A POINTFREE ANALOGUES OF LATTICE-VALUED BITOPOLOGICAL SPACES

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Abstract. The concept of coupled semi-quantales is introduced. An adjunction between the category of coupled semi-quantales and the category of lattice-valued biquasi-topological spaces is established. The topological and the lattice-theoretic concepts of regularity and compactness are extended to both lattice-valued biquasi-topological spaces and coupled semi-quantales, respectively. Some relations among these axioms are established.

1. Introduction

In 1986 Mulvey [9], proposed the concept quantale as a non-commutative extension of frame (or pointfree topology) with aim to develop the concept of non-commutative topology [6] and provide a constructive foundations for both quantum mechanics and non-commutative logic [17]. Nowadays, the concepts of quantales and semi-quantales (as a generalization of quantales [14]) can boast many areas of applications, e.g., the area of non-commutative topology [5, 10, 11]. Further details about quantales can be found in [15].

In 2015 Höhle [7], established a non-commutative extension of the well known Papert-Papert-Isbell adjunction [8, 12] between the category of locales and the category of topological spaces to one between the category of quantales and the category of many valued topological spaces.

In [4], El-Saady extended the Höhle’s adjunction to a more general one between the category of semi-quantales and the category of lattice-valued quasi-topological spaces.

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In this paper we aim to introduce the concept of coupled semi-quantales as the pointfree analogues of lattice-valued bitopological spaces and extend the dual adjunction between the category of semi-quantales and the category of lattice-valued quasi-topological spaces to one between the category of coupled semi-quantales and the category of lattice valued biquasi-topological spaces. Also, the topological and the lattice-theoretic concepts of regularity and compactness are extended to lattice-valued biquasi-topological spaces and coupled semi-quantales, respectively. Some relations among these axioms are established.

2. Preliminaries

By a complete join-semilattice (or V-semilattice) we mean a partially ordered set $(L, \leq)$ having arbitrary sups.

**Definition 2.1.** [14] A semi-quantale $(L, \leq, \otimes)$ is a complete join-semilattice $(L, \leq)$ equipped with a binary operation $\otimes : L \times L \to L$, with no additional assumptions, called a tensor product.

**Definition 2.2.** [14] Let $L$ and $M$ be semi-quantales. A function $h : L \to M$ is said to be:

1. a semi-quantale morphism if it preserves $\otimes$ and arbitrary sups;
2. a strong semi-quantale morphism if it preserves $\otimes$, arbitrary sups and $\top$.

By $\text{SQuant}$ (resp. $\text{SSQuant}$), we mean the category of all semi-quantales and semi-quantale morphisms (resp. strong semi-quantale morphism).

**Definition 2.3.** A semi-quantale $(L, \leq, \otimes)$ is said to be:

1. a quantale [15] if whose multiplication $\otimes$ is associative and distributes across $\lor$ from both sides. $\text{Quant}$ denotes the full subcategory of $\text{SQuant}$ of all quantales.
2. a unital semi-quantale [14] if whose multiplication $\otimes$ has an identity element $e \in L$ called the unit. $\text{USQuant}$ denotes the category all unital semi-quantales together with all semi-quantale morphisms preserving the unit $e$.
3. a commutative semi-quantate [14] if whose multiplication $\otimes$ satisfies that $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1, q_2 \in L$. $\text{CSQuant}$ denotes the full subcategory of $\text{SQuant}$ of all commutative semi-quantales.
4. a distributive semi-quantate [16] if whose multiplication $\otimes$ distributes across finite $\lor$ from both sides. $\text{DSQuant}$ is the category of distributive semi-quantales.

**Definition 2.4.** [4] Let $L \in |\text{SQuant}|$, $M \subseteq L$, and $a, b \in M$. An element $a$ is said to be well-inside of $b$ (w.r.t. $M$), denoted $a \preceq b$, if exists $c \in M$ with $a \otimes c = \bot$ and $c \lor b = \top$.

An $L \in |\text{SQuant}|$ is said to be regular [4], if for each $a \in L$ there exists $D \subseteq I_a$, where $I_a = \{b \in L : b \preceq a\}$ such that $a = \lor D$. 

Definition 2.5. [3] Let \( L = (L, \leq, \otimes) \) be a semi-quantale. A subset \( K \subseteq L \) is a subsemi-quantale of \( L \) if and only if the inclusion \( K \hookrightarrow L \) is a semi-quantale morphism, i.e., \( K \) is closed under \( \otimes \) and arbitrary sups. A subsemi-quantale \( K \) of \( L \) is said to be strong if and only if \( \top \) belongs to \( K \). If \( L \) is a unital semi-quantale with the identity \( e \), then a subsemi-quantale \( K \) of \( L \) is called a unital subsemi-quantale of \( L \) if and only if \( e \) belongs to \( K \).

Let \( L = (L, \leq, \otimes) \) be a semi-quantale. For any non-empty set \( X \), let \( L^X \) be the set of all \( L \)-valued maps \( X \rightarrow L \). We can extend the algebraic and lattice-theoretic structure from \( L \) to \( L^X \) pointwisely, i.e., for all \( x \in X, f, g \in L^X \) and \( \{f_j : j \in J\} \subseteq L^X \), we have

\[
\begin{align*}
  f \leq g & \iff f(x) \leq g(x), \\
  (f \otimes g)(x) & = f(x) \otimes g(x), \\
  \left( \bigvee_{j \in J} f_j \right)(x) & = \bigvee_{j \in J} (f_j(x)).
\end{align*}
\]

Then \( L^X \) is again a semi-quantale with respect to the multiplication \( \otimes \). If \( L \) is a unital semi-quantale with unit \( e \), then \( L^X \) becomes a unital semi-quantale with the unit \( e \) (a mapping from \( X \) to \( L \), defined by \( e(x) = e \) for all \( x \in X \)), where \( e \) is the unit of \( \otimes \) in \( L \).

For an ordinary mapping \( f : X \rightarrow Y \), the forward and backward powerset operators [13, 14]:

\[
\begin{align*}
  f^+_L : L^X & \rightarrow L^Y \text{ and } f^-_L : L^Y \rightarrow L^X,
\end{align*}
\]

defined by

\[
\begin{align*}
  f^+_L(A)(y) & = \bigvee\{A(x) : x \in X, f(x) = y\} \text{ and } f^-_L(B) = B \circ f,
\end{align*}
\]
respectively.

Theorem 2.1. [14] Let \( L \in |\text{SQuant}| \), \( X, Y \) be a nonempty ordinary sets and \( f : X \rightarrow Y \) be an ordinary mapping, then we have:

1. \( f^+_L \) preserves arbitrary \( \bigvee \);
2. \( f^-_L \) preserves arbitrary \( \bigvee \), \( \otimes \), and all constant maps;
3. \( f^-_L \) preserves the unit if \( L \in |\text{USQuant}| \).

For a fixed \( L \in |\text{SQuant}| \) and a set \( X \), an \( L \)-quasi-topology on \( X \) [14] is a subsemi-quantale \( \tau \) of \( L^X = (L^X, \leq, \otimes) \), i.e., satisfying the following conditions.

\( (T_1) \) For all \( A, B \in L^X \), if \( A, B \in \tau \) then \( A \otimes B \in \tau \).
\( (T_2) \) For all \( \{A_j : j \in J\} \subseteq L^X \), if \( \{A_j : j \in J\} \subseteq \tau \) then \( \bigvee_{j \in J} A_j \in \tau \).

An \( L \)-quasi-topology \( \tau \) is said to be strong [3] if and only if it is strong as a subsemi-quantale of \( L^X \), i.e., \( \tau \) satisfies the additional axiom:

\( (T_3) \) \( \top \in \tau \).
If $L \in |\text{USQuant}|$ with unit $e$, a unital subsemi-quantale $\tau$ of $L^X$ is called an $L$-topology on $X$ [14], i.e., $\tau$ satisfies $(T_1), (T_2)$ and the following:

$(T_3)$ $\emptyset \subseteq \tau$.

If $\tau \subseteq L^X$ is an $L$-quasi-topology (resp. $L$-topology), then the pair $(X, \tau)$ is said to be an $L$-quasi-topological (resp. $L$-topological) space. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $L$-continuous (resp. $L$-open) [13] if $(f_\sigma^T)^\tau: \tau \leftarrow \sigma$ (resp. $(f_\sigma^T)^\tau: \tau \rightarrow \sigma$).

An $L$-continuous bijection $f : (X, \tau) \rightarrow (Y, \sigma)$ is an $L$-homeomorphism [13] if $f^{-1}$ is $L$-continuous.

It is clear that $L$-quasi-topological (resp. strong $L$-quasi-topological, $L$-topological) spaces and $L$-continuous maps form a category denoted by $L\text{-QTop}$ (resp. $L\text{-SQTop}$, $L\text{-Top}$).

One can easily prove that each of $L\text{-QTop}$, $L\text{-SQTop}$ and $L\text{-Top}$ is a topological category over the category $\text{Set}$.

**Definition 2.6.** [4] An $(X, \tau) \in |L\text{-QTop}|$ is called

1. $L\text{-QT}_0$ if for every $x, y \in X$ with $x \neq y$ there exists $\mu \in \tau$ with $\mu(x) \neq \mu(y)$;
2. $L\text{-qsober}$ if and only if $\eta_X : (X, \tau) \rightarrow (L\text{PT}(\tau), \Phi_L^{\tau}(\tau))$ is bijective.

3. **Coupled Semi-quantales and Lattice-valued Biquasi-topological Spaces**

Before we go on, this section, we begin our study by the following.

**Lemma 3.1.** If $\{A_j : j \in J\}$ is any collection of subsemi-quantales of a semi-quantale $Q$, then $\bigcap_j A_j$ is also a subsemi-quantale of $Q$, provided $\bigcap_j A_j \neq \phi$.

**Proof.** Let $M = \bigcap_j A_j$ and $a, b \in M$. Then $a, b \in A_j \Rightarrow a \otimes b \in A_j$ for each subsemi-quantale $A_j \Rightarrow a \otimes b \in M \Rightarrow M$ is closed under $\otimes$. Also, one can easily prove that $M$ is closed under sups. \qed

For a fixed $Q \in |\text{SQuant}|$, it follows, as a consequence of the above lemma, that the family of all subsemi-quantales of $Q$, ordered by inclusion, forms a complete lattice, with the meet $Q_1 \wedge Q_2 = Q_1 \cap Q_2$ (the set-intersection), and the join $Q_1 \vee Q_2$ is the least subsemi-quantale of $Q$ containing $Q_1$ and $Q_2$ (which is not their set-theoretical union). The supremum (joins) of a set $\{A_j : j \in J\}$ of subsemi-quantales of $Q$, is the intersection of subsemi-quantales of $Q$ which contains the union $\bigcup_j A_j$. More generally there is for each subset $K \subseteq Q$ of a semi-quantale $Q$ a smallest subsemi-quantale of $Q$ (sometimes denoted by $[K]$) which contains $K$ and is the subsemi-quantale generated by $K$.

**Definition 3.1.** (The category of coupled semi-quantales)

1. A coupled semi-quantale is a triple $Q = (Q_0, Q_1, Q_2)$ in which $Q_0$ is a semi-quantale, $Q_1$ and $Q_2$ are subsemi-quantales of $Q_0$ such that $Q_1 \cup Q_2$ generates $Q_0$. 

(2) A map \( h : Q \to P \) between coupled semi-quantales is a semi-quantale morphism \( Q_0 \to P_0 \) for which the restrictions \( h|_{Q_i} : Q_i \to P_i \) are semi-quantale morphisms i.e., \( h(Q_i) \subseteq P_i \) for \( i = 1, 2 \).

(3) The resulting category will be denoted by \( \text{CSQuant} \).

We refer to \( Q_0 \) as the total part of \( Q \), and \( Q_1, Q_2 \) as its first and second parts, respectively.

**Definition 3.2.** A coupled semi-quantale \( Q = (Q_0, Q_1, Q_2) \) is said to be:

1. **unital** if and only if \( Q_0 \) is unital and \( e \) belongs to both \( Q_1 \) and \( Q_2 \).
   - \( \text{UnCSQuant} \) is the full subcategory of \( \text{CSQuant} \) of all unital coupled semi-quantales.
2. **coupled quantal** \([1]\) if \( Q_0 \) is a quantale and both \( Q_1 \) and \( Q_2 \) are subquantales.
   - \( \text{CQuant} \) is the full subcategory of \( \text{CSQuant} \) of all coupled quantales.
3. **strong coupled quantal** if both \( Q_1 \) and \( Q_2 \) are strong subquantales of \( Q_0 \).
4. **symmetric** if and only if \( Q_0 = Q_1 = Q_2 \).
5. **right-sided** (resp. left-sided) if and only if \( a \otimes \top \leq a \) (resp. \( \top \otimes a \leq a \)) for all \( a \in Q_0 \).
6. **idempotent** if and only if the total part \( Q_0 \) is idempotent, i.e., \( a \otimes a = a \) for all \( a \in Q_0 \).
7. **commutative** if the operation \( \otimes \) is commutative, i.e., \( q_1 \otimes q_2 = q_2 \otimes q_1 \) for every \( q_1 \in Q_i \) and \( q_2 \in Q_k \).
   - \( \text{ComCSQuant} \) is the full subcategory of \( \text{CSQuant} \) of all commutative coupled semi-quantales.

**Example 3.1.** For a fixed \( L \in |\text{SQuant}| \) and a non-empty set \( X \). For \( i = 1, 2 \), let \( \tau_i \subseteq L^X \) be a subsemi-quantale of \( L^X \), i.e., \( L \)-quasi-topologies on \( X \). The triple \( (\tau_1 \lor \tau_2, \tau_1, \tau_2) \) is a coupled semi-quantale where \( \tau_1 \lor \tau_2 \) is the coarsest \( L \)-quasi-topology finer than both \( \tau_1 \) and \( \tau_2 \).

**Example 3.2.** Let \( Q = \{\bot, a, b, \top\} \) be the four Boolean lattice and let \( \otimes : Q \times Q \to Q \) defined by

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It is clear that \( Q \) is a coupled quantales with \( Q_0 = \{\bot, a, b, \top\} \) as the total part, \( Q_1 = \{\bot, a, \top\} \) as the first part and \( Q_2 = \{\bot, b, \top\} \) as the second part.

**Example 3.3.** Any biframe \( A = (A_0, A_1, A_2) \) \([2]\) is a commutative coupled quantale provided that \( \otimes = \land \) and any element of \( a \in A_0 \) can be expressed as \( a = \lor \{a_1 \otimes a_2 : a_1 \in A_1, a_2 \in A_2\} \).

**Definition 3.3.** (The category of \( L \)-biquasi-topological spaces)
(1) An \(L\)-biquasi-topological space is a triple \((X, \tau_1, \tau_2)\) consisting of a non-empty set \(X\) and two \(L\)-quasi-topologies \(\tau_1\) and \(\tau_2\) on \(X\).

(2) A morphism \(f : X \to Y\) between \(L\)-biquasi-topological spaces \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) is a function between their underlying sets for which
\[
f : (X, \tau_1) \to (Y, \sigma_1) \text{ and } f : (X, \tau_2) \to (Y, \sigma_2)
\]
are \(L\)-continuous.

(3) The category of \(L\)-biquasi-topological spaces and their morphisms will be denoted by \(L\text{-BiQT op}\).

Between the category \(L\text{-QT op}\) and \(L\text{-BiQT op}\) there is a faithful functor \(E_S : L\text{-BiQT op} \to L\text{-QT op}\), which we describe as follows. If \(X = (X, \tau_1, \tau_2) \in |L\text{-BiQT op}|\), then
\[
E_S(X) = (X, \tau_1 \vee \tau_2),
\]
where \(\tau_1 \vee \tau_2\) is the coarsest \(L\)-quasi topology finer than both \(\tau_1\) and \(\tau_2\), \(E_S(f) = f\).

The left adjoint of \(S\) is the functor \(E_d : L\text{-QT op} \to L\text{-BiQT op}\), by the following correspondences:
\[
E_d(X, \tau) = (X, \tau, \tau), \quad E_d(f) = f.
\]

One notes that since \(E_S\) embeds \(L\text{-QT op}\) in \(L\text{-BiQT op}\), then we will regard the constructions in \(L\text{-BiQT op}\) as extensions of the constructions in the category \(L\text{-QT op}\).

For \(L \in |S\text{Quant}|\) and \((X, \tau_1, \tau_2) \in |L\text{-BiQT op}|\). The functor \(\mathcal{O}_L : L\text{-BiQT op} \to \text{CS\text{Quant} op}\)
is defined as follows
\[
\mathcal{O}_L(X, \tau_1, \tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2).
\]
For the \(L\)-biquasi-topological space \((X, \tau_1, \tau_2)\), the triple \((\tau_1 \vee \tau_2, \tau_1, \tau_2)\) is a coupled semi-quantale where \(\tau_1 \vee \tau_2\) is the coarsest \(L\)-quasi-topology finer than both \(\tau_1\) and \(\tau_2\), and
\[
\mathcal{O}_L(f : (X, \tau_1, \tau_2) \to (Y, \theta_1, \theta_2)) = [(f^{-1}_L)|_{\theta_i}]^\text{op} : \tau_i \to \theta_i, \quad i = 1, 2.
\]
Now, we will introduce some ideas needed to define a functor in the opposite direction. For a coupled semi-quantale \(Q = (Q_0, Q_1, Q_2)\), let
\[
LPT(Q_0) = \{p : Q_0 \to L : p \in |S\text{Quant}|\}.
\]
Also, we define a coupled semi-quantale map
\[
\Phi_L : (Q_0, Q_1, Q_2) \to (L^{LPT(Q_0)}, L^{LPT(Q_0)}, L^{LPT(Q_0)})
\]
such that
(1) \(\Phi_L : Q_0 \to L^{LPT(Q_0)}\) is a semi-quantale map, where \(\Phi_L(a)(p) = p(a)\);
(2) \(\Phi_L^+(Q_1) \subseteq L^{LPT(Q_0)}\);
(3) \(\Phi_L^-(Q_2) \subseteq L^{LPT(Q_0)}\).
As given in [4] the function $\Phi_L$ preserves $\otimes$ and arbitrary $\lor$, where these are inherited by the codomain of $\Phi_L$ from $L$. Also, for $i = 1, 2$, we have $\Phi_L^i(Q_i)$ is closed under these operations and hence is an $L$-quasi topology on $LPT(Q_0)$. Thus we have

$$LPT : L\text{-BiQTop} \rightarrow \text{CSQuant}^{op},$$

defined by

$$(Q_0, Q_1, Q_2) \rightarrow (LPT(Q_0), \Phi_L^1(Q_1), \Phi_L^1(Q_2)),$$

where $LPT(f : A \rightarrow B) = [f]^{op}$, that is, $LPT(f)(p) = p \circ f^{op}$, $f^{op} : B \rightarrow A$, is a $L$-quasi-topology in $\text{CSQuant}$. It is clear that $\{\Phi_L(a_i) : a_i \in Q_i, i = 1, 2\}$ is an $L$-continuous map in $LPT(Q_0)$ and, therefore, we have $(LPT(Q_0), \Phi_L^1(Q_1), \Phi_L^1(Q_2)) \in [L\text{-BiQTop}]$.

**Proposition 3.1.** For a fixed $L \in |\text{SQant}|$ and $Q, P \in |\text{CSQuant}|$, the mapping

$$LPT(f) : (LPT(Q_0), \Phi_L^1(Q_1), \Phi_L^1(Q_2)) \rightarrow (LPT(P_0), \Phi_L^1(P_1), \Phi_L^1(P_2))$$

is $L$-bicontinuous.

**Proof.** We need to check the $L$-continuity of both the functions

1. $LPT(f) : (LPT(Q_0), \Phi_L^1(Q_1)) \rightarrow (LPT(P_0), \Phi_L^1(P_1))$ and
2. $LPT(f) : (LPT(Q_0), \Phi_L^1(Q_2)) \rightarrow (LPT(P_0), \Phi_L^1(P_2))$.

The first function is $L$-continuous since for all $q_2 \in P_0, p \in LPT(Q_0)$, we have

$$LPT(f)^r(\Phi_L(q_2)(p)) = \Phi_L(q_2)(LPT(f)(p))$$

$$= \Phi_L(q_2)(p \circ f^{op})$$

$$= \Phi_L(f^{op}(q_2))(p).$$

Similarly, we can check the $L$-continuity of the second function and this completes the proof. \qed

Then we have the spectrum or point functor

$$LPT : \text{CSQuant}^{op} \rightarrow L\text{-BiQTop}.$$ 

To study the adjunction between the functors

$$LPT : \text{CSQuant}^{op} \rightarrow L\text{-BiQTop}$$

and

$$\mathcal{O}_L : L\text{-BiQTop} \rightarrow \text{CSQuant}^{op}.$$ 

we give the following definitions.

For fixed $L \in |\text{SQant}|$, $(X, \tau_1, \tau_2) \in |L\text{-BiQTop}|$ and $Q \in |\text{CSQuant}|$ define the maps:

1. $\eta_X : (X, \tau_1, \tau_2) \rightarrow (LPT(\tau_1 \lor \tau_2), \Phi_L^1(\tau_1), \Phi_L^1(\tau_2))$, by setting, for all $x \in X$ and $\mu \in \mathcal{O}_L(X)$, $\eta_X(x)(\mu) = \mu(x)$;
2. $\varepsilon_Q^{op} : Q \rightarrow \mathcal{O}_L(LPT(Q))$, by setting $\varepsilon_Q^{op} = \Phi_L \phi^{op}(Q_0)$.

It is clear that by definition $\varepsilon_Q^{op}$ always surjective.
Lemma 3.2. Let \( L \in |S\text{Quant}| \), \( (X, \tau_1, \tau_2) \in |L\text{-BiQTop}| \) and \( Q \in |\text{CSQuant}| \). Then

1. the map \( \eta_X : (X, \tau_1, \tau_2) \to (LPT(\tau_1 \lor \tau_2), \Phi^+_L(\tau_1), \Phi^+_L(\tau_2)) \), is \( L \)-bicontinuous, and pairwise \( L \)-open w.r.t. its range in \( (LPT(\tau_1 \lor \tau_2), \Phi^+_L(\tau_1), \Phi^+_L(\tau_2)) \)

2. the map \( \varepsilon^Q_\circ : Q \to \mathcal{O}_L(LPT(Q)) \) is a coupled semi-quantale morphism.

Proof. (1) To prove that the mapping \( \eta_X \) is \( L \)-bicontinuous, it suffices to prove that both the mappings \( \eta_X : (X, \tau_1) \to (LPT(\tau_1 \lor \tau_2), \Phi^+_L(\tau_1)) \) and \( \eta_X : (X, \tau_2) \to (LPT(\tau_1 \lor \tau_2), \Phi^+_L(\tau_2)) \) are \( L \)-continuous and \( L \)-open with respect to their respective ranges.

(i) \( L \)-continuity: for \( i \in \{1, 2\} \), for all \( \mu \in \Phi^+_L(\tau_i) \), and for all \( x \in X \), there exists \( \rho \in \tau_i \) such that \( \Phi^+_L(\rho) = \mu \), \( \eta_X^L(\mu)(x) = \eta_X^L(\Phi^+_L(\rho))(x) = \rho(x) \), that is, \( \eta_X^L(\mu) \in \tau_i \). Hence \( \eta_X \) is \( L \)-bicontinuous.

(ii) Openness: in fact, for \( \nu \in \tau_i \), \( i \in \{1, 2\} \), and \( \rho \in LPT(\tau_1 \lor \tau_2) \):

\[
(\eta_X^L(\nu)(p)) = \bigvee \{ \nu(x) : \eta_X(x) = p \} \\
= \bigvee \{ \eta_X(x)(\nu) : \eta_X(x) = p \} \\
= p(\nu) = \Phi^+_L(\nu)(p).
\]

Now, \( \Phi^+_L(\nu) \in \Phi^+_L(\tau_1) \), the \( L \)-quasi-topology on \( LPT(\tau_1 \lor \tau_2) \), and it follows that \( (\eta_X^L(\nu) = \Phi^+_L(\nu) \), that is, \( (\eta_X^L(\nu) = (\eta_X^L(\Phi^+_L(\rho))(x) = (\eta_X^L(\Phi^+_L(\rho))(x) \)

Thus \( (\eta_X^L(\nu) \) is open w.r.t. the subspace topology of \( (\eta_X^L(\tau_1)) \) induced from \( LPT(\tau_1 \lor \tau_2) \), that is, \( \eta_X \) is a pairwise \( L \)-open map.

(2) As given in [4], we note that the mapping \( \varepsilon^Q_\circ : Q_0 \to \mathcal{O}_L(LPT(Q_0)) \) is a semi-quantale homomorphism and so the mappings \( \varepsilon^Q_\circ : Q_0 \to \mathcal{O}_L(LPT(Q_0)) \), for \( i = 1, 2 \). Thus we have that the mapping \( \varepsilon^Q_\circ : Q \to \mathcal{O}_L(LPT(Q)) \) is a coupled semi-quantale morphism.

\[ \square \]

Theorem 3.1. The functor

\[ LPT : L\text{-BiQTop} \leftarrow \text{CSQuant}^{op} \]

is a right adjoint of the functor

\[ \mathcal{O}_L : L\text{-BiQTop} \to \text{CSQuant}^{op} \]

with unit \( \eta_X : X \to LPT(\mathcal{O}_L(X, \tau_1, \tau_2)) \) and counit \( \varepsilon_Q : Q \leftarrow \mathcal{O}_L(LPT(Q)) \).

Proof. It will be enough to show that for every \( Q \in |\text{CSQuant}| \) and an \( L\text{-BiQTop}\)-morphism \( (X, \tau_1, \tau_2) \overset{f}{\to} LPT(Q) \), there exists uniquely a \( \text{CSQuant} \)-morphism \( Q \overset{\varepsilon_Q}{\to} \mathcal{O}_L(X, \tau_1, \tau_2) \) such that the left diagram of the following diagram in Figure 1 is commutative, where by \( \tau_0 \) we mean the coarsest \( L \)-quasi-topology \( \tau_1 \lor \tau_2 \).

To prove the existence, let \( f^* = \mathcal{O}_L(f) \circ \varepsilon_Q \). From the definitions of \( \mathcal{O}_L(f) \) and \( \varepsilon_Q \), one can easily prove that \( f^* : Q \to \Omega(X, \tau_1, \tau_2) \) is a \( \text{CSQuant} \)-morphism. For commutativity of the above-mentioned left diagram notice that for \( x \in X \) and \( a \in Q_0 \), we have
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\[
\begin{array}{c}
LPT(Q) \\
\downarrow \text{f} \\
X \\
\downarrow \text{f} \\
LPT\left( f^* \right) \\
\downarrow \text{f} \\
\rightleftharpoons \end{array}
\]

\text{Figure 1.}

\[pt(f^*) \circ \eta_X(x)(a) = \eta_X(x)(f^*(a))
= \eta_X(x)(O_L(f) \circ \varepsilon_Q(a))
= (O_L(f)(\Phi_L(a)))(x)
= (f^*_L(\Phi_L(a)))(x)
= (\Phi_L(a) \circ f)(x)
= f(x)(a).\]

Uniqueness of the function \( f^* \) follows from the observation that given another \text{CSQuant}-morphism \( Q \xrightarrow{g} \Omega(X, \tau_1, \tau_2) \) with the same property: for all \( x \in X \), and for all \( a \in Q_0 \), we have

\[f(x)(a) = \eta_X(x)(g(a))
= \eta_X(x)(O_L(g) \circ \varepsilon_L(a))
= (g^*_L(\Phi_L(a)))(x)
= (\Phi_L(a) \circ g)(x)
= g(a)(a).\]

Hence for all \( x \in X \) and for all \( a \in Q_0 \), we have \( f^*(a) = g(a) \), i.e., \( f^* = g \). \hfill \Box

\textbf{Definition 3.4.} An \((X, \tau_1, \tau_2) \in |L-\text{BiQTop}|\) is said to be pairwise \( L-QT_0 \) (i.e., fulfills the \( T_0 \)-axiom) if and only if for every pair \((x, y) \in X \times X \) with \( x \neq y \), there exists \( \mu \in \tau_1 \vee \tau_2 \) such that \( \mu(x) \neq \mu(y) \).

By \( L-T_0\text{-BiQTop} \), we mean a full subcategory of \( L-\text{BiQTop} \) consisting of those \( L-\text{BiQTop} \) objects, which are pairwise \( L-QT_0 \).

As a consequence of \textbf{Definition 2.6}, we have the following easily established proposition.
Proposition 3.2. \((X, \tau_1, \tau_2) \in |L-T_0\text{BiQTop}|\) if and only if \(S(X, \tau_1, \tau_2) = (X, \tau_1 \lor \tau_2)\) is \(L-QT_0\).

Proposition 3.3. An \((X, \tau_1, \tau_2) \in |L\text{-BiQTop}|\) is pairwise \(L-QT_0\) if and only if the mapping 
\[
\eta_x : (X, \tau_1, \tau_2) \to (LPT(\tau_1 \lor \tau_2), \Phi^+_L(\tau_1), \Phi^-_L(\tau_2))
\]

is pairwise \(L\)-embedding.

Proof. First, suppose that \((X, \tau_1, \tau_2) \in |L\text{-BiQTop}|\) is pairwise \(L-QT_0\), then for \(x \neq y \in X\), there exists \(\mu \in \tau_1 \lor \tau_2\) such that \(\mu(x) \neq \mu(y)\). Therefore, \(\eta_x(x)(\mu) = \mu(x) \neq \mu(y) = \eta_x(y)(\mu)\), that is, the mapping \(\eta_x\) is injective. Also, since the mapping \(\eta_x\) is pairwise \(L\)-continuous and \(L\)-open (see Lemma 3.2), then \(\eta_x\) is \(L\)-embedding. □

Now, we will introduce the concept of sobriety of objects in the category \(L-\text{BiQTop}\).

Definition 3.5. An \((X, \tau_1, \tau_2) \in |L\text{-BiQTop}|\) is \(L\)-sober if and only if the mapping
\[
\eta_x : X \to LPT_{\rightarrow}(O_L(X, \tau_1, \tau_2))
\]
is bijective.

By \(L\text{-SobBiQTop}\), we mean the full subcategory of \(L\text{-BiQTop}\) of all sober objects.

Lemma 3.3. An \((X, \tau_1, \tau_2) \in |L\text{-BiQTop}|\) is \(L\)-sober if and only if the mapping
\[
\eta_x : (X, \tau_1, \tau_2) \to (LPT(\tau_1 \lor \tau_2), \Phi^+_L(\tau_1), \Phi^-_L(\tau_2))
\]
is a pairwise homomorphism.

Proof. \(L\)-sobriety of an \((X, \tau_1, \tau_2) \in |L\text{-BiQTop}|\) is equivalent to the fact of bijectivity of the mapping
\[
\eta_x : (X, \tau_1, \tau_2) \to (LPT(\tau_1 \lor \tau_2), \Phi^+_L(\tau_1), \Phi^-_L(\tau_2)).
\]
Also, the mapping \(\eta_x\) is pairwise \(L\)-continuous and \(L\)-open (see Lemma 3.2), and this is equivalent to the fact that \(\eta_x\) is pairwise \(L\)-homomorphism. □

By the above and Definition 2.6, one have the following easily established result.

Proposition 3.4. An \((X, \tau_1, \tau_2) \in |L\text{-BiQTop}|\) is \(L\)-sober if and only if \((X, \tau_1 \lor \tau_2)\) is \(L\)-qsober.

Definition 3.6. The coupled semi-quantales \(Q = (Q_0, Q_1, Q_2)\) is spatial if and only if the total part \(Q_0\) is spatial. Equivalently the map
\[
\varepsilon^{op}_Q : Q_0 \to O_L(LPT(Q_0))
\]
is a semi-quantale isomorphism [4].

By \(\text{SpatCSQuant}\), we mean the full subcategory of the spatial coupled semi-quantales in \(\text{CSQuant}\).

Lemma 3.4. For all \(Q = (Q_0, Q_1, Q_2) \in |\text{CSQuant}|\), \(Q = (Q_0, Q_1, Q_2)\) is spatial if and only if the mapping
\[ \varepsilon_{Q}^{op} : (Q_0, Q_1, Q_2) \to O_L(LPT(Q_0, Q_1, Q_2)) \]

is a coupled semi-quantale isomorphism.

**Proof.** Let \( Q = (Q_0, Q_1, Q_2) \) be a spatial coupled semi-quantale. Then, by the definition, the total part \( Q_0 \) is spatial, and this is equivalent to the fact that the map

\[ \varepsilon_{Q}^{op} : Q_0 \to O_L(LPT(Q_0)) \]

is a semi-quantale isomorphism, and this implies that the map

\[ \varepsilon_{Q}^{op} : (Q_0, Q_1, Q_2) \to O_L(LPT(Q_0, Q_1, Q_2)) \]

is a coupled semi-quantale isomorphism. \( \square \)

**Lemma 3.5.** For all \((X, \tau_1, \tau_2) \in |L-BiQTop|\) and for all \(Q \in |CSQuant|\), then

(i) \( O_L(X, \tau_1, \tau_2) = (\tau_1 \lor \tau_2, \tau_1, \tau_2) \) is spatial;

(ii) \( LPT(Q_0, Q_1, Q_2) = (LPT(Q_0), \Phi^+_L(Q_1), \Phi^+_L(Q_2)) \) is \(L\)-sober.

**Proof.** As to (i), clearly, the map

\[ \varepsilon_{\tau_1 \lor \tau_2}^{op} : (\tau_1 \lor \tau_2) \to O_L(LPT(\tau_1 \lor \tau_2)) = \Phi^+_L(\tau_1 \lor \tau_2) \]

is a semi-quantale isomorphism, which implies that \( \tau_1 \lor \tau_2 \) is a spatial semi-quantale and, therefore, the coupled semi-quantale \( O_L(X, \tau_1, \tau_2) = (\tau_1 \lor \tau_2, \tau_1, \tau_2) \) is spatial.

As to (ii), by definition, it suffices to prove that the mapping

\[ \eta_X : LPT(Q) \to LPT(O_L(LPT(Q))) = LPT((\Phi^+_L(Q_1) \lor \Phi^+_L(Q_2)), \Phi^+_L(Q_1), \Phi^+_L(Q_2)) \]

is bijective. Now, we have the following.

(a) \( \eta_X \) is one-to-one. For all \( p_1, p_2 \in LPT(Q_0) \) with \( p_1 \neq p_2 \), there exist some \( a \in Q_0 \) with \( p_1(a) \neq p_2(a) \), and this implies that

\[ \eta_X(p_1)(\Phi^+_L(a)) = \Phi^+_L(a)(p_1) = p_1(a) \neq p_2(a) = \eta_X(p_2)(\Phi^+_L(a)). \]

Hence \( \eta_X \) is one-to-one.

(b) \( \eta_X \) is onto. For all \( q \in LPT(\Phi^+_L(Q_1 \lor Q_2)) \), let \( p = q \circ \Phi^+_L : Q_0 \to \Phi^+_L(Q_0) \to L \), then \( p \in LPT(Q_0) \) and \( a \in Q_0 \). We have \( \eta_X(p)(\Phi^+_L(a)) = \Phi^+_L(a)(p) = p(a) = q(\Phi^+_L(a)). \) Hence \( \eta_X(p) = q \), that is, \( \eta_X \) is onto. From (a) and (b), it follows that \( \eta_X \) is bijective, and this completes the proof. \( \square \)

**Proposition 3.5.** The following functors are valid:

(i) \( O_L : L-BiQTop \to SpatCSQuant^{op} ; \)

(ii) \( LPT : L-SobBiQTop \leftarrow CSQuant^{op} \).

The equivalence between the categories \( L-SobBiQTop \) and \( SpatCSQuant \) is proven as follows.

**Theorem 3.2.** For all \( L \in |SQuant| \), \( L-SobBiQTop \approx SpatCSQuant^{op} \).
Proof. The categorical equivalence \( L\text{-SobBiQTTop} \approx \text{SpatCSQuant}^{\text{op}} \) follows directly from the adjunction \( O_L \dashv \text{LPT} \) and the fact that both the unit and counit are isomorphisms in the categories \( L\text{-SobBiQTTop} \) and \( \text{SpatCSQuant}^{\text{op}} \), respectively. \( \square \)

4. Regularity and Pairwise Compactness

Now, we will define the regularity and compactness for a certain \( L\text{-BiQTTop} \) and \( \text{CSQuant} \) objects.

Definition 4.1. Let \( Q = (Q_0, Q_1, Q_2) \in |\text{CSQuant}| \) and \( a, b \in Q_i, \ i = 1, 2 \). An element \( a \) is said to be well inside of \( b \) (w.r.t. \( Q_i \)) and denoted by \( a \preceq_i b \), if and only if exists \( c \in Q_k, k \neq i \), such that \( a \otimes c = \bot \) and \( c \lor b = \top \).

Lemma 4.1. For any strong \( \text{CSQuant} \)-morphism \( h : Q \to P \)
\[ a \preceq_i b \Rightarrow h(a) \preceq_i h(b). \]

Proof. Let \( a, b \in Q_i \) with \( a \preceq_i b \), then exists \( c \in Q_k, k \neq i \), with \( c \otimes a = \bot \), \( c \lor b = \top \). Since \( h : Q \to P \) is a strong semi-quantale homomorphism, then \( h(c \otimes a) = h(c) \otimes h(a) = \bot \) and \( h(c) \lor h(b) = h(c) \lor (h(b) = h(\top) = \top) \). So exists \( h(c) \in P_k, k \neq i \), such that \( h(c) \otimes h(a) = \bot \) and \( h(c) \lor h(b) = \top \) which means that \( h(a) \preceq_i h(b) \). \( \square \)

Definition 4.2. An \( Q = (Q_0, Q_1, Q_2) \in |\text{CSQuant}| \) is said to be regular if and only if both \( Q_1 \) and \( Q_2 \) are regular subsemi-quantales. Or equivalently

for all \( a \in Q_i \), exists \( D \subseteq \{b \in Q_i : b \preceq_i a\} \) such that \( a = \lor D, i = 1, 2 \).

By \( \text{RegCSQuant} \), we mean the full subcategory of \( \text{CSQuant} \) of regular objects.

A coupled semi-quantale map \( h : Q \to P \) is said to be surjective if and only if \( h|_{Q_i} : Q_i \to P_i \) is surjective for \( i = 1, 2 \).

Lemma 4.2. If \( h : Q \to P \) is a surjective strong coupled semi-quantale homomorphism and \( Q \in |\text{RegCSQuant}| \), then \( P \in |\text{RegCSQuant}| \).

Proof. For \( i = 1, 2 \), let \( x \in P_i \). Then \( x = h(a) \) for some \( a \in Q_i \). Regularity of \( Q \) means that exists \( D \subseteq \{b \in Q_i : b \preceq_i a\} \), \( a = \lor D, i = 1, 2 \). Therefore there exists \( E \subseteq \{h(b) \in P_i : b \preceq_i a\} \) such that \( E = h(D) \). Since \( a \preceq_i b \) implies \( x = h(a) \preceq_i h(b) = y \). Hence \( E \subseteq \{y \in P_i : y \preceq_i x\} \) and \( x = \lor E \). Thus \( P \in |\text{RegCSQuant}| \). \( \square \)

Definition 4.3. Let \( L \in |\text{SQuant}| \). An \( (X, \tau_1, \tau_2) \) is regular if and only if \( O_L(X, \tau_1, \tau_2) \in |\text{RegCSQuant}| \).

By \( L\text{-RegBiQTTop} \), we mean the full subcategory of \( L\text{-BiQTTop} \) of regular objects.

Proposition 4.1. For \( Q = (Q_0, Q_1, Q_2) \in |\text{DCSQuant}| \) and \( (X, \tau_1, \tau_2) \in |L\text{-BiQTTop}| \).

(1) An \( Q = (Q_0, Q_1, Q_2) \) is regular if and only if
Then

\begin{align*}
a &= \bigvee \{ b \in Q_i : b \preceq_i a \} \text{ for all } a \in Q_i.
\end{align*}

(2) For \( L \in \mathcal{DSQuant} \). An \((X, \tau_1, \tau_2)\) is regular if and only if

\[ \mu = \bigvee \{ \nu \in \tau_i : \nu \preceq_i \mu \} \text{ for all } \mu \in \tau_i. \]

Proof. (1) Let \( Q = (Q_0, Q_1, Q_2) \in \mathcal{DSQuant} \). Distributivity and \( b \preceq_i a \) imply \( a \leq b \). Let \( D \subseteq \{ b \in Q_i : b \preceq_i a \} \) such that \( a = \bigvee D \). Then,

\[ \bigvee D \leq \bigvee \{ b \in Q_i : b \preceq_i a \} \leq \bigvee \{ b \in Q_i : b \leq a \} = a = \bigvee D. \]

This shows \( a = \bigvee D = \bigvee \{ b \in Q_i : b \preceq_i a \} \) and from this follows the claims. (2) Follows from (1). \( \square \)

As the preceding proposition offers the preserving of the regular axiom under the functor

\[ LPT : \text{L-BiQTop} \dashv \text{CSQuant}^{op}, \]

and with the aid of Definition 4.3, we have the following easily established proposition.

**Proposition 4.2.** The following functors holds:

\[ \mathcal{O}_L : \text{L-RegBiQTop} \to \text{RegCSQuant}^{op}, \]

\[ LPT : \text{L-RegBiQTop} \leftarrow \text{RegCSQuant}^{op}. \]

**Definition 4.4.** An \((X, \tau_1, \tau_2) \in \mathcal{L-BiQTop}\) is said to be pairwise compact if \( E_\alpha(X, \tau_1, \tau_2) = (X, \tau_1 \lor \tau_2) \) is compact.

**Theorem 4.1.** Let \( L \in \mathcal{SQuant} \), \( Q \in \mathcal{CSQuant} \) and \((X, \tau_1, \tau_2) \in \mathcal{L-BiQTop}\). Then

(1) \((X, \tau_1, \tau_2)\) is pairwise compact if and only if \( \mathcal{O}_L(X, \tau_1, \tau_2) = (\tau_1 \lor \tau_2, \tau_1, \tau_2) \) is compact;

(2) if \( Q \) is spatial, then \( Q \) is compact if and only if \( LPT(Q_0, Q_1, Q_2) \) is pairwise compact.

Proof. As to (1), if \((X, \tau_1, \tau_2)\) is a compact object of \( \text{L-BiQTop} \), that is, for all \( S \subseteq (\tau_1 \lor \tau_2) \), \( \forall S = \exists_\alpha \), \( \exists F(\text{finite}) \subseteq S \), \( \forall F = \exists_\alpha \) if and only if \((\tau_1 \lor \tau_2)\) is a compact semi-quantale if and only if \((\tau_1 \lor \tau_2, \tau_1, \tau_2)\) is a compact coupled semi-quantale.

As to (2), let \( Q = (Q_0, Q_1, Q_2) \) be spatial, then the mapping

\[ \zeta^\alpha_Q : Q \to \mathcal{O}_L(LPT(Q_0, Q_1, Q_2)) \]

is a coupled semi-quantale isomorphism, that is, \( Q \approx \Phi^\alpha_L(Q) \).

Compactness of \((Q_0, Q_1, Q_2) \leftrightarrow Q_0 \) is compact

\[ \iff \quad \text{LPT}(Q_0) = (\text{LPT}(Q_0), \Phi^\alpha_L(Q_0)) \text{ is compact} \]

\[ \iff \quad (\text{LPT}(Q_0), \Phi^\alpha_L(Q_1) \lor \Phi^\alpha_L(Q_2)) \text{ is compact.} \]

\[ \iff \quad \text{LPT}(Q) = (\text{LPT}(Q_0), \Phi^\alpha_L(Q_1), \Phi^\alpha_L(Q_2)) \]

is pairwise compact and this completes the proof. \( \square \)
5. Conclusion

The concept of coupled semi-quantales is introduced as a pointfree analogues of lattice-valued bitopological (or biquasi-topological spaces). An adjunction between the category of coupled semi-quantales and the category of lattice-valued biquasi-topological spaces is established. Through such adjunction topological and the lattice-theoretic concepts of regularity and compactness are defined and studied for both lattice-valued biquasi-topological spaces and coupled semi-quantales, respectively.

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