Abstract. Assume that $R$ is a commutative ring with nonzero identity. In this paper, we introduce and investigate zero-annihilator graph of $R$ denoted by $\text{ZA}(R)$. It is the graph whose vertex set is the set of all nonzero nonunit elements of $R$ and two distinct vertices $x$ and $y$ are adjacent whenever $\text{Ann}_R(x) \cap \text{Ann}_R(y) = \{0\}$.

1. Introduction

Throughout this paper all rings are commutative with nonzero identity. In [6], Beck associated to a ring $R$ its zero-divisor graph $G(R)$ whose vertices are the zero-divisors of $R$ (including 0), and two distinct vertices $x$ and $y$ are adjacent if $xy = 0$. Later, in [3], Anderson and Livingston studied the subgraph $\Gamma(R)$ of $G(R)$ (whose vertices are the nonzero zero-divisors of $R$). In the recent years, several researchers have done interesting and enormous works on this field of study. For instance, see [4, 5, 9]. The concept of co-annihilating ideal graph of a ring $R$, denoted by $\text{A}_R$ was introduced by Akbari et al. in [1]. As in [1], co-annihilating ideal graph of $R$, denoted by $\text{A}_R$, is a graph whose vertex set is the set of all non-zero proper ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent whenever $\text{Ann}_R(I) \cap \text{Ann}_R(J) = \{0\}$. In the present paper, we introduce zero-annihilator graph of $R$ denoted by $\text{ZA}(R)$. It is the graph whose vertex set is the set of all nonzero nonunit elements of $R$ and two distinct vertices $x$ and $y$ are adjacent whenever $\text{Ann}_R(Rx + Ry) = \text{Ann}_R(x) \cap \text{Ann}_R(y) = \{0\}$. Note that $\text{ZA}(R)$ is an induced subgraph of $\text{A}_R$.

Let $G$ be a simple graph with the vertex set $V(G)$ and edge set $E(G)$. For every vertex $v \in V(G)$, $N_G(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$. The degree of a vertex $v$ is defined as $\deg_G(v) = |N_G(v)|$. The minimum degree of $G$ is denoted
by \( \delta(G) \). Recall that a graph \( G \) is connected if there is a path between every two distinct vertices. For distinct vertices \( x \) and \( y \) of a connected graph \( G \), let \( d_G(x, y) \) be the length of the shortest path from \( x \) to \( y \). The diameter of a connected graph \( G \) is \( \operatorname{diam}(G) = \sup\{d_G(x, y) \mid x \text{ and } y \text{ are distinct vertices of } G\} \). The girth of \( G \), denoted by \( \operatorname{girth}(G) \), is defined as the length of the shortest cycle in \( G \) and \( \operatorname{girth}(G) = \infty \) if \( G \) contains no cycles. A bipartite graph is a graph all of whose vertices can be partitioned into two parts \( U \) and \( V \) such that every edge joins a vertex in \( U \) to a vertex in \( V \). A complete bipartite graph \( G \) is a bipartite graph with parts \( U, V \) such that every vertex in \( U \) is adjacent to every vertex in \( V \). A graph in which all vertices have degree \( k \) is called a \( k \)-regular graph. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. Also, if a graph \( G \) contains one vertex to which all other vertices are joined and \( G \) has no other edges, is called a star graph. A clique in a graph \( G \) is a subset of pairwise adjacent vertices and the number of vertices in a maximum clique of \( G \), denoted by \( \omega(G) \), is called the clique number of \( G \). The chromatic number of \( G \), denoted by \( \chi(G) \), is the minimum number of colors needed to color the vertices of \( G \) so that no two adjacent vertices have the same color. Obviously, \( \chi(G) \geq \omega(G) \).

2. Some Properties of \( \mathcal{ZA}(R) \)

Recall that, an empty graph is a graph with no edges. A Bézout ring is a ring in which all finitely generated ideals are principal.

**Theorem 2.1.** Let \( R \) be a ring. If \( \mathcal{ZA}(R) \) is an empty graph, then \( R \) is a local ring and \( \text{Ann}_R(x) \neq \{0\} \) for every nonunit element \( x \in R \). The converse is true if \( R \) is a Bézout ring.

*Proof.* Assume that \( \mathcal{ZA}(R) \) is empty. Let \( m_1, m_2 \) be two distinct maximal ideals of \( R \). Then \( m_1 + m_2 = R \) implies that there exist \( x \in m_1 \) and \( x_2 \in m_2 \) such that \( x + y = 1 \). So \( x \) and \( y \) are adjacent, which is a contradiction. Hence \( R \) is a local ring. Let \( m \) be the maximal ideal of \( R \) and \( x \) be an element of \( m \). Suppose that \( \text{Ann}_R(x) = \{0\} \). Then \( \{x^n \mid n \in \mathbb{N}\} \) is an infinite clique in \( \mathcal{ZA}(R) \) that is a contradiction. So \( \text{Ann}_R(x) \neq \{0\} \).

Suppose that \( R \) is a local Bézout ring and \( \text{Ann}_R(x) \neq \{0\} \) for every nonunit element \( x \in R \). Let \( x, y \) be two vertices in \( \mathcal{ZA}(R) \). Then \( x, y \in m \). Hence \( Rx + Ry = Rz \) for some nonzero nonunit element \( z \in R \). So \( x, y \) are not adjacent which shows that \( \mathcal{ZA}(R) \) is empty. \( \square \)

**Remark 2.1.** Suppose that \( R \) has a nontrivial idempotent element \( e \). Then \( e + (1 - e) = 1 \) implies that \( e \) and \( 1 - e \) are adjacent. Hence \( \deg_{\mathcal{ZA}(R)}(e) \geq 1 \) and so \( \mathcal{ZA}(R) \) is not an empty graph.

**Remark 2.2.** Let \( R \) be a ring. Notice that if \( R \) is an Artinian ring or a Boolean ring, then \( \dim(R) = 0 \). By [2, Theorem 3.4], \( \dim(R) = 0 \) if and only if for every \( x \in R \) there exists a positive integer \( n \) such that \( x^{n+1} \) divides \( x^n \). Therefore, every nonzero
nonunit element of a zero-dimensional ring has a nonzero annihilator. Hence, if \( R \) is a zero-dimensional chained ring, then \( Z^*(R) \) is an empty graph.

Let \( Z^+(R) \) denote the zero divisors of \( R \) and \( Z(R) = Z^*(R) \cup \{0\} \).

**Theorem 2.2.** Let \( R \) be a ring and \( S \) be a multiplicative closed subset of \( R \) such that \( S \cap Z(R) = \{0\} \). Then \( Z^*(R) \approx Z^*(R_S) \).

*Proof.* Define the vertex map \( \Phi : V(Z^*(R)) \to V(Z^*(R_S)) \) by \( x \mapsto \frac{x}{y} \). We can easily verify that \( x = y \) if and only if \( \frac{x}{y} \in S \). Also, it is easy to see that \( \text{Ann}_R(x) \cap \text{Ann}_R(y) = \{0\} \) if and only if \( \text{Ann}_{R_S}(\frac{x}{y}) \cap \text{Ann}_{R_S}(\frac{y}{x}) = \{0\} \). \( \square \)

**Theorem 2.3.** Let \( R \) be a Bézout ring with \( |\text{Max}(R)| < \infty \) such that \( \delta(Z^*(R)) > 0 \). Then \( Z^*(R) \) is a finite graph if and only if every vertex of \( Z^*(R) \) has finite degree.

*Proof.* The “only if” part is evident.

Suppose that each vertex of \( Z^*(R) \) has finite degree. If \( \text{Ann}_R(x) = \{0\} \) for some nonzero nonunit element \( x \in R \), then \( x \) is adjacent to all vertices of \( Z^*(R) \) that implies \( Z^*(R) \) is a finite graph. Assume that \( \text{Ann}_R(x) \neq \{0\} \) for each nonzero nonunit element \( x \in R \). We claim that \( \text{Jac}(R) = \{0\} \). On the contrary, assume that there exists a nonzero element \( a \in \text{Jac}(R) \). Since \( Z^*(R) \) has no isolated vertex, \( a \) is adjacent to another vertex, say \( b \). Since \( R \) is a Bézout ring, \( Ra + Rb \) is generated by a nonzero nonunit element \( c \) of \( R \) and so \( \text{Ann}_R(Ra + Rb) = \text{Ann}_R(c) \neq \{0\} \), which is impossible. So \( \text{Jac}(R) = \{0\} \). Hence by Chinese Remainder Theorem we have \( R \approx F_1 \times F_2 \times \cdots \times F_n \), where \( F_i \)'s are fields and \( n = |\text{Max}(R)| \). Let \( 0 \neq u \in F_1 \). Then \((u,0,\ldots,0)\) and \((0,1,\ldots,1)\) are adjacent. Since \((0,1,\ldots,1)\) has finite degree, so \( F_1 \) is a finite field. Similarly we can show that \( F_i \)'s are finite fields. Consequently \( R \) has finitely many nonzero nonunit elements and the proof is complete. \( \square \)

**Theorem 2.4.** Let \( R \) be a Bézout ring with \( |\text{Max}(R)| < \infty \). Then the following conditions are equivalent:

(a) \( Z^*(R) \) is a bipartite graph with \( \delta(Z^*(R)) > 0 \);
(b) \( Z^*(R) \) is a complete bipartite graph;
(c) \( R \approx F_1 \times F_2 \) where \( F_1 \) and \( F_2 \) are two fields.

*Proof.* (a)\(\Rightarrow\)(c) Suppose that \( Z^*(R) \) is a bipartite graph with \( \delta(Z^*(R)) > 0 \). If \( \text{Ann}_R(x) = \{0\} \) for some nonzero nonunit element \( x \) of \( R \), then \( \{x^n \mid n \in \mathbb{N}\} \) is an infinite clique that is a contradiction. Then, for every nonzero nonunit element \( x \) of \( R \) we have \( \text{Ann}_R(x) \neq \{0\} \). Similar to the proof of Theorem 2.3 we can show that \( R = F_1 \times F_2 \times \cdots \times F_n \), where \( F_i \)'s are fields and \( n = |\text{Max}(R)| \). Clearly \( n \neq 1 \). If \( n \geq 3 \), then \( \{(0,1,\ldots,1),(1,0,1,\ldots,1),(1,1,0,1,\ldots,1)\} \) is a clique in \( Z^*(R) \), a contradiction. So \( R \approx F_1 \times F_2 \).

(c)\(\Rightarrow\)(b) Suppose that \( R \approx F_1 \times F_2 \) where \( F_1 \) and \( F_2 \) are two fields. Every vertex in \( Z^*(R) \) is of the form \((u,0)\) or \((0,v)\) where \( 0 \neq u \in F_1 \) and \( 0 \neq v \in F_2 \). Also, two vertices \((u,0)\) and \((0,v)\) are adjacent. On the other hand, every two vertices \((u_1,0),(u_2,0)\) cannot be adjacent.
(b)⇒(a) is clear.

**Theorem 2.5.** Let \( R \) be a ring and \( n \geq 2 \) be a natural number. Then
\[
\text{girth}(\mathcal{Z}(M_n(R))) = 3.
\]

**Proof.** For \( n = 2 \), the following matrices are pairwise adjacent in \( \mathcal{Z}(M_2(R)) \):
\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix}.
\]
For \( n \geq 3 \), the following matrices are pairwise adjacent in \( \mathcal{Z}(M_n(R)) \):
\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]

3. **When is \( \mathcal{Z}(R) \) Connected?**

A ring \( R \) is called *semiprimitive* if \( \text{Jac}(R) = 0 \), [7]. A ring \( R \) is semiprimitive if and only if it is a subdirect product of fields, [8, p. 179].

**Theorem 3.1.** Let \( R \) be a semiprimitive ring. If at least one of the maximal ideals of \( R \) is principal, then \( \mathcal{Z}(R) \) is a connected graph with \( \text{diam}(\mathcal{Z}(R)) \leq 4 \).

**Proof.** Suppose that \( m \) is a maximal ideal of \( R \) where \( m = Rt \) for some \( t \in R \). Let \( x, y \) be two different nonzero nonunit elements of \( R \). Consider the following cases.

**Case 1.** Let \( x, y \notin m \). Then \( Rx + m = R \) and \( Ry + m = R \). Hence \( x, y \) are adjacent to \( t \). So \( d_{\mathcal{Z}(R)}(x, y) \leq 2 \).

**Case 2.** Let \( x \in m \) and \( y \notin m \). Notice that \( y \) is adjacent to \( t \). Since \( \text{Jac}(R) = \{0\} \), there exists a maximal ideal \( m' \) different from \( m \) such that \( x \notin m' \). So \( Rx + m' = R \), and thus there exist elements \( r \in R \) and \( z \in m' \) such that \( rx + z = 1 \). Therefore \( \text{Ann}_R(x) \cap \text{Ann}_R(z) = \{0\} \). So \( x \) is adjacent to \( z \). Clearly \( z \notin m \). Then \( z \) is adjacent to \( t \). Hence \( d_{\mathcal{Z}(R)}(x, y) \leq 3 \).

**Case 3.** Let \( x, y \in m \). A manner similar to Case 2 shows that \( d_{\mathcal{Z}(R)}(x, t) \leq 2 \) and \( d_{\mathcal{Z}(R)}(y, t) \leq 2 \). Therefore \( d_{\mathcal{Z}(R)}(x, y) \leq 4 \).
Consequently $ZA(R)$ is a connected graph with $\text{diam}(ZA(R)) \leq 4$. □

**Theorem 3.2.** Let $R$ be a Bézout ring. If $ZA(R)$ is connected, then one of the following conditions holds:

(a) there exists a nonzero nonunit element $x$ of $R$ such that $\text{Ann}_R(x) = \{0\}$;

(b) $\text{Jac}(R) = \{0\}$;

(c) $\text{Jac}(R) = \{0, x\}$ where $x$ is the only nonzero nonunit element of $R$.

**Proof.** Assume that for every nonzero nonunit element $u$ of $R$, $\text{Ann}_R(u) \neq \{0\}$ and also $\text{Jac}(R) \neq \{0\}$. Let $x$ be a nonzero element in $\text{Jac}(R)$. Suppose that $ZA(R)$ has a vertex $y$ different from $x$. Thus $Rx + Ry = Rz$ for some $z \in R$, because $R$ is a Bézout ring. Notice that $y \in m$ for some maximal ideal $m$ of $R$. Hence $z$ is nonzero nonunit and so by assumption $\text{Ann}_R(z) \neq \{0\}$, which shows that $x$ and $y$ are not adjacent. This contradiction implies that $|V(ZA(R))| = 1$, and so $\text{Jac}(R) = \{0, x\}$. □

As a direct consequence of Theorem 3.1 and Theorem 3.2 we have the following result.

**Corollary 3.1.** Let $R$ be a Bézout ring such that at least one of the maximal ideals of $R$ is principal. Then $ZA(R)$ is connected if and only if one of the following conditions holds:

(a) there exists a nonzero nonunit element $x$ of $R$ such that $\text{Ann}_R(x) = \{0\}$;

(b) $\text{Jac}(R) = \{0\}$;

(c) $\text{Jac}(R) = \{0, x\}$ where $x$ is the only nonzero nonunit element of $R$.

**Theorem 3.3.** Let $R = F_1 \times F_2 \times \cdots \times F_n$ where $F_i$’s are fields. Then $ZA(R)$ is a connected graph with

$$\text{diam}(ZA(R)) = \begin{cases} 
1, & \text{if } n = 2 \text{ and } |F_1| = |F_2| = 2, \\
2, & \text{if } n = 2 \text{ and either } |F_1| > 2 \text{ or } |F_2| > 2, \\
3, & \text{if } n \geq 3.
\end{cases}$$

**Proof.** Let $n = 2$. In this case every vertex in $ZA(R)$ is of the form $(u, 0)$ or $(0, v)$ where $u \neq 0$ and $v \neq 0$. Furthermore, two vertices $(u, 0)$ and $(0, v)$ are adjacent.

In the case when $n = 2$ and $|F_1| = |F_2| = 2$, we have $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. So $ZA(R) \simeq K_2$.

Let $n = 2$ and $|F_1| > 2$. In this case, every two different vertices $(u_1, 0)$ and $(u_2, 0)$ cannot be adjacent. On the other hand $(u_1, 0)$ and $(u_2, 0)$ are adjacent to $(0, 1)$. So $d_{ZA(R)}((u_1, 0), (u_2, 0)) = 2$. Hence $\text{diam}(ZA(R)) = 2$.

Now, let $n \geq 3$. Assume that $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are two different vertices. There exist two indexes $i, j$ such that $u_i \neq 0$ and $v_j \neq 0$. So $u = (u_1, u_2, \ldots, u_n)$ is adjacent to $(1, \ldots, 1, 0, 1, \ldots, 1)$. Also $v = (v_1, v_2, \ldots, v_n)$ is adjacent to $(1, \ldots, 1, 0, 1, \ldots, 1)$. If $i \neq j$, then the vertex $(1, \ldots, 1, 0, 1, \ldots, 1)$
is adjacent to \((1, \ldots, 1, 0, 1, \ldots, 1)\). Thus \(ZA(R)\) is connected and \(d_{ZA(R)}(u, v) \leq 3\).

In special case, we have the following path
\[(0, 1, 0, \ldots, 0) \rightarrow (1, 0, 1, \ldots, 1) \rightarrow (0, 1, \ldots, 1) \rightarrow (1, 0, \ldots, 0).
\]
Consequently \(\text{diam}(ZA(R)) = 3\).

\[\square\]

**4. When is \(ZA(R)\) Star?**

**Lemma 4.1.** Let \(R\) be a ring. If \(ZA(R)\) is a star, then \(|\text{Max}(R)| \leq 2\).

**Proof.** Suppose that \(ZA(R)\) is a star. If \(m\) and \(m'\) are two different maximal ideals of \(R\), then for every \(x \in m \setminus m'\) we have \(Rx + m' = R\). Hence there exist elements \(r \in R\)
and \(y \in m' \setminus m\) such that \(rs + y = 1\). Therefore \(\text{Ann}_R(x) \cap \text{Ann}_R(y) = \{0\}\). So \(x\) and \(y\) are adjacent. Let \(m_1, m_2\) and \(m_3\) be three different maximal ideals of \(R\). Then there are elements \(a \in m_1 \setminus (m_2 \cup m_3), b \in m_2 \setminus (m_1 \cup m_3)\) and \(c \in m_3 \setminus (m_1 \cup m_2)\). Then either \(a, b, c\) are pairwise adjacent or there exist at least two disjoint edges in \(ZA(R)\), which is a contradiction. Consequently \(|\text{Max}(R)| \leq 2\).

\[\square\]

**Theorem 4.1.** Let \(R\) be a Bézout ring that is not a field. Then \(ZA(R)\) is a star if and only if one of the following conditions holds:

1. \((R, m)\) when \(m = \{0, x\}\) in which \(x\) is a nonzero element of \(R\) with \(x^2 = 0\);
2. \(R \simeq \mathbb{Z}_2 \times F\) where \(F\) is a field.

**Proof.** \((\Rightarrow)\) Suppose that \(ZA(R)\) is a star. Hence \(|\text{Max}(R)| \leq 2\), by Lemma 4.1. Notice that if \(\text{Ann}_R(t) = \{0\}\) for some element \(t\) of a maximal ideal \(m\), then \(\{t^n \mid n \in \mathbb{N}\}\) is an infinite clique that is impossible. Consider the following cases:

**Case 1.** \(\text{Max}(R) = \{m\}\). Let \(x\) be a nonzero element in \(m\). Then by Theorem 2.1, \(ZA(R)\) is empty and so \(m = \{0, x\}\). On the other hand, by Nakayama’s Lemma we have that \(x^2 = 0\).

**Case 2.** \(\text{Max}(R) = \{m_1, m_2\}\). Since \(m_1 + m_2 = R\), there exist \(x \in m_1\) and \(y \in m_2\) such that \(x + y = 1\). Hence \(x\) and \(y\) are adjacent. Now, if there exists \(0 \neq z \in m_1 \cap m_2\), then \(z\) is not adjacent to \(x\) and \(y\), because \(R\) is a Bézout ring and \(\text{Ann}_R(t) = \{0\}\) for every nonzero nonunit element \(t\) of \(R\). This contradiction shows that \(m_1 \cap m_2 = \{0\}\). Hence by Chinese Remainder Theorem we deduce that \(R \simeq R/m_1 \oplus R/m_2\). If there exist nonzero elements \(a_1, a_2 \in R/m_1\) and \(b_1, b_2 \in R/m_2\), then we have the following path
\[(a_1, 0) - (0, b_1) - (a_2, 0) - (0, b_2),
\]
a contradiction. Hence we can assume that \(R/m_1 = \mathbb{Z}_2\).

\((\Leftarrow)\) If (a) holds, then clearly \(ZA(R)\) is a star. Assume that (b) holds. Notice that \((1, 0)\) is adjacent to all vertices \((0, u)\) where \(u\) is a nonzero element of \(F\). Also, for every two different elements \(u_1, u_2 \in F\), \((0, u_1)\) and \((0, u_2)\) are not adjacent. Consequently \(ZA(R)\) is a star.

\[\square\]
5. When is $ZA(R)$ Complete?

**Proposition 5.1.** Let $R$ be a ring. If $ZA(R)$ is a complete graph, then $A_R$ is a complete graph.

**Proof.** Assume that $ZA(R)$ is a complete graph. Let $I, J$ be two nonzero proper ideals of $R$. Then there are two different nonzero nonunit elements $x, y \in R$ such that $x \in I$ and $y \in J$. Hence $Ann_R(I) \cap Ann_R(J) \subseteq Ann_R(x) \cap Ann_R(y) = \{0\}$. Therefore $I$ and $J$ are adjacent. □

The following remark shows that the converse of Proposition 5.1 is not true.

**Remark 5.1.** Consider the ring $R = \mathbb{Z}_5 \times \mathbb{Z}_5$. By [1, Theorem 6], $A_R(= K_2)$ is a complete graph. But $ZA(R)$ is a 4-regular graph that is not a complete graph.

![Figure 1. $ZA(R)$](image)

**Theorem 5.1.** Let $R$ be a ring. Then $ZA(R)$ is a complete graph if and only if one of the following conditions holds:

(a) $R$ has exactly one nonzero nonunit element;
(b) $R$ is an integral domain;
(c) $R = \mathbb{Z}_2 \times \mathbb{Z}_2$.

**Proof.** ($\Rightarrow$) Assume that $ZA(R)$ is a complete graph. Then, by Proposition 5.1, $A_R$ is a complete graph. Suppose that $R$ is not an integral domain. So there exists a nonzero nonunit element $x \in R$ such that $Ann_R(x) \neq \{0\}$. Therefore, [1, Theorem 6] implies that either $R$ has exactly one nonzero proper ideal or $R$ is a direct product of two fields. Suppose that the former case holds. If $y$ is a nonzero nonunit element of $R$ different from $x$, then $Rx = Ry$. So $Ann_R(x) \cap Ann_R(y) = Ann_R(x) \neq \{0\}$, which is a contradiction. Therefore $R$ has exactly one nonzero nonunit element. Now, let $R$ be a direct product of two fields, say $R = F_1 \times F_2$. If there exist two different nonzero elements $u, v$ in $F_1$, then $(u, 0)$ and $(v, 0)$ cannot be adjacent. Hence $F_1 = \mathbb{Z}_2$. Similarly, we can show that $F_2 = \mathbb{Z}_2$. Consequently $R = \mathbb{Z}_2 \times \mathbb{Z}_2$.

($\Leftarrow$) Clearly, if (a) or (b) holds, then $ZA(R)$ is a complete graph. Assume that (c) holds. Then $ZA(R) \simeq K_2$ and we are done. □

6. Chromatic Number and Clique Number of $ZA(R)$

Recall that, a ring $R$ is said to be reduced if it has no nonzero nilpotent elements.
Theorem 6.1. If \( R \) is a reduced Noetherian ring, then the chromatic number of \( \mathcal{Z}(R) \) is infinite or \( R \) is a direct product of finitely many fields.

Proof. The proof is similar to that of [1, Theorem 16].

Lemma 6.1. Let \( P_1 \) and \( P_2 \) be two prime ideals of a ring \( R \) with \( P_1 \cap P_2 = \{0\} \). Then every two nonzero elements \( x \in P_1 \) and \( y \in P_2 \) are adjacent.

Proof. Suppose that \( r \in \operatorname{Ann}_R(x) \cap \operatorname{Ann}_R(y) \). Since \( rx = 0 \in P_2 \) and \( x \notin P_2 \), then \( r \in P_2 \). Similarly it turns out that \( r \in P_1 \). Hence \( r \in P_1 \cap P_2 = \{0\} \).

Theorem 6.2. Let \( R \) be a ring and \( n \geq 2 \) be a natural number. If either \( \lvert \operatorname{Min}(R) \rvert = n \) or \( R = R_1 \times R_2 \times \cdots \times R_n \) where \( R_i \)'s are rings, then \( \omega(\mathcal{Z}(R)) \geq n \).

Proof. Assume that \( \operatorname{Min}(R) = \{p_1, p_2, \ldots, p_s\} \) where \( p_i \)'s are nonzero. So, by Lemma 6.1, \( n \leq \omega(\mathcal{Z}(R)) \). Now, suppose that \( R = R_1 \times R_2 \times \cdots \times R_n \) where \( R_i \)'s are rings. Then \( \{(1, \ldots, 1, 0, 1, \ldots, 1) \mid 1 \leq i \leq n\} \) is a clique in \( \mathcal{Z}(R) \) and the result follows.

7. When is \( \mathcal{Z}(R) \) \( k \)-regular?

Recall that a finite field of order \( q \) exists if and only if the order \( q \) is a prime power \( p^s \). A finite field of order \( p^s \) is denoted by \( \mathbb{F}_{p^s} \).

Theorem 7.1. Let \( R \) be a Bézout ring with \( \lvert \operatorname{Max}(R) \rvert < \infty \). Then \( \mathcal{Z}(R) \) is a \( k \)-regular graph \((0 < k < \infty)\) if and only if \( R \simeq \mathbb{F}_{p^s} \times \mathbb{F}_{k+1} \).

Proof. The “if” part has a routine verification. Let \( \mathcal{Z}(R) \) be a \( k \)-regular graph \((0 < k < \infty)\). If \( \operatorname{Ann}_R(x) = \{0\} \) for some nonzero nonunit element \( x \) of \( R \), then \( \{x^n \mid n \in \mathbb{N}\} \) is an infinite clique that is a contradiction. Then, for every nonzero nonunit element \( x \) of \( R \) we have \( \operatorname{Ann}_R(x) \neq \{0\} \). Similar to the manner that described in the proof of Theorem 2.3, we have \( R \simeq F_1 \times F_2 \times \cdots \times F_n \) where \( F_i \)'s are fields and \( n = \lvert \operatorname{Max}(R) \rvert \).

Since \( \operatorname{Ann}_R((1,0,\ldots,0)) = 0 \times F_2 \times F_3 \times \cdots \times F_n \) and \( \operatorname{Ann}_R((0,1,0,\ldots,0)) = F_1 \times 0 \times F_3 \times \cdots \times F_n \), then

\[
\mathcal{N}_{\mathcal{Z}(R)}((1,0,\ldots,0)) = \{(0,u_2,\ldots,u_n) \mid u_i \in F_i \setminus \{0\} \text{ for } 2 \leq i \leq n\}
\]

and

\[
\mathcal{N}_{\mathcal{Z}(R)}((0,1,0,\ldots,0)) = \{(u_1,0,u_3,\ldots,u_n) \mid u_i \in F_i \setminus \{0\} \text{ for } 1 \leq i \leq n, i \neq 2\}.
\]

So

\[
(|F_2| - 1)(|F_3| - 1)\cdots(|F_n| - 1) = (|F_1| - 1)(|F_3| - 1)\cdots(|F_n| - 1),
\]

because \( \mathcal{Z}(R) \) is \( k \)-regular. Hence \( |F_1| = |F_2| \). Similarly we can show that \( |F_1| = |F_2| = \cdots = |F_n| \). Let \( n \geq 3 \). Note that \( \mathcal{N}_{\mathcal{Z}(R)}((1,1,0,\ldots,0)) \) is the union of the following sets

\[
\{(u_1,0,u_3,\ldots,u_n) \mid u_i \in F_i \setminus \{0\} \text{ for } 1 \leq i \leq n, i \neq 2\},
\]
\{(0, u_2, \ldots, u_n) \mid u_i \in F_i \setminus \{0\} \text{ for } 2 \leq i \leq n\}

and

\{(0, 0, u_3, \ldots, u_n) \mid u_i \in F_i \setminus \{0\} \text{ for } 3 \leq i \leq n\}.

Therefore,

\(|F_1| - 1)|^{n-1} = 2(|F_1| - 1)^{n-1} + (|F_1| - 1)^{n-2},

since \(ZA(R)\) is \(k\)-regular. Thus \(|F_1| = 0\) which is a contradiction. Consequently \(n = 2\).

If there exist two different nonzero elements \(u, u'\) in \(F_1\), then \((u, 0)\) and \((u', 0)\) cannot be adjacent. On the other hand for every nonzero elements \(u \in F_1\) and \(v \in F_2\), \((u, 0)\) and \((0, v)\) are adjacent. So \(\deg_{ZA(R)}((u, 0)) = |F_1| - 1 = k\). Therefore \(R \cong F_{k+1} \times F_{k+1}\).

\textbf{Corollary 7.1.} Let \(R\) be a Bézout ring with \(|\text{Max}(R)| < \infty\). If \(ZA(R)\) is a \(k\)-regular graph \((0 < k < \infty)\), then \(k + 1\) is a prime power.

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\textbf{References}


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