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A POINTFREE ANALOGUES OF LATTICE-VALUED BITOPOLOGICAL SPACES

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Abstract. The concept of coupled semi-quantales is introduced. An adjunction between the category of coupled semi-quantales and the category of lattice-valued biquasi-topological spaces is established. The topological and the lattice-theoretic concepts of regularity and compactness are extended to both lattice-valued biquasi-topological spaces and coupled semi-quantales, respectively. Some relations among these axioms are established.

1. Introduction

In 1986 Mulvey [9], proposed the concept quantale as a non-commutative extension of frame (or pointfree topology) with aim to develop the concept of non-commutative topology [6] and provide a constructive foundations for both quantum mechanics and non-commutative logic [17]. Nowadays, the concepts of quantales and semi-quantales (as a generalization of quantales [14]) can boast many areas of applications, e.g., the area of non-commutative topology [5, 10, 11]. Further details about quantales can be found in [15].

In 2015 Höhle [7], established a non-commutative extension of the well known Papert-Papert-Isbell adjunction [8,12] between the category of locales and the category of topological spaces to one between the category of quantales and the category of many valued topological spaces.

In [4], El-Saady extended the Höhle's adjunction to a more general one between the category of semi-quantales and the category of lattice-valued quasi-topological spaces.

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In this paper we aim to introduce the concept of coupled semi-quantales as the pointfree analogues of lattice-valued bitopological spaces and extend the dual adjunction between the category of semi-quantales and the category of lattice-valued quasi-topological spaces to one between the category of coupled semi-quantales and the category of lattice valued biquasi-topological spaces. Also, the topological and the lattice-theoretic concepts of regularity and compactness are extended to lattice-valued biquasi-topological spaces and coupled semi-quantales, respectively. Some relations among these axioms are established.

2. Preliminaries

By a complete join-semilattice (or $\bigvee$-semilattice) we mean a partially ordered set $(L, \leq)$ having arbitrary sups.

Definition 2.1. [14] A semi-quantale $(L, \leq, \otimes)$ is a complete join-semilattice $(L, \leq)$ equipped with a binary operation $\otimes : L \times L \to L$, with no additional assumptions, called a tensor product.

Definition 2.2. [14] Let L and M be semi-quantales. A function $h : L \to M$ is said to be:

(1) a semi-quantale morphism if it preserves $\otimes$ and arbitrary sups;
(2) a strong semi-quantale morphism if it preserves $\otimes$, arbitrary sups and $\top$.

By **SQuant**(resp. **SSQuant**), we mean the category of all semi-quantales and semi-quantale morphisms (resp. strong semi-quantale morphism).

Definition 2.3. A semi-quantale $(L, \leq, \otimes)$ is said to be:

(1) a quantale [15] if whose multiplication $\otimes$ is associative and distributes across $\bigvee$ from both sides. **Quant** denotes the full subcategory of **SQuant** of all quantales.
(2) a unital semi-quantale [14] if whose multiplication $\otimes$ has an identity element $e \in L$ called the unit. **USQuant** denotes the category all unital semi-quantales together with all semi-quantales morphisms preserving the unit e.
(3) a commutative semi-quantate [14] if whose multiplication $\otimes$ satisfies that $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1, q_2 \in L$. **CSQuant** denotes the full subcategory of **SQuant** of all commutative semi-quantales.
(4) a distributive semi-quantate [16] if whose multiplication $\otimes$ distributes across finite $\vee$ from both sides. **DSQuant** is the category of distributive semi-quantales.

Definition 2.4. [4] Let $L \in |\mathbf{SQuant}|$, $M \subseteq L$, and $a, b \in M$. An element a is said to be well-inside of b (w.r.t. M), denoted $a \preceq b$, if

$$\text{exists } c \in M \text{ with } a \otimes c = \bot \text{ and } c \vee b = \top.$$

An $L \in |\mathbf{SQuant}|$ is said to be *regular* [4], if for each $a \in L$ there exists $D \subseteq I_a$, where $I_a = \{b \in L : b \preceq a\}$ such that $a = \bigvee D$.

Definition 2.5. [3] Let $L = (L, \leq, \otimes)$ be a semi-quantale. A subset $K \subseteq L$ is a subsemi-quantale of L if and only if the inclusion $K \hookrightarrow L$ is a semi-quantale morphism, i.e., K is closed under $\otimes$ and arbitrary sups. A subsemi-quantale K of L is said to be strong if and only if $\top$ belongs to K. If L is a unital semi-quantale with the identity e, then a subsemi-quantale K of L is called a unital subsemi-quantale of L if and only if e belongs to K.

Let $L = (L, \leq, \otimes)$ be a semi-quantale. For any non-empty set X, let L^X be the set of all L-valued maps $X \xrightarrow{f} L$. We can extend the algebraic and lattice-theoretic structure from L to L^X pointwisely, i.e., for all $x \in X$, $f, g \in L^X$ and $\{f_j : j \in J\} \subseteq L^X$, we have

$$f \leq g \Leftrightarrow f(x) \leq g(x),$$
$$(f \otimes g)(x) = f(x) \otimes g(x),$$
$$\left(\bigvee_{j \in J} f_j\right)(x) = \bigvee_{j \in J} (f_j(x)).$$

Then L^X is again a semi-quantale with respect to the multiplication $\otimes$. If L is a unital semi-quantale with unit e, then L^X becomes a unital semi-quantale with the unit $\underline{e}$ (a mapping from X to L, defined by $\underline{e}(x) = e$ for all $x \in X$), where e is the unit of $\otimes$ in L.

For an ordinary mapping $f : X \to Y$, the forward and backward powerset operators [13, 14]:

$$f_L^{\rightarrow} : L^X \to L^Y \text{ and } f_L^{\leftarrow} : L^Y \to L^X,$$

defined by

$$f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) : x \in X, f(x) = y\} \text{ and } f_L^{\leftarrow}(B) = B \circ f,$$

respectively.

Theorem 2.1. [14] *Let $L \in |\mathbf{SQuant}|$, X, Y be a nonempty ordinary sets and $f : X \to Y$ be an ordinary mapping, then we have:*

(1) $f_L^{\rightarrow}$ *preserves arbitrary* $\bigvee$*;*
(2) $f_L^{\leftarrow}$ *preserves arbitrary* $\bigvee$*,* $\otimes$*, and all constant maps;*
(3) $f_L^{\leftarrow}$ *preserves the unit if* $L \in |\mathbf{USQuant}|$*.*

For a fixed $L \in |\mathbf{SQuant}|$ and a set X, an L-quasi-topology on X [14] is a subsemi-quantale τ of $L^X = (L^X, \leq, \otimes)$, i.e., satisfying the following conditions.

(T_1) For all $A, B \in L^X$, if $A, B \in \tau$ then $A \otimes B \in \tau$.
(T_2) For all $\{A_j : j \in J\} \subseteq L^X$, if $\{A_j : j \in J\} \subseteq \tau$ then $\bigvee_j A_j \in \tau$.

An L-quasi-topology τ is said to be strong [3] if and only if it is strong as a subsemi-quantale of L^X, i.e., τ satisfies the additional axiom:

(T_3) $\underline{\top} \in \tau$.

If $L \in |\mathbf{USQuant}|$ with unit e, a unital subsemi-quantale τ of L^X is called an L-topology on X [14], i.e., τ satisfies $(T_1), (T_2)$ and the following:

(T_4) $\underline{e} \in \tau$.

If $\tau \subseteq L^X$ is an L-quasi-topology (resp. L-topology), then the pair (X, τ) is said to be an L-quasi-topological (resp. L-topological) space. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be L-*continuous* (resp. L-*open*) [13] if $(f_L^{\leftarrow})_{|\rho} : \tau \leftarrow \sigma$ (resp. $(f_L^{\rightarrow})_{|\tau} : \tau \to \sigma$). An L-continuous bijection $f : (X, \tau) \to (Y, \sigma)$ is an L-*homeomorphism* [13] if f^{-1} is L-continuous.

It is clear that L-quasi-topological (resp. strong L-quasi-topological, L-topological) spaces and L-continuous maps form a category denoted by L-**QTop** (resp. L-**SQTop**, L-**Top**).

One can easily prove that each of L-**QTop**, L-**SQTop** and L-**Top** is a topological category over the category **Set**.

Definition 2.6. [4] An $(X, \tau) \in |L\text{-}\mathbf{QTop}|$ is called

(1) L-QT_0 if for every $x, y \in X$ with $x \neq y$ there exists $\mu \in \tau$ with $\mu(x) \neq \mu(y)$;
(2) L-qsober if and only if $\eta_X : (X, \tau) \to (LPT(\tau), \Phi_L^{\rightarrow}(\tau))$ is bijective.

3. Coupled Semi-quantales and Lattice-valued Biquasi-topological Spaces

Before we go on, this section, we begin our study by the following.

Lemma 3.1. *If $\{A_j : j \in J\}$ is any collection of subsemi-quantales of a semi-quantale Q, then $\bigcap_j A_j$ is also a subsemi-quantale of Q, provided $\bigcap_j A_j \neq \phi$.*

Proof. Let $M = \cap_j A_j$ and $a, b \in M$. Then $a, b \in A_j \Rightarrow a \otimes b \in A_j$ for each subsemi-quantale $A_j \Rightarrow a \otimes b \in M \Rightarrow M$ is closed under $\otimes$. Also, one can easily prove that M is closed under sups. □

For a fixed $Q \in |\mathbf{SQuant}|$, it follows, as a consequence of the above lemma, that the family of all subsemi-quantales of Q, ordered by inclusion, forms a complete lattice, with the meet $Q_1 \wedge Q_2 = Q_1 \cap Q_2$ (the set-intersection), and the join $Q_1 \vee Q_2$ is the least subsemi-quantale of Q containing Q_1 and Q_2 (which is not their set-theoretical union). The supremum (joins) of a set $\{A_j : j \in J\}$ of subsemi-quantales of Q, is the intersection of subsemi-quantales of Q which contains the union $\cup_j A_j$. More generally there is for each subset $K \subseteq Q$ of a semi-quantale Q a smallest subsemi-quantale of Q (sometimes denoted by $[K]$) which contains K and is the subsemi-quantale generated by K.

Definition 3.1. (The category of coupled semi-quantales)

(1) A coupled semi-quantale is a triple $Q = (Q_0, Q_1, Q_2)$ in which Q_0 is a semi-quantale, Q_1 and Q_2 are subsemi-quantales of Q_0 such that $Q_1 \cup Q_2$ generates Q_0.

(2) A map $h : Q \to P$ between coupled semi-quantales is a semi-quantale morphism $Q_0 \to P_0$ for which the restrictions $h|_{Q_i} : Q_i \to P_i$ are semi-quantale morphisms i.e., $h(Q_i) \subseteq P_i$ for $i = 1, 2$.
(3) The resulting category will be denoted by **CSQuant**.

We refer to Q_0 as the total part of Q, and Q_1, Q_2 as its first and second parts, respectively.

Definition 3.2. A coupled semi-quantale $Q = (Q_0, Q_1, Q_2)$ is said to be:

(1) *unital* if and only if Q_0 is unital and e belongs to both Q_1 and Q_2. **UnCSQuant** is the full subcategory of **CSQuant** of all unital coupled semi-quantales.
(2) *coupled quantal* [1] if Q_0 is a quantale and both Q_1 and Q_2 are subquantales. **CQuant** is the full subcategory of **CSQuant** of all coupled quantales.
(3) *strong coupled quantal* if both Q_1 and Q_2 are strong subquantales of Q_0.
(4) symmetric if and only if $Q_0 = Q_1 = Q_2$.
(5) right-sided (resp. left-sided) if and only if $a \otimes \top \leq a$ (resp. $\top \otimes a \leq a$) for all $a \in Q_0$.
(6) idempotent if and only if the total part Q_0 is idempotent, i.e., $a \otimes a = a$ for all $a \in Q_0$.
(7) commutative if the operation $\otimes$ is commutative, i.e., $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1 \in Q_i$ and $q_2 \in Q_k$. **ComCSQuant** is the full subcategory of **CSQuant** of all commutative coupled semi-quantales.

Example 3.1. For a fixed $L \in |\mathbf{SQuant}|$ and a non-empty set X. For $i = 1, 2$, let $\tau_i \subset L^X$ be a subsemi-quantale of L^X, i.e., L-quasi-topologies on X. The triple $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is a coupled semi-quantale where $\tau_1 \vee \tau_2$ is the coarsest L-quasi-topology finer than both τ_1 and τ_2.

Example 3.2. Let $Q = \{\bot, a, b, \top\}$ be the four Boolean lattice and let $\otimes : Q \times Q \to Q$ defined by

$\otimes$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	a
b	$\bot$	$\bot$	b	b
$\top$	$\bot$	a	b	$\top$

.

It is clear that Q is a coupled quantales with $Q_0 = \{\bot, a, b, \top\}$ as the total part, $Q_1 = \{\bot, a, \top\}$ as the first part and $Q_2 = \{\bot, b, \top\}$ as the second part.

Example 3.3. Any biframe $A = (A_0, A_1, A_2)$ [2] is a commutative coupled quantale provided that $\otimes = \wedge$ and any element of $a \in A_0$ can be expressed as $a = \bigvee\{a_1 \otimes a_2 : a_1 \in A_1, a_2 \in A_2\}$.

Definition 3.3. (The category of L-biquasi-topological spaces)

(1) An L-biquasi-topological space is a triple (X, τ_1, τ_2) consisting of a non-empty set X and two L-quasi-topologies τ_1 and τ_2 on X.
(2) A morphism $f : X \to Y$ between L-biquasi-topological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) is a function between their underlying sets for which
$$f : (X, \tau_1) \to (Y, \sigma_1) \text{ and } f : (X, \tau_2) \to (Y, \sigma_2)$$
are L-continuous.
(3) The category of L-biquasi-topological spaces and their morphisms will be denoted by L-**BiQTop**.

Between the category L-**QTop** and L-**BiQTop** there is a faithful functor
$$E_S : L\text{-}\mathbf{BiQTop} \to L\text{-}\mathbf{QTop}\ ,$$
which we describe as follows. If $X = (X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$, then $E_S(X) = (X, \tau_1 \vee \tau_2)$, where $\tau_1 \vee \tau_2$ is the coarsest L-quasi topology finer than both τ_1 and τ_2, $E_S(f) = f$.

The left adjoint of S is the functor
$$E_d : L\text{-}\mathbf{QTop} \to L\text{-}\mathbf{BiQTop},$$
by the following correspondences:
$$E_d(X, \tau) = (X, \tau, \tau),\ E_d(f) = f.$$

One notes that since E_S embeds L-**QTop** in L-**BiQTop**, then we will regard the constructions in L-**BiQTop** as extensions of the constructions in the category L-**QTop**.

For $L \in |\mathbf{SQuant}|$ and $(X, \tau_1, \tau_2) \in |\boldsymbol{L}\text{-}\mathbf{BiQTop}|$. The functor
$$\mathcal{O}_L : L\text{-}\mathbf{BiQTop} \to \mathbf{CSQuant}^{op}$$
is defined as follows
$$\mathcal{O}_L(X, \tau_1, \tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2).$$
For the L-biquasi-topological space (X, τ_1, τ_2), the triple $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is a coupled semi-quantale where $\tau_1 \vee \tau_2$ is the coarsest L-quasi-topology finer than both τ_1 and τ_2 , and
$$\mathcal{O}_L(f : (X, \tau_1, \tau_2) \to (Y, \theta_1, \theta_2)) = [(f_L^{\leftarrow})|_{\theta_i}]^{op} : \tau_i \to \theta_i, \quad i = 1, 2.$$
Now, we will introduce some ideas needed to define a functor in the opposite direction. For a coupled semi-quantale $Q = (Q_0, Q_1, Q_2)$, let
$$LPT(Q_0) = \{p : Q_0 \to L : p \in |\mathbf{SQuant}|\}.$$
Also, we define a coupled semi-quantale map
$$\Phi_L : (Q_0, Q_1, Q_2) \to (L^{LPT(Q_0)}, L^{LPT(Q_0)}, L^{LPT(Q_0)})$$
such that

(1) $\Phi_L : Q_0 \to L^{LPT(Q_0)}$ is a semi-quantale map, where $\Phi_L(a)(p) = p(a)$;
(2) $\Phi_L^{\rightarrow}(Q_1) \subseteq L^{LPT(Q_0)}$;
(3) $\Phi_L^{\rightarrow}(Q_2) \subseteq L^{LPT(Q_0)}$.

As given in [4] the function Φ_L preserves $\otimes$ and arbitrary $\bigvee$, where these are inhertied by the codomain of Φ_L from L. Also, for $i = 1, 2$, we have $\Phi_L^{\rightarrow}(Q_i)$ is closed under these operations and hence is an L-quasi topology on $LPT(Q_0)$. Thus we have

$$LPT : L\text{-}\mathbf{BiQTop} \leftarrow \mathbf{CSQuant}^{op},$$

defined by

$$(Q_0, Q_1, Q_2) \to (LPT(Q_0), \Phi_L^{\rightarrow}(Q_1), \Phi_L^{\rightarrow}(Q_2)),$$

where $LPT(f : A \to B) = [f]^{op}$, that is, $LPT(f)(p) = p \circ f^{op}$, $f^{op} : B \to A$, is a concrete map in **CSQuant**. It is clear that $\{\Phi_L(a_i) : a_i \in Q_i,\ i = 1, 2\}$ is an L-quasi-topology on $LPT(Q_0)$ and, therefore, we have $(LPT(Q_0), \Phi_L^{\rightarrow}(Q_1), \Phi_L^{\rightarrow}(Q_2)) \in |L\text{-}\mathbf{BiQTop}|$.

Proposition 3.1. *For a fixed $L \in |\mathbf{SQuant}|$ and $Q, P \in |\mathbf{CSQuant}|$, the mapping*

$$LPT(f) : (LPT(Q_0), \Phi_L^{\rightarrow}(Q_1), \Phi_L^{\rightarrow}(Q_2)) \to (LPT(P_0), \Phi_L^{\rightarrow}(P_1), \Phi_L^{\rightarrow}(P_2))$$

is L-bicontinuous.

Proof. We need to check the L-continuity of both the functions

(1) $LPT(f) : (LPT(Q_0), \Phi_L^{\rightarrow}(Q_1)) \to (LPT(P_0), \Phi_L^{\rightarrow}(P_1))$ and
(2) $LPT(f) : (LPT(Q_0), \Phi_L^{\rightarrow}(Q_2)) \to (LPT(P_0), \Phi_L^{\rightarrow}(P_2))$.

The first function is L-continuous since for all $q_2 \in P_0$, $p \in LPT(Q_0)$, we have

$$\begin{aligned} LPT(f)^{\leftarrow}(\Phi_L(q_2)(p)) &= \Phi_L(q_2)(LPT(f)(p)) \\ &= \Phi_L(q_2)(p \circ f^{op}) \\ &= \Phi_L(f^{op}(q_2))(p). \end{aligned}$$

Similarly, we can check the L-continuity of the second function and this completes the proof. □

Then we have the spectrum or point functor

$$LPT : \mathbf{CSQuant}^{op} \to L\text{-}\mathbf{BiQTop}.$$

To study the adjunction between the functors

$$LPT : \mathbf{CSQuant}^{op} \to L\text{-}\mathbf{BiQTop}$$

and

$$\mathcal{O}_L : L\text{-}\mathbf{BiQTop} \to \mathbf{CSQuant}^{op}.$$

we give the following definitions.

For fixed $L \in |\mathbf{SQuant}|$, $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ and $Q \in |\mathbf{CSQuant}|$ define the maps:

(1) $\eta_X : (X, \tau_1, \tau_2) \to (LPT(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_1), \Phi_L^{\rightarrow}(\tau_2))$, by setting, for all $x \in X$ and $\mu \in \mathcal{O}_L(X)$, $\eta_X(x)(\mu) = \mu(x)$;
(2) $\varepsilon_Q^{op} : Q \to \mathcal{O}_L(LPT(Q))$, by setting $\varepsilon_{Q_0}^{op} = \Phi_L|_{\Phi_L^{\rightarrow}(Q_0)}$.

It is clear that by definition ε_Q^{op} always surjective.

Lemma 3.2. *Let* $L \in |\mathbf{SQuant}|$, $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ *and* $Q \in |\mathbf{CSQuant}|$. *Then*

1. *the map* $\eta_X : (X, \tau_1, \tau_2) \to (LPT(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_1), \Phi_L^{\rightarrow}(\tau_2))$, *is* L*-bicontinuous, and pairwise* L*-open w.r.t. its range in* $(LPT(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_1), \Phi_L^{\rightarrow}(\tau_2))$ *and*
2. *the map* $\varepsilon_Q^{op} : Q \to \mathcal{O}_L(LPT(Q))$ *is a coupled semi-quantale morphism.*

Proof. (1) To prove that the mapping η_X is L-bicontinuous and pairwise L-open, it suffices to prove that both the mappings $\eta_X : (X, \tau_1) \to (LPT(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_1))$ and $\eta_X : (X, \tau_2) \to (LPT(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_2))$ are L-continuous and L-open with respect to their respective ranges.

(i) L-continuity: for $i \in \{1, 2\}$, for all $\mu \in \Phi_L^{\rightarrow}(\tau_i)$, and for all $x \in X$, there exists $\rho \in \tau_i$ such that $\Phi_L(\rho) = \mu$, $(\eta_X)_L^{\leftarrow}(\mu)(x) = (\eta_X)_L^{\leftarrow}(\Phi_L(\rho))(x) = \rho(x)$, that is, $(\eta_X)_L^{\leftarrow}(\mu) \in \tau_i$. Hence η_X is L-bicontinuous.

(ii) Openness: in fact, for $\nu \in \tau_i$, $i \in \{1, 2\}$, and $p \in LPT(\tau_1 \vee \tau_2)$:

$$\begin{aligned}(\eta_X)_L^{\rightarrow}(\nu)(p) &= \bigvee_{x \in X} \{\nu(x) : \eta_X(x) = p\} \\ &= \bigvee_{x \in X} \{\eta_X(x)(\nu) : \eta_X(x) = p\} \\ &= p(\nu) = \Phi_L^{\rightarrow}(\nu)(p).\end{aligned}$$

Now, $\Phi_L(\nu) \in \Phi_L^{\rightarrow}(\tau_i)$, the L-quasi-topology on $LPT(\tau_1 \vee \tau_2)$, and it follows that $(\eta_X)_L^{\rightarrow}(\nu) = \Phi_L(\nu)$, that is, $(\eta_X)_L^{\rightarrow}(\nu)|_{(\eta_X)_L^{\rightarrow}(X)} = \Phi_L(\nu)|_{(\eta_X)_L^{\rightarrow}(X)}$. Thus $(\eta_X)_L^{\rightarrow}(\nu)$ is open w.r.t. the subspace topology of $(\eta_X)_L^{\rightarrow}(X)$ induced from $LPT(\tau_1 \vee \tau_2)$, that is, η_X is a pairwise L-open map.

(2) As given in [4], we note that the mapping $\varepsilon_{Q_0}^{op} : Q_0 \to \mathcal{O}_L(LPT(Q_0))$ is a semi-quantale homomorphism and so the mappings $\varepsilon_Q^{op}|_{Q_i} : Q_0 \to \mathcal{O}_L(LPT(Q_0))$, for $i = 1, 2$. Thus we have that the mapping $\varepsilon_Q^{op} : Q \to \mathcal{O}_L(LPT(Q))$ is a coupled semi-quantale morphism. □

Theorem 3.1. *The functor*

$$LPT : L\text{-}\mathbf{BiQTop} \leftarrow \mathbf{CSQuant}^{op}$$

is a right adjoint of the functor

$$\mathcal{O}_L : L\text{-}\mathbf{BiQTop} \to \mathbf{CSQuant}^{op}$$

with unit $\eta_X : X \to LPT^{\rightarrow}(\mathcal{O}_L(X, \tau_1, \tau_2))$ *and counit* $\varepsilon_Q : Q \leftarrow \mathcal{O}_L(LPT(Q))$.

Proof. It will be enough to show that for every $Q \in |\mathbf{CSQuant}|$ and an L-$\mathbf{BiQTop}$-morphism $(X, \tau_1, \tau_2) \xrightarrow{f} LPT(Q)$, there exists uniquely a $\mathbf{CSQuant}$-morphism $Q \xrightarrow{f^*} \mathcal{O}_L(X, \tau_1, \tau_2)$ such that the left diagram of the following diagram in Figure 1 is commutative, where by τ_0 we mean the coarsest L-quasi-topology $\tau_1 \vee \tau_2$.

To prove the existence, let $f^* = \mathcal{O}_L(f) \circ \varepsilon_Q$. From the definitions of $\mathcal{O}_L(f)$ and ε_{Q_0} one can easily prove that $f^* : Q \to \Omega(X, \tau_1, \tau_2)$ is a $\mathbf{CSQuant}$-morphism. For commutativity of the above-mentioned left diagram notice that for $x \in X$ and $a \in Q_0$, we have

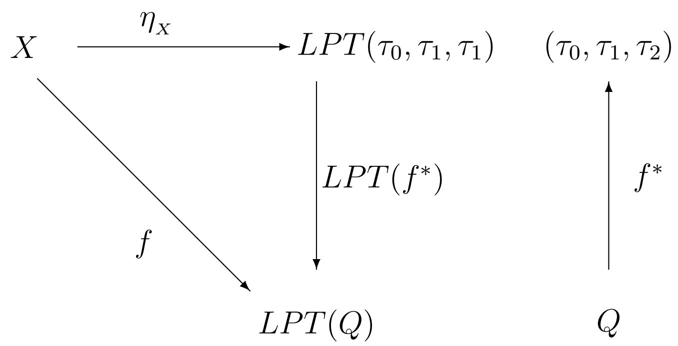

FIGURE 1.

$$\begin{aligned} pt(f^*) \circ \eta_X(x)(a) &= \eta_X(x)(f^*(a)) \\ &= \eta_X(x)(\mathcal{O}_L(f) \circ \varepsilon_Q(a)) \\ &= (\mathcal{O}_L(f)(\Phi_L(a)))(x) \\ &= (f_L^{\leftarrow}(\Phi_L(a))))(x) \\ &= (\Phi_L(a) \circ f))(x) \\ &= f(x)(a). \end{aligned}$$

Uniqueness of the function f^* follows from the observation that given another **CSQuant**-morphism $Q \xrightarrow{g} \Omega(X, \tau_1, \tau_2)$ with the same property: for all $x \in X$, and for all $a \in Q_0$, we have

$$\begin{aligned} f(x)(a) &= \eta_X(x)(g(a)) \\ &= \eta_X(x)(\mathcal{O}_L(g) \circ \varepsilon_L(a)) \\ &= (g_L^{\leftarrow} \Phi_L(a))(x) \\ &= (\Phi_L(a) \circ g)(x) \\ &= g(a)(x). \end{aligned}$$

Hence for all $x \in X$ and for all $a \in Q_0$, we have $f^*(a) = g(a)$, i.e., $f^* = g$. □

Definition 3.4. An $(X, \tau_1, \tau_2) \in |L$-**BiQTop**$|$ is said to be pairwise L-QT_0 (i.e., fulfills the T_0-axiom) if and only if for every pair $(x, y) \in X \times X$ with $x \neq y$, there exists $\mu \in \tau_1 \vee \tau_2$ such that $\mu(x) \neq \mu(y)$.

By L-$\mathbf{T}_0$**BiQTop**, we mean a full subcategory of L-**BiQTop** consisting of those L-**BiQTop** objects, which are pairwise L-QT_0.

As a consequence of Definition 2.6, we have the following easily established proposition.

Proposition 3.2. $(X, \tau_1, \tau_2) \in |L\text{-}\boldsymbol{T}_0\mathbf{BiQTop}|$ *if and only if* $S(X, \tau_1, \tau_2) = (X, \tau_1 \vee \tau_2)$ *is* L-QT_0.

Proposition 3.3. *An* $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ *is pairwise* L-QT_0 *if and only if the mapping*

$$\eta_X : (X, \tau_1, \tau_2) \to (LPT(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_1), \Phi_L^{\rightarrow}(\tau_2))$$

is pairwise L*-embedding.*

Proof. First, suppose that $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ is pairwise L-QT_0, then for $x \neq y \in X$, there exists $\mu \in \tau_1 \vee \tau_2$ such that $\mu(x) \neq \mu(y)$. Therefore, $\eta_X(x)(\mu) = \mu(x) \neq \mu(y) = \eta_X(y)(\mu)$, that is, the mapping η_X is injective. Also, since the mapping η_X is pairwise L-continuous and L-open (see Lemma 3.2), then η_X is L-embedding. $\square$

Now, we will introduce the concept of sobriety of objects in the category $L-\mathbf{BiQTop}$.

Definition 3.5. An $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ is L-sober if and only if the mapping

$$\eta_X : X \to LPT_{\rightarrow}(\mathcal{O}_L(X, \tau_1, \tau_2))$$

is bijective.

By L-**SobBiQTop**, we mean the full subcategory of L-**BiQTop** of all sober objects.

Lemma 3.3. *An* $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ *is* L*-sober if and only if the mapping*

$$\eta_X : (X, \tau_1, \tau_2) \to (LPT(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_1), \Phi_L^{\rightarrow}(\tau_2))$$

is a pairwise homomorphism.

Proof. L-sobriety of an $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ is equivalent to the fact of bijectivity of the mapping

$$\eta_X : (X, \tau_1, \tau_2) \to (LPT(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_1), \Phi_L^{\rightarrow}(\tau_2)).$$

Also, the mapping η_X is pairwise L-continuous and L-open (see Lemma 3.2), and this is equivalent to the fact that η_X is pairwise L-homomorphism. $\square$

By the above and Definition 2.6, one have the following easily established result.

Proposition 3.4. *An* $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ *is* L*-sober if and only if* $(X, \tau_1 \vee \tau_2)$ *is* L*-qsober.*

Definition 3.6. The coupled semi-quantales $Q = (Q_0, Q_1, Q_2)$ is spatial if and only if the total part Q_0 is spatial. Equivalently the map

$$\varepsilon_Q^{op} : Q_0 \to \mathcal{O}_L(LPT(Q_0))$$

is a semi-quantale isomorphism [4].

By **SpatCSQuant**, we mean the full subcategory of the spatial coupled semi-quantales in **CSQuant**.

Lemma 3.4. *For all* $Q = (Q_0, Q_1, Q_2) \in |\mathbf{CSQuant}|$, $Q = (Q_0, Q_1, Q_2)$ *is spatial if and only if the mapping*

$$\varepsilon_Q^{op} : (Q_0, Q_1, Q_2) \to \mathcal{O}_L(LPT(Q_0, Q_1, Q_2))$$

is a coupled semi-quantale isomorphism.

Proof. Let $Q = (Q_0, Q_1, Q_2)$ be a spatial coupled semi-quantale. Then, by the definition, the total part Q_0 is spatial, and this is equivalent to the fact that the map

$$\varepsilon_Q^{op} : Q_0 \to \mathcal{O}_L(LPT(Q_0))$$

is a semi-quantale isomorphism, and this implies that the map

$$\varepsilon_Q^{op} : (Q_0, Q_1, Q_2) \to \mathcal{O}_L(LPT(Q_0, Q_1, Q_2))$$

is a coupled semi-quantale isomorphism. □

Lemma 3.5. *For all* $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ *and for all* $Q \in |\mathbf{CSQuant}|$*, then*

(i) $\mathcal{O}_L(X, \tau_1, \tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2)$ *is spatial;*
(ii) $LPT(Q_0, Q_1, Q_2) = (LPT(Q_0), \Phi_L^{\to}(Q_1), \Phi_L^{\to}(Q_2)$ *is L-sober.*

Proof. As to (i), clearly, the map

$$\varepsilon_{\tau_1 \vee \tau_2}^{op} : (\tau_1 \vee \tau_2) \to \mathcal{O}_L(LPT(\tau_1 \vee \tau_2)) = \Phi_L^{\to}(\tau_1 \vee \tau_2)$$

is a semi-quantale isomorphism, which implies that $\tau_1 \vee \tau_2$ is a spatial semi-quantale and, therefore, the coupled semi-quantale $\mathcal{O}_L(X, \tau_1, \tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is spatial.

As to (ii), by definition, it suffices to prove that the mapping

$$\eta_X : LPT(Q) \to LPT(\mathcal{O}_L(LPT(Q))) = LPT((\Phi_L^{\to}(Q_1) \vee \Phi_L^{\to}(Q_2)), \Phi_L^{\to}(Q_1), \Phi_L^{\to}(Q_2))$$

is bijective. Now, we have the following.

(a) η_X is one-to-one. For all $p_1, p_2 \in LPT(Q_0)$ with $p_1 \neq p_2$, there exist some $a \in Q_0$ with $p_1(a) \neq p_2(a)$, and this implies that
$$\eta_X(p_1)(\Phi_L^{\to}(a)) = \Phi_L^{\to}(a)(p_1) = p_1(a) \neq p_2(a) = \eta_X(p_2)(\Phi_L^{\to}(a)).$$
Hence η_X is one-to-one.

(b) η_X is onto. For all $q \in LPT(\Phi_L^{\to}(Q_1 \vee Q_2))$, let $p = q \circ \Phi_L^{\to} : Q_0 \to \Phi_L^{\to}(Q_0) \to L$, then $p \in LPT(Q_0)$ and $a \in Q_0$. We have $\eta_X(p)(\Phi_L^{\to}(a)) = \Phi_L^{\to}(a)(p) = p(a) = q(\Phi_L^{\to}(a))$. Hence $\eta_X(p) = q$, that is, η_X is onto. From (a) and (b), it follows that η_X is bijective, and this completes the proof. □

Proposition 3.5. *The following functors are valid:*

(i) $\mathcal{O}_L : L\text{-}\mathbf{BiQTop} \to \mathbf{SpatCSQuant}^{op}$;
(ii) $LPT : L\text{-}\mathbf{SobBiQTop} \leftarrow \mathbf{CSQuant}^{op}$.

The equivalence between the categories L-**SobBiQTop** and **SpatCSQuant** is proven as follows.

Theorem 3.2. *For all* $L \in |\mathbf{SQuant}|$, $L\text{-}\mathbf{SobBiQTop} \approx \mathbf{SpatCSQuant}^{op}$.

Proof. The categorical equivalence L-**SobBiQTop** $\approx$ **SpatCSQuant**^{op} follows directly from the adjunction $\mathcal{O}_L \dashv LPT$ and the fact that both the unit and counit are isomorphisms in the categories L-**SobBiQTop** and **SpatCSQuant**^{op}, respectively. □

4. Regularity and Pairwise Compactness

Now, we will define the regularity and compactness for a certain L-**BiQTop** and **CSQuant** objects.

Definition 4.1. Let $Q = (Q_0, Q_1, Q_2) \in |\mathbf{CSQuant}|$ and $a, b \in Q_i$, $i = 1, 2$. An element a is said to be well inside of b (w.r.t. Q_i) and denoted by $a \preceq_i b$, if and only if exists $c \in Q_k$, $k \neq i$, such that $a \otimes c = \bot$ and $c \vee b = \top$.

Lemma 4.1. *For any strong* **CSQuant**-*morphism* $h : Q \to P$

$$a \preceq_i b \Rightarrow h(a) \preceq_i h(b).$$

Proof. Let $a, b \in Q_i$ with $a \preceq_i b$, then exists $c \in Q_k$, $k \neq i$, with $c \otimes a = \bot$, $c \vee b = \top$. Since $h : Q \to P$ is a strong semi-quantale homomorphism, then $h(c \otimes a) = h(c) \otimes h(a) = \bot$ and $h(c \vee b) = h(c) \vee h(b) = h(\top) = \top$. So exists $h(c) \in P_k$, $k \neq i$, such that $h(c) \otimes h(a) = \bot$ and $h(c) \vee h(b) = \top$ which means that $h(a) \preceq_i h(b)$. □

Definition 4.2. An $Q = (Q_0, Q_1, Q_2) \in |\mathbf{CSQuant}|$ is said to be regular if and only if both Q_1 and Q_1 are regular subsemi-quantales. Or equivalently

$$\text{for all } a \in Q_i, \text{ exists } D \subseteq \{b \in Q_i : b \preceq_i a\} \text{ such that } a = \bigvee D,\ i = 1, 2.$$

By **RegCSQuant**, we mean the full subcategory of **CSQuant** of regular objects.

A coupled semi-quantale map $h : Q \to P$ is said to be *surjective* if and only if $h|_{Q_i} : Q_i \to P_i$ is surjective for $i = 1, 2$.

Lemma 4.2. *If* $h : Q \to P$ *is a surjective strong coupled semi-quantale homomorphism and* $Q \in |\mathbf{RegCSQuant}|$, *then* $P \in |\mathbf{RegCSQuant}|$.

Proof. For $i = 1, 2$, let $x \in P_i$. Then $x = h(a)$ for some $a \in Q_i$. Regularity of Q means that exists $D \subseteq \{b \in Q_i : b \preceq_i a\}$, $a = \bigvee D$, $i = 1, 2$. Therefore there exists $E \subseteq \{h(b) \in P_i : b \preceq_i a\}$ such that $E = h(D)$. Since $a \preceq_i b$ implies $x = h(a) \preceq_i h(b) = y$. Hence $E \subseteq \{y \in P_i : y \preceq_i x\}$ and $x = \bigvee E$. Thus $P \in |\mathbf{RegCSQuant}|$. □

Definition 4.3. Let $L \in |\mathbf{SQuant}|$. An (X, τ_1, τ_2) is regular if and only if $\mathcal{O}_L(X, \tau_1, \tau_2) \in |\mathbf{RegCSQuant}|$.

By L-**RegBiQTop**, we mean the full subcategory of L-**BiQTop** of regular objects.

Proposition 4.1. *For* $Q = (Q_0, Q_1, Q_2) \in |\mathbf{DCSQuant}|$ *and* $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$.

(1) *An* $Q = (Q_0, Q_1, Q_2)$ *is regular if and only if*

$$a = \bigvee\{b \in Q_i : b \preceq_i a\} \text{ for all } a \in Q_i.$$

(2) *For $L \in |\mathbf{DSQuant}|$. An (X, τ_1, τ_2) is regular if and only if*

$$\mu = \bigvee\{\nu \in \tau_i : \nu \preceq_i \mu\} \text{ for all } \mu \in \tau_i.$$

Proof. (1) Let $Q = (Q_0, Q_1, Q_2) \in |\mathbf{DCSQuant}|$. Distributivity and $b \preceq_i a$ imply $a \leq b$. Let $D \subseteq \{b \in Q_i : b \preceq_i a\}$ such that $a = \bigvee D$. Then,

$$\bigvee D \leq \bigvee\{b \in Q_i : b \preceq_i a\} \leq \bigvee\{b \in Q_i : b \leq a\} = a = \bigvee D.$$

This shows $a = \bigvee D = \bigvee\{b \in Q_i : b \preceq_i a\}$ and from this follows the claims.

(2) Follows from (1). □

As the preceding proposition offers the preserving of the regular axiom under the functor

$$LPT : L\text{-}\mathbf{BiQTop} \leftarrow \mathbf{CSQuant}^{op}$$

and with the aid of Definition 4.3, we have the following easily established proposition.

Proposition 4.2. *The following functors holds:*

$$\mathcal{O}_L : L\text{-}\mathbf{RegBiQTop} \to \mathbf{RegCSQuant}^{op},$$
$$LPT : L\text{-}\mathbf{RegBiQTop} \leftarrow \mathbf{RegCSQuant}^{op}.$$

Definition 4.4. An $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ is said to be pairwise compact if $E_s(X, \tau_1, \tau_2) = (X, \tau_1 \vee \tau_2)$ is compact.

Theorem 4.1. *Let $L \in |\mathbf{SQuant}|$, $Q \in |\mathbf{CSQuant}|$ and $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$. Then*

(1) *(X, τ_1, τ_2) is pairwise compact if and only if $\mathcal{O}_L(X, \tau_1, \tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is compact;*

(2) *if Q is spatial, then Q is compact if and only if $LPT(Q_0, Q_1, Q_2)$ is pairwise compact.*

Proof. As to (1), if (X, τ_1, τ_2) is a compact object of L-**BiQTop**, that is, for all $S \subseteq (\tau_1 \vee \tau_2)$, $\bigvee S = \top$, exists F(finite)$\subseteq S$, $\bigvee F = \top$ if and only if $(\tau_1 \vee \tau_2)$ is a compact semi-quantale if and only if $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is a compact coupled semi-quantale.

As to (2), let $Q = (Q_0, Q_1, Q_2)$ be spatial, then the mapping

$$\varepsilon_Q^{op} : Q \to \mathcal{O}_L(LPT(Q_0, Q_1, Q_2))$$

is a coupled semi-quantale isomorphism, that is, $Q \approx \Phi_L^{\rightarrow}(Q)$.

$$\begin{aligned} \text{Compactness of } (Q_0, Q_1, Q_2) &\Leftrightarrow Q_0 \textit{ is } \text{compact} \\ &\Leftrightarrow LPT(Q_0) = (LPT(Q_0), \Phi_L^{\rightarrow}(Q_0)) \text{ is compact} \\ &\Leftrightarrow (LPT(Q_0), \Phi_L^{\rightarrow}(Q_1) \vee \Phi_L^{\rightarrow}(Q_2)) \text{ is compact.} \\ &\Leftrightarrow LPT(Q) = (LPT(Q_0), \Phi_L^{\rightarrow}(Q_1), \Phi_L^{\rightarrow}(Q_2)) \end{aligned}$$

is pairwise compact and this completes the proof. □

5. Conclusion

The concept of coupled semi-quantales is introduced as a pointfree analogues of lattice-valued bitopological (or biquasi-topological spaces). An adjunction between the category of coupled semi-quantales and the category of lattice-valued biquasi-topological spaces is established. Through such adjunction topological and the lattice-theoretic concepts of regularity and compactness are defined and studied for both lattice-valued biquasi-topological spaces and coupled semi-quantales, respectively.

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