Because, let
\[
x = \left( \begin{array}{cc} a & b \\ 0 & 0 \end{array} \right), \quad y = \left( \begin{array}{cc} c & d \\ 0 & 0 \end{array} \right), \quad m = \left( \begin{array}{cc} s & t \\ 0 & 0 \end{array} \right) \quad \text{and} \quad n = I,
\]
where \(I\) is the identity matrix. Then \((x, y), (m, n) \in \mathcal{U}\), but
\[\varphi(xm, yn) \neq \varphi(x, y)\varphi(m, n).\]

3. Characterization of 3-bi-Jordan Homomorphisms

Clearly, the Banach algebra \(\mathcal{U}\) is unital if and only if both \(\mathcal{A}\) and \(\mathcal{B}\) are unital. Without any confusion we denote by \(e\), the unit element of both \(\mathcal{A}\) and \(\mathcal{B}\).

Lemma 3.1. Let \(\mathcal{U}\) be a unital commutative Banach algebra, and \(\varphi : \mathcal{U} \to \mathbb{C}\) be a 3-bi-Jordan homomorphism. Then for all \((a, b) \in \mathcal{U}\),
\[
(a) \quad \varphi(a^3, b^2 + b) = \varphi(a, b)\varphi(a, e) + \varphi(a, b)\varphi(a, e)^2,
(b) \quad \varphi(a^2 + a, b^3) = \varphi(a, b)\varphi(e, b) + \varphi(a, b)\varphi(e, b)^2.
\]
Proof. The proof is straightforward. \(\square\)

Lemma 3.2. By the hypotheses of above Lemma, for all \((a, b), (x, y) \in \mathcal{U}\),
(a) $3\varphi(ax^2, b) = \varphi(a, b)\varphi(x, e)^2 + 2\varphi(a, e)\varphi(x, e)\varphi(x, b)$,
(b) $3\varphi(ab^2) = \varphi(a, b)\varphi(e, y)^2 + 2\varphi(e, b)\varphi(e, y)\varphi(a, y)$.

**Proof.** We prove (a), that the assertion (b) can be proved similarly. Let $\varphi : U \to \mathbb{C}$ be a 3-bi-Jordan homomorphism. Then for all $(a, b) \in U$,

(3.1) $\varphi(a^3, b^3) = \varphi(a, b)^3$.

Replacing $b$ by $b + y$ in (3.1), gives

(3.2) $\varphi(a^3, b^2y + by^2) = \varphi(a, b)^2\varphi(a, y) + \varphi(a, b)\varphi(a, y)^2$,

for all $(a, b), (a, y) \in U$. Replacing $y$ by $-y$ in (3.2), we get

(3.3) $\varphi(a^3, -b^2y + by^2) = -\varphi(a, b)^2\varphi(a, y) + \varphi(a, b)\varphi(a, y)^2$.

By (3.2) and (3.3) we have

(3.4) $\varphi(a^3, by^2) = \varphi(a, b)\varphi(a, y)^2$.

Replacing $y$ by $e$ in (3.4), gives

(3.5) $\varphi(a^3, b) = \varphi(a, b)\varphi(a, e)^2$.

Replacing $a$ by $a + x$ in (3.5), to obtain

(3.6) $3\varphi(ax^2 + a^2x, b) = I + J$,

where

$I = \varphi(x, b)\varphi(a, e)^2 + 2\varphi(a, b)\varphi(a, e)\varphi(x, e)$

and

$J = \varphi(a, b)\varphi(x, e)^2 + 2\varphi(a, e)\varphi(x, b)\varphi(x, e)$. Replacing $x$ by $-x$ in (3.6), we get

(3.7) $3\varphi(ax^2 - a^2x, b) = -I + J$.

By (3.6) and (3.7) we have

$3\varphi(ax^2, b) = \varphi(a, b)\varphi(x, e)^2 + 2\varphi(a, e)\varphi(x, b)\varphi(x, e)$,

for all $(a, b), (x, e) \in U$, as required. \qed

Now we state and prove the main Theorem of this section.

**Theorem 3.1.** Suppose that $\varphi$ is a 3-bi-Jordan homomorphism from unital commutative Banach algebra $U$ into $\mathbb{C}$. Then $\varphi$ is a 3-bi-homomorphism.

**Proof.** Let $\varphi : U \to \mathbb{C}$ be a 3-bi-Jordan homomorphism. Then

(3.8) $\varphi(a^3, b^3) = \varphi(a, b)^3$, \quad $(a, b) \in U$.

Replacing both of $a$ and $b$ by $e$, gives $\varphi(e, e) = \varphi(e, e)^3$. Since $\varphi(e, e) \neq 0$, so $\varphi(e, e) = 1$ or $\varphi(e, e) = -1$. We first assume that $\varphi(e, e) = 1$. Replacing $a$ by $a + e$ and $b$ by $b + e$ in (3.8), and simplifies the result by Lemma 3.1, we get

(3.9) $9\varphi(a^2 + a, b^2 + b) = 3\{\varphi(a, b)^2 + \varphi(a, b) + P + Q + R + S\}$,
where
\[ P = 2\varphi(a, b)\varphi(a, e) + \varphi(e, b)\varphi(a, e)^2, \quad Q = 2\varphi(a, b)\varphi(e, b) + \varphi(a, e)\varphi(e, b)^2, \]
\[ R = 2\varphi(a, e)\varphi(e, b), \quad S = 2\varphi(a, e)\varphi(a, b)\varphi(e, b). \]

It follows from preceding Lemma that for all \((a, b) \in \mathcal{U}\),
\begin{equation}
(3.10) \quad P = 3\varphi(a^2, b), \quad Q = 3\varphi(a, b^2), \quad R = 2\varphi(a, b) \quad \text{and} \quad S = 2\varphi(a, b)^2.
\end{equation}

By (3.9) and (3.10) we obtain
\[ \varphi(a^2, b^2) = \varphi(a, b)^2, \]
for all \((a, b) \in \mathcal{U}\). Hence, \(\varphi\) is bi-Jordan homomorphism and so it is bi-homomorphism by Theorem 2.1. Thus, \(\varphi\) is 3-bi-homomorphism.

Now suppose that \(\varphi(e, e) = -1\). Then by a similar argument we have
\[ \varphi(a^2, b^2) = -\varphi(a, b)^2, \quad (a, b) \in \mathcal{U}. \]

Therefore by Theorem 2.2, \(\varphi\) is co-bi-homomorphism. That is,
\[ \varphi(ax, by) = -\varphi(a, b)\varphi(x, y), \]
for all \((a, b), (x, y) \in \mathcal{U}\). Thus,
\begin{align*}
\varphi(axu, byv) &= -\varphi(a, b)[\varphi(xu, yv)] \\
&= -\varphi(a, b)[-\varphi(x, y)\varphi(u, v)] \\
&= \varphi(a, b)\varphi(x, y)\varphi(u, v),
\end{align*}
for all \((a, b), (x, y), (u, v) \in \mathcal{U}\). So \(\varphi\) is 3-bi-homomorphism, as claimed. \(\square\)

As a consequence of Theorem 3.1 we have the next result.

**Corollary 3.1.** Suppose that \(\varphi\) is a 3-bi-Jordan homomorphism from unital commutative Banach algebra \(\mathcal{U}\) into a semisimple commutative Banach algebra \(\mathcal{D}\). Then \(\varphi\) is a 3-bi-homomorphism.

In view of Theorem 1.1 and Theorem 2.1, the following question suggests itself: does Theorem 2.1 hold without commutativity of \(\mathcal{U}\)?

**Acknowledgements.** The author gratefully acknowledges the helpful comments of the anonymous referees.

This research was partially supported by the grant from Ayatollah Borujerdi University with No. 15664–137285.
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