Kragujevac Journal of Mathematics
Volume 42(4) (2018), Pages 569–577.

# SOME INEQUALITIES FOR THE NUMERICAL RADIUS AND RHOMBIC NUMERICAL RADIUS

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Abstract. In this paper, the definition Rhombic numerical radius is introduced and we present several numerical radius inequalities. Some applications of these inequalities are considered as well. Particular, it is shown that, if $A \in \mathcal{B}(\mathcal{H})$ with the Cartesian decomposition $A = C + iD$ and $r \geq 1$, then

$$\omega^r(A) \leq \frac{\sqrt{2}}{2}\left\| |C+D|^{2r} + |C-D|^{2r} \right\|^{\frac{1}{2}}.$$

## 1. Introduction and Preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. The numerical radius of $A \in \mathcal{B}(\mathcal{H})$ is given by

$$\omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle| .$$

It is well-known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$ which is equivalent to the usual operator norm $\|A\| = \sup_{\|x\|=1} \|Ax\|$. In fact for $A \in \mathcal{B}(\mathcal{H})$ we have

$$\frac{1}{2}\|A\| \leq \omega(A)) \leq \|A\|.$$

Several numerical radius inequalities that provide alternative lower and upper bounds for $\omega(A)$ have received much attention from many authors. We refer the readers to [2] for the history and significance, and [3] for recent developments this area. Kittaneh

*Key words and phrases.* Rhombic numerical radius, numerical radius, usual operator norm.
2010 *Mathematics Subject Classification.* Primary: 47A12; secondary 47A30, 47A63.
*Received*: March 28, 2017.
*Accepted*: July 11, 2017.

[4] proved that for $A \in \mathcal{B}(\mathcal{H})$

$$\omega(A) \leq \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{\frac{1}{2}}\right).$$

So it is clear that if $A^2 = 0$, then

$$\omega(A) = \frac{1}{2}\|A\|.$$

Popescu in [11] define the Euclidean numerical radius. Note that in [11], the author has introduced the concept for an $n$-tuple of operators and pointed out its main properties. In the following Dragomir [1] considered the Euclidean operator radius of a pair $(C, D)$ of bounded linear operators defined on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ as follows:

$$\omega_e(C, D) = \sup_{\|x\|=1}\left(|\langle Cx, x\rangle|^2 + |\langle Dx, x\rangle|^2\right)^{\frac{1}{2}}.$$

It is worth to mention here that $\omega_e : \mathcal{B}^2(\mathcal{H}) \to [0, \infty)$ is a norm (see [10]) and the following inequality holds

$$\frac{\sqrt{2}}{4}\|C^*C + D^*D\|^{\frac{1}{2}} \leq \omega_e(C, D) \leq \|C^*C + D^*D\|^{\frac{1}{2}}, \tag{1.1}$$

where the constants $\frac{\sqrt{2}}{4}$ and 1 are best possible in (1.1). We observe that, if $C$ and $D$ are self-adjoint operators, then (1.1) becomes

$$\frac{\sqrt{2}}{4}\left\|C^2 + D^2\right\|^{\frac{1}{2}} \leq \omega_e(C, D) \leq \left\|C^2 + D^2\right\|^{\frac{1}{2}}.$$

We observe also that if $A \in \mathcal{B}(\mathcal{H})$ and $A = B + iC$ is the Cartesian decomposition of $A$, then

$$\omega_e^2(B, C) = \omega^2(A).$$

The main aim of the present paper is to introduce the notion of Rhombic numerical radius. Correspondingly, we establish some of the basic properties of the Rhombic numerical radius.

We obtain an upper bounds for the numerical radius of a Cartesian decomposition. More precisely, we prove that

$$\omega^r(A) \leq \frac{\sqrt{2}}{2}\left\||C + D|^{2r} + |C - D|^{2r}\right\|^{\frac{1}{2}},$$

where $A = C + iD$, be the Cartesian decomposition $A$ and $r \geq 1$. Besides, our result gives a sharper estimation for numerical radius than the corresponding result obtained in [6].

## 2. Main Results

Let $C, D$ be two bounded linear operators on $\mathcal{H}$, the Rhombic numerical radius is defined by

$$\omega_R(C,D) = \sup_{\|x\|=1} \left( |\langle Cx, x\rangle| + |\langle Dx, x\rangle| \right).$$

We can also consider the following norm on $\mathcal{B}^2(\mathcal{H}) := \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$, by

$$\|(C,D)\|_R = \sup_{\|x\|=\|y\|=1} \left( |\langle Cx, y\rangle| + |\langle Dx, y\rangle| \right).$$

We remark that $\|(C,D)\|_R = \|(C^*, D^*)\|_R$. It is not hard to see that $\omega_R(\cdot,\cdot)$ is a norm on $\mathcal{B}^2(\mathcal{H})$.

The next results represent some of the basic properties and sharp lower bound for the Rhombic numerical radius may be stated.

**Theorem 2.1.** *The Rhombic numerical radius $\omega_R : \mathcal{B}^2(\mathcal{H}) \longrightarrow [0,\infty)$ for two operators satisfies the following properties:*

(i) $\omega_R(C,D) = 0$ *if and only if* $C = D = 0$*;*
(ii) $\omega_R(\lambda C, \lambda D) = |\lambda| \omega_R(C,D)$*;*
(iii) $\omega_R(C+E, D+F) \leq \omega_R(C,D) + \omega_R(E,F)$*;*
(iv) $\omega_R(U^*CU, U^*DU) = \omega_R(C,D)$ *for any unitary operator* $U \in \mathcal{B}(\mathcal{H})$*;*
(v) $\omega_R(X^*CX, X^*DX) \leq \|X\|^2 \omega_R(C,D)$ *for any operator* $X \in \mathcal{B}(\mathcal{H})$*:*
(vi) *If* $A \in \mathcal{B}(\mathcal{H})$ *and* $A = C + iD$ *is the Cartesian decomposition of* $A$*, then* $\omega(A) \leq \omega_R(C,D)$*;*
(vii) $\omega_R(C,C) = 2\omega(C)$*;*
(viii) $\frac{1}{2}\|(C,D)\|_R \leq \omega_R(C,D) \leq \|(C,D)\|_R$.

*Proof.* The first seven properties can be easily deduced using the definition of $\omega_R$. Now, since

$$\{|\langle Cx, x\rangle| + |\langle Dx, x\rangle|,\ x \in \mathcal{H}\} \subseteq \{|\langle Cx, y\rangle| + |\langle Dx, y\rangle|,\ x, y \in \mathcal{H}\},$$

by taking the supremum when $x, y \in \mathcal{H},\ \|x\| = \|y\| = 1$, we have

$$\omega_R(C,D) \leq \|(C,D)\|_R.$$

To prove the other inequality, we use in the following polarization principle, if $T \in \mathcal{B}(\mathcal{H})$, then

$$\begin{aligned} 4\langle Tx, y\rangle &= \langle T(x+y), (x+y)\rangle - \langle T(x-y), (x-y)\rangle \\ &\quad + i\langle T(x+iy), (x+iy)\rangle - i\langle T(x-iy), (x-iy)\rangle, \end{aligned}$$

for any $x, y \in \mathcal{H}$. And for any $x \in \mathcal{H}$, we have

$$|\langle Cx, x\rangle| + |\langle Dx, x\rangle| \leq \omega_R(C,D)\|x\|^2,$$

Hence,

$$\begin{aligned}4\left(|\langle Cx,y\rangle|+|\langle Dx,y\rangle|\right) &\le \omega_R\left(C,D\right)\left(\|x+y\|^2+\|x-y\|^2+\|x+iy\|^2+\|x-iy\|^2\right)\\ &= 4\omega_R\left(C,D\right)\left(\|x\|^2+\|y\|^2\right)\quad \text{(by the Parallelogram Law).}\end{aligned}$$

Choosing $\|x\|=\|y\|=1$, we have

$$4\left(|\langle Cx,y\rangle|+|\langle Dx,y\rangle|\right)\le 8\ \omega_R(C,D),$$

which implies

$$\|(C,D)\|_R\le 2\omega_R(C,D).$$

Therefore, we deduce (viii). □

**Theorem 2.2.** *Let $C, D : \mathcal{H}\to\mathcal{H}$ be two bounded linear operators on the Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle)$. Then*

$$\omega(C\oplus D)\le\omega_R(C,D)\le 2\ \omega(C\oplus D).$$

*Proof.* Since

$$|\langle (C+D)x,x\rangle|\le|\langle Cx,x\rangle|+|\langle Dx,x\rangle|,$$

we have

$$\begin{aligned}|\langle Cx,x\rangle| &\le |\langle Cx,x\rangle|+|\langle Dx,x\rangle|,\\ |\langle Dx,x\rangle| &\le |\langle Dx,x\rangle|+|\langle Dx,x\rangle|.\end{aligned}$$

Taking now the supremum over all $x\in\mathcal{H}$ with $\|x\|=1$, we obtain the first inequality, therefore,

$$\omega(C\oplus D)\le\omega_R(C,D).$$

To prove the second inequality we have

$$\begin{aligned}\omega_R\left(C,D\right) &= \sup_{\|x\|=1}\ \left(|\langle Cx,x\rangle|+|\langle Dx,x\rangle|\right)\\ &\le \sup_{\|x\|=1}\ |\langle Cx,x\rangle|+\sup_{\|x\|=1}\ |\langle Dx,x\rangle|\\ &= \omega\left(C\right)+\omega\left(D\right)\\ &\le 2\max\left(\omega\left(C\right),\omega\left(D\right)\right)\\ &= 2\omega\left(C\oplus D\right).\end{aligned}$$

Consequently, we obtain the second inequality. □

In particular, for any two self-adjoint bounded linear operators on the Hilbert space $\mathcal{H}$, we have

$$\|(C\oplus D)\|\le\omega_R(C,D)\le 2\|(C\oplus D)\|.$$

To prove our generalized Rhombic numerical radius inequality, we need several well known lemmas. The first lemma is known as the generalized mixed Schwartz inequality, which has been proved in [7].

**Lemma 2.1.** *Let $C \in \mathcal{B}(\mathcal{H})$, then*

$$|\langle Cx, y\rangle|^2 \leq \left\langle |C|^{2\alpha} x, x\right\rangle \left\langle |C^*|^{2(1-\alpha)} y, y\right\rangle,$$

*for all $x, y \in \mathcal{H}$ and for all $\alpha$ with $0 \leq \alpha \leq 1$.*

The second lemma is a simple consequence of the classical Jensen inequality concerning the convexity or concavity of certain power function. This is a special case of Schlomilchs inequality for weighted means of non negative real numbers. For generalization of this lemma, we refer to [5].

**Lemma 2.2.** *For $a, b \geq 0$, $0 < \alpha < 1$, and $r \neq 0$, let $M_r(a, b, \alpha) = (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}$ and let $M_0(a, b, \alpha) = a^\alpha b^{1-\alpha}$. Then $M_r(a, b, \alpha) \leq M_s(a, b, \alpha)$, for $r \leq s$.*

The third lemma is an immediate consequence of the spectral theorem for self-adjoint positive operators and Jensen inequality. For generalization of this lemma, we refer to [7].

**Lemma 2.3.** *Let $A \in \mathcal{B}(\mathcal{H})$ be positive, and let $x \in \mathcal{H}$ be any unit vector. Then*
(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ *for $r \geq 1$;*
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for $0 < r \leq 1$.*

The forth lemma is an immediate consequence of the spectral theorem for self-adjoint operators. For generalization of this lemma, we refer to [7].

**Lemma 2.4.** *Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $x \in \mathcal{H}$ be any vector. Then*

$$|\langle Ax, x\rangle| \leq \langle |A| x, x\rangle.$$

Now we are ready to state the main result of this section. We use some similar strategies as in [3] to prove the next result.

**Theorem 2.3.** *Let $C, D \in \mathcal{B}(\mathcal{H})$, $0 < \alpha < 1$ and $r \geq 1$. Then*

$$\omega_R^r (C, D) \leq 2^{r-2} \left\| |C|^{2\alpha r} + |C^*|^{2(1-\alpha)r} + |D|^{2\alpha r} + |D^*|^{2(1-\alpha)r} \right\|.$$

*Proof.* For any unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned} |\langle Cx, x\rangle| &\leq \left\langle |C|^{2\alpha} x, x\right\rangle^{\frac{1}{2}} \left\langle |C^*|^{2(1-\alpha)} x, x\right\rangle^{\frac{1}{2}} \quad \text{(by Lemma 2.1)} \\ &\leq \left( \frac{\left\langle |C|^{2\alpha} x, x\right\rangle^r + \left\langle |C^*|^{2(1-\alpha)} x, x\right\rangle^r}{2} \right)^{\frac{1}{r}} \quad \text{(by Lemma 2.2)} \\ &\leq \left( \frac{\left\langle |C|^{2\alpha r} x, x\right\rangle + \left\langle |C^*|^{2(1-\alpha)r} x, x\right\rangle}{2} \right)^{\frac{1}{r}} \quad \text{(by Lemma 2.3 (i))}. \end{aligned}$$

Thus,

$$|\langle Cx, x\rangle|^r \leq \frac{1}{2} \left\langle |C|^{2\alpha r} + |C^*|^{2(1-\alpha)r} x, x\right\rangle,$$

by convexity of the function$f(t) = t^r on$ $[0, \infty)$, we have

$$(|\langle Cx, x\rangle| + |\langle Dx, x\rangle|)^r \leq 2^{r-1} \left(|\langle Cx, x\rangle|^r + |\langle Dx, x\rangle|^r\right)$$
$$\leq 2^{r-2} \left\langle \left(|C|^{2\alpha r} + |C^*|^{2(1-\alpha)r} + |D|^{2\alpha r} + |D^*|^{2(1-\alpha)r}\right) x, x \right\rangle .$$

Now taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$\omega_R^r (C, D) \leq 2^{r-2} \left\| |C|^{2\alpha r} + |C^*|^{2(1-\alpha)r} + |D|^{2\alpha r} + |D^*|^{2(1-\alpha)r} \right\| ,$$

as required. □

Using this observation we give the following corollary.

**Corollary 2.1.** *If $C \in \mathcal{B}(\mathcal{H})$, then*

$$\omega^r(C) \leq \frac{1}{2} \| |C|^{2\alpha r} + |C^*|^{2(1-\alpha)r} \|.$$

*Proof.* If in Theorem 2.3, we choose $C = D$, then by Theorem 2.1 (vii) we get

$$\omega_R^r(C, C) = 2^r \omega^r(C),$$

which implies the desired result. □

*In particular, if we choose $r = 2$, $\alpha = \frac{1}{2}$, we have*

$$\omega^2(C) \leq \frac{1}{2} \|C^*C + CC^*\|. \tag{2.1}$$

We remark that, in [9], the authors proved the inequality (2.1).

**Corollary 2.2.** *Let $A = C + iD$ be the Cartesian decomposition of $A$ and $r \geq 1$. Then*

$$\omega^r(A) \leq 2^{r-2} \| |C|^{2\alpha r} + |C|^{2(1-\alpha)r} + |D|^{2\alpha r} + |D|^{2(1-\alpha)r} \|.$$

*Proof.* By Theorem 2.1 (vi), reached the desired result. □

**Corollary 2.3.** *For any bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ and $\beta, \gamma \in \mathbb{C}$,*

$$(|\gamma| + |\beta|)^r \omega^r(A) \leq 2^{r-2} \|(|\gamma|^{2\alpha r} + |\beta|^{2\alpha r})|A|^{2\alpha r} + (|\gamma|^{2(1-\alpha)r} + |\beta|^{2(1-\alpha)r})|A^*|^{2(1-\alpha)r}\|.$$

*Proof.* If in Theorem 2.3, we choose $C = \gamma A$ and $D = \beta A^*$, then we get

$$\omega_R^r(C, D) = (|\gamma| + |\beta|)^r \omega^r(A),$$

which implies the desired result. □

*Remark* 2.1. If in Corollary 2.3, we choose $\gamma = 1$, $\beta = i$ and $\alpha = \frac{1}{2}$, $r = 1$, then we get

$$\omega(A) \leq \frac{1}{2} \| |A| + |A^*| \|. \tag{2.2}$$

Notice that, in [8], the authors proved the inequality (2.2).

In the following theorem we compare the Rhombic numerical radius and Euclidean numerical radius.

**Theorem 2.4.** *Let $C, D \in \mathcal{B}(\mathcal{H})$, then*

$$\omega(C^2 + D^2) \leq \omega_R^2(C, D) \leq 2\ \omega_e^2(C, D).$$

*Proof.* For the first inequality, we have

$$\begin{aligned} 2\omega_R^2\left(C, D\right) &\geq \omega^2\left(C + D\right) + \omega^2\left(C - D\right) \\ &\geq \omega(C + D)^2 + \omega(C - D)^2 \\ &\geq \omega\left((C + D)^2 + (C - D)^2\right) \\ &= 2\omega\left(C^2 + D^2\right). \end{aligned}$$

Thus,

$$\omega_R^2(C, D) \geq \omega(C^2 + D^2).$$

Notice that for any unit vectors $x \in \mathcal{H}$, by the convexity function $f(t) = t^r, r \geq 1$ we have

$$\left(|\langle Cx, x\rangle| + |\langle Dx, x\rangle|\right)^2 \leq 2\left(|\langle Cx, x\rangle|^2 + |\langle Dx, x\rangle|^2\right).$$

Thus,

$$\omega_R^2(C, D) \leq 2\ \omega_e^2(C, D). \qquad \square$$

**Corollary 2.4.** *For any self-adjoint bounded linear operators $C, D \in \mathcal{B}(\mathcal{H})$, we have*

$$\|C^2 + D^2\| \leq \omega_R^2(C, D) \leq 2\ \omega_e^2(C, D).$$

To prove Theorem 2.5, we need the following lemma.

**Lemma 2.5.** *Let $a_i$ be a positive real number for $i = 1, 2, \ldots, n$. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1} \sum_{i=1}^{n} {a_i}^r,$$

*for all $r \geq 1$.*

This lemma concerned with positive real number, and it is a consequence of the convexity of the function $f(t) = t^r$, $r \geq 1$.

The main aim of the expression following Theorem is to obtain an upper bound for numerical radius by means the Cartesian decomposition of operators. We use some similar strategies as in [3] to prove the next result.

**Theorem 2.5.** *Let $A_j \in \mathcal{B}(\mathcal{H})$ have the Cartesian decomposition $A_j = C_j + iD_j$, for $j = 1, \ldots, n$ and $r \geq 1$. Then*

$$\omega^r\left(\sum_{j=1}^{n} A_j\right) \leq n^{r-1} 2^{-\frac{r}{2}} \sum_{j=1}^{n} \omega_R^{\frac{r}{2}}\left(|C_j + D_j|^2, |C_j - D_j|^2\right).$$

*Proof.* For every unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}\left|\sum_{j=1}^{n}\langle A_j x, x\rangle\right|^r &\leq \left(\sum_{j=1}^{n}\left(\langle C_j x, x\rangle^2 + \langle D_j x, x\rangle^2\right)^{\frac{1}{2}}\right)^r \\ &\leq \left(\sum_{j=1}^{n}\frac{1}{2}\left(\langle (C_j + D_j)\, x, x\rangle^2 + \langle (C_j - D_j)\, x, x\rangle^2\right)^{\frac{1}{2}}\right)^r \\ &\leq n^{r-1}2^{-\frac{r}{2}}\sum_{j=1}^{n}\left(\langle (C_j + D_j)\, x, x\rangle^2 + \langle (C_j - D_j)\, x, x\rangle^2\right)^{\frac{r}{2}} \\ &\leq n^{r-1}2^{-\frac{r}{2}}\sum_{j=1}^{n}\left(\left\langle |C_j + D_j|^2 x, x\right\rangle + \left\langle |C_j - D_j|^2 x, x\right\rangle\right)^{\frac{r}{2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\omega^r\left(\sum_{j=1}^{n} A_j\right) &= \sup\left\{\left|\sum_{j=1}^{n}\langle A_j x, x\rangle\right|^r : x \in \mathcal{H}, \|x\| = 1\right\} \\ &\leq n^{r-1}2^{-\frac{r}{2}}\sum_{j=1}^{n}\omega_R^{\frac{r}{2}}\left(|C_j + D_j|^2, |C_j - D_j|^2\right).\end{aligned}$$

□

**Corollary 2.5.** *Let $A_j \in \mathcal{B}(\mathcal{H})$, have the Cartesian decomposition $A_j = C_j + iD_j$ for $j = 1, \ldots, n$, $0 < \alpha < 1$ and $r \geq 1$. Then*

$$\begin{aligned}\omega^r\left(\sum_{j=1}^{n} A_j\right) \leq \frac{1}{2}n^{r-1}\sum_{j=1}^{n}\Big\| &|C_j + D_j|^{4\alpha r} + |C_j + D_j|^{4(1-\alpha)r} \\ &+ |C_j - D_j|^{4\alpha r} + |C_j - D_j|^{4(1-\alpha)r}\Big\|^{\frac{1}{2}}.\end{aligned}$$

*Proof.* If applied Theorem 2.3 for the Cartesian decomposition of $A_j$, we reached desired result. □

In particular for $n = 1$ and $\alpha = \frac{1}{2}$, we have the following result.

**Corollary 2.6.** *Let $A = C + iD$ be the Cartesian decomposition $A$ and $r \geq 1$. Then*

$$\omega^r(A) \leq \frac{\sqrt{2}}{2}\left\||C + D|^{2r} + |C - D|^{2r}\right\|^{\frac{1}{2}}. \tag{2.3}$$

In [6] the authors proved that

$$\omega^r(A) \leq 2^{\frac{r}{2}-1}\left\|(C + D)^{2r} + (C - D)^{2r}\right\|^{\frac{1}{2}}. \tag{2.4}$$

*Remark* 2.2. For $r > 1$ the inequality (2.3) is sharper than the (2.4).

**Acknowledgements.** The authors express their gratitude to the anonymous referees for their careful reading and detailed comments which have considerably improved the paper.

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