On the Computation of Foldings

H. Stachel
Professor emeritus
Institute of Discrete Mathematics
and Geometry
Vienna University of Technology
Austria

The process of determining the development (or net) of a polyhedron or of a developable surface is called unfolding and has a unique result, apart from the placement of different components in the plane. The reverse process called folding is much more complex. In the case of polyhedra it leads to a system of algebraic equations. A given development can correspond to several or even to infinitely many incongruent polyhedra. The same holds also for smooth surfaces. In the paper two examples of such foldings are presented.
In both cases the spatial realisations bound solids, for which mathematical models are required. In the first example, the cylinders with curved creases are given. In this case the involved curves can be exactly described. In the second example, even the ruling of the involved developable surface is unknown. Here, the obtained model is only an approximation.

Keywords: *folding, curved folding, developable surfaces, revolute surfaces of constant curvature.*

1. UNFOLDING AND FOLDING

In Descriptive Geometry there are standard procedures available for the construction of the *development* (*net* or *unfolding*) of polyhedra or piecewise linear surfaces, i.e., polyhedral structures. The same holds for developable smooth surfaces. These are surfaces with vanishing Gaussian curvature, composed from cylinders, cones or toruses; the latter are surfaces swept out by the tangent lines of spatial curves.

Of course, the development can also be computed by methods of Analytic Geometry or Differential Geometry. However, the planar counterparts of the spatial parametrized bounding curves need not have a parametrization in terms of elementary functions. An oblique cylinder with a circular basis serves as an example [11].

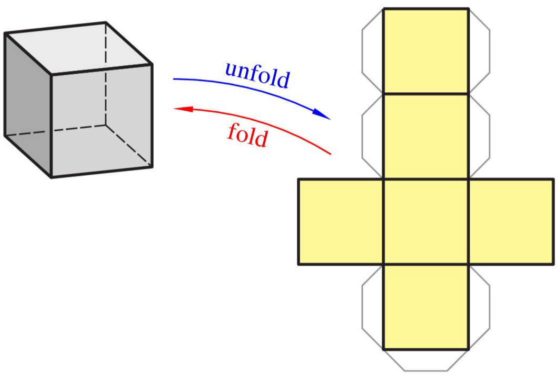

Figure 1. Unfolding and folding

The result of the procedure of unfolding, i.e., the development Φ_0 of a given polyhedral or smooth surface Φ is unique, apart from the placement of the components in the plane. The unfolding induces an *isometry* $\Phi \to \Phi_0$: each curve c on Φ has the same length as its planar counterpart $c_0 \subset \Phi_0$. Hence, the development shows in the plane the *interior metric* of the original spatial structure.

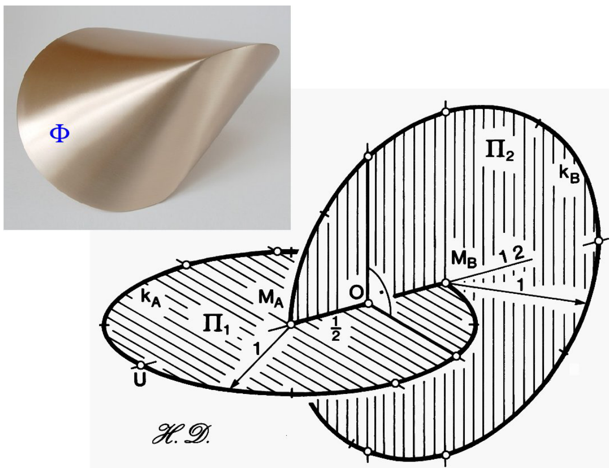

Figure 2. The *Oloid* is the convex hull of two circles in orthogonal planes such that each circle passes through the center of the other

As an example, in Fig. 2 an *Oloid* is depicted. This is the convex hull of a particular pair of congruent circles. The Oloid's bounding surface Φ is of course developable. For its development (see Fig. 3), there is even an explicit arc-length parametrization of the bounding curves available [5, Theorems 2 and 3]. Furthermore, by virtue of the isometry, not only the lengths of curves are preserved during the unfolding, but also the surface area of Φ, which equals that of the unit sphere [5, Theorem 5], provided that the circles defining the Oloid are unit circles.

Remark: The book [4] presents a wide variety of mathema-tical problems around the unfolding and folding of polyhedra.

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Correspondence to: Hellmuth Stachel
Institute of Discrete Mathematics and Geometry
Vienna University of Technology, Wien, Austria
E-mail: stachel@dmg.tuwien.ac.at
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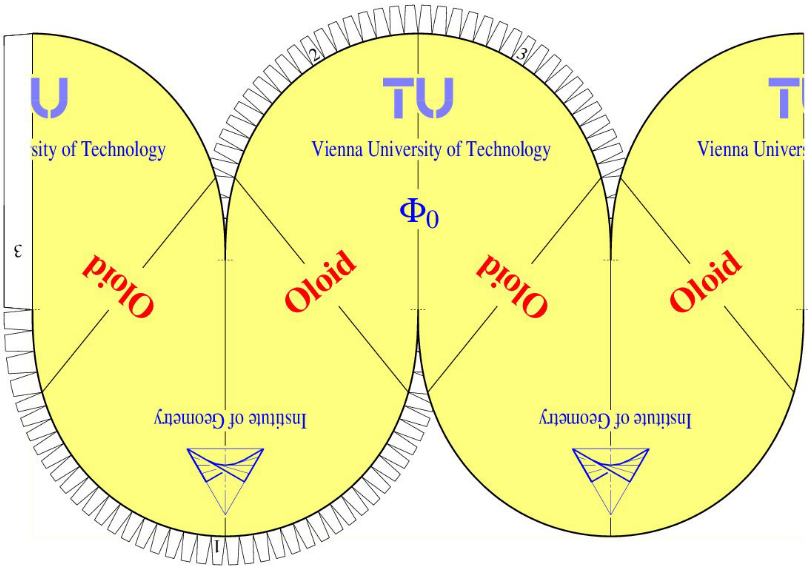

Figure 3. The development of the *Oloid's* bounding surface

The inverse problem, i.e., the determination of a folded structure from a given development is more complex. In the smooth case we obtain a continuum of bent poses, in general. This is easy to visualize by bending a sheet of paper. In the polyhedral case the computation leads to a system of algebraic equations, and also here the shape of the corresponding spatial object needs not be unique. Sometimes even infinitely many spatial objects with the same development are possible. Then the structure is called *flexible*. If the system of algebraic equations has an isolated solution with higher multiplicity, one speaks of an *infinitely flexible* realisation. Otherwise it is called *locally rigid* (for details see [16], [12] or [13] and the references there).

If a given development allows two realisations sufficiently close together, a physical model can flip from one into the other. Their seeming flexibility results from slight bendings of the faces or clearances at the hinges. A famous example in this respect is a polyhedron called *"Vierhorn"* (Fig. 4). It is locally rigid, but can flip between its spatial shape and two flat poses in the planes of symmetry. At the science exposition *"Phänomena"* 1984 in Zürich this polyhedron was exposed and falsely stated that it is flexible (note [18]).

If a polyhedron bounds a *convex* solid then the result of the folding is unique. We owe this result to the Russian mathematician Aleksandr Danilovich Alexandrov who stated in his famous *Uniqueness Theorem* (1941): For any convex intrinsic metric there is a unique convex polyhedron [1]. In this respect, an intrinsic metric is called convex, if for each vertex the sum of intrinsic angles for all adjacent surfaces is smaller than 360°. By the same token, A. I. Bobenko and I. Izmestiev created 2006 an algorithm for the construction of the convex polyhedron with given intrinsic metric [2].

It is of course possible that such a convex metric admits beside the convex realisation still other realisations. Take, e.g., a cube and replace one face by a right pyramid with the bounding square as basis and a sufficiently small height. Then the development remains the same, whether the apex of this pyramid lies in the exterior or interior of the original cube.

The smooth counterpart of Alexandrov's Uniqueness Theorem is the theorem stating the *rigidity of ovaloids*, which are defined as compact two-dimensional Riemannian spaces of positive Gaussian curvature (see, e.g., [3, 15]).

In the sequel we present two examples of smooth foldings. In one case the ruling is given; the involved surfaces are cylinders. In the other, much more complex case the ruling of the involved developable surface is unknown.

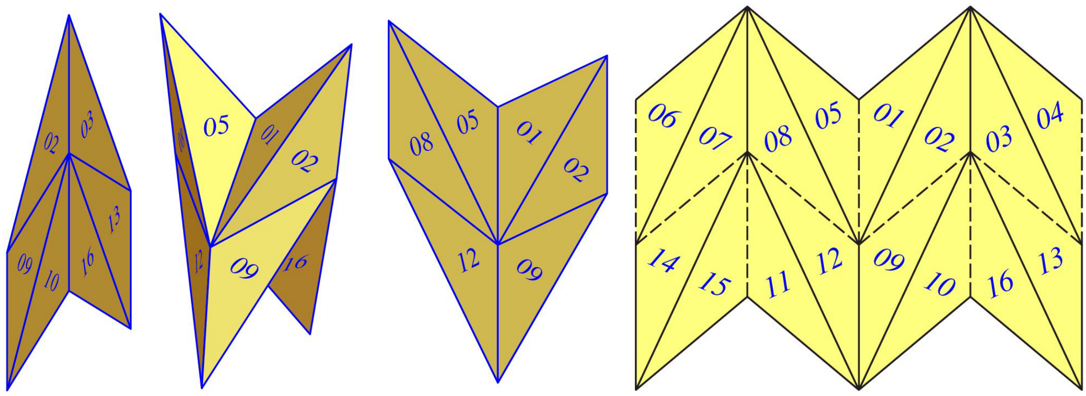

Figure 4. This polyhedron called "Vierhorn" flips between its spatial shape and two flat realizations. Dashes in the development below indicate valley folds.

2. FOLDING CYLINDERS WITH A COMMON CURVED EDGE

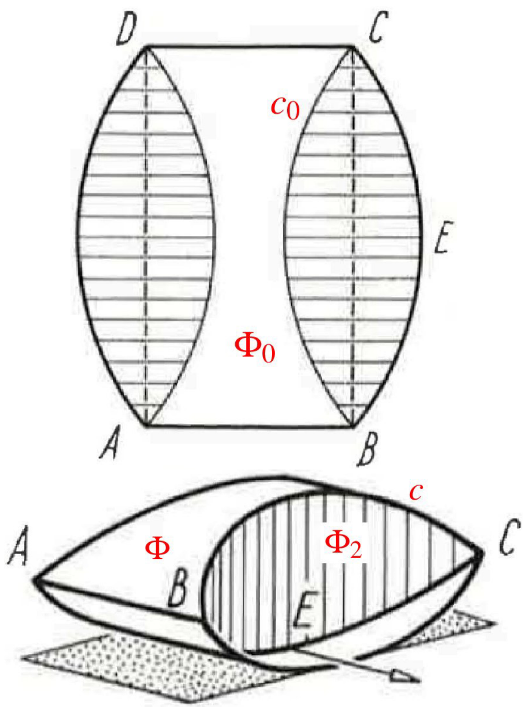

Figure 5. Wunderlich's original figure in [17]: development with crease c_0 (top) and spatial form (bottom)

A very common way of producing small boxes in shops or in fast-food restaurants is to push up appropriate planar cardbord forms with prepared creases. In the case of creases along circular arcs (see Fig. 5) W. Wunderlich proved in [17] that at the spatial form the creases between the cylinders are again planar. They belong to a family $\mathcal{F}$ of non-elementary curves which are well-known in Differential Geometry since C. F. Gauß: the curves are meridians of surfaces of revolution with constant Gaussian curvature. The family $\mathcal{F}$ includes circular arcs, since spheres have a constant curvature, too. Below, we prove a slight generalization of Wunderlich's result.

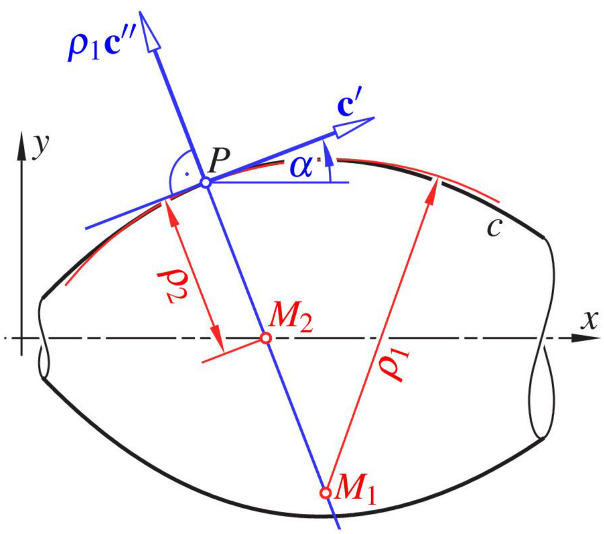

Figure 6. Given a surface of revolution, $\kappa_1 = 1/\rho_1$ and $\kappa_2 = 1/\rho_2$ are the principal curvatures at the point $P \in c$

To begin with, we determine a differential equation which characterizes the curves of $\mathcal{F}$: Let the meridian c in the xy-plane with the twice-differentiable arc-length parametrization

$$\mathbf{c}(s) = (x(s), y(s)) \text{ for } s_1 \le s \le s_2$$

rotate about the x-axis (Fig. 6). If primes indicate the differentiation with respect to (w.r.t., in brief) the arc-length s then $\mathbf{c}' = (x', y') = (\cos\alpha, \sin\alpha)$ is the unit tangent vector, and $\mathbf{c}'' = (x'', y'') = \kappa_1(y', -x')$ is the curvature vector.

At all surfaces of revolution the meridians and parallel circles are the principal curvature lines. Therefore, the signed principal curvatures at the point $P = \mathbf{c}(s)$ are

$$\kappa_1 = -\frac{y''}{\cos\alpha}, \ \kappa_2 = \frac{\cos\alpha}{y}.$$

The Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ is constant if and only if the arc-length parametrization of the meridian c satisfies the differential equations

$$y'' + K\,y = 0, \quad x' = \sqrt{1 - y'^2} \qquad (1)$$

with $K = \text{const.}$, provided that $\cos\alpha \ne 0$.

In the case $K = 0$ the meridians are lines; the corresponding surfaces of revolution are cones or cylinders. In the remaining cases $K \ne 0$ we obtain the general solutions

$$\begin{aligned} K > 0:\ & y = a\cos s\sqrt{K} + b\sin s\sqrt{K} \quad \text{or} \\ K < 0:\ & y = a\cosh s\sqrt{-K} + b\sinh s\sqrt{-K} \end{aligned} \qquad (2)$$

with constants $a, b \in \mathbb{R}$ and $x = \int\sqrt{1 - y'^2}\,ds$.

After specifying an appropriate initial point $s = 0$ the arc-length parametrization, there remain – up to similarities – six different cases. This classification dates back to C. F. Gauß (1827) and F. A. Minding (1839) (note [3, p. 169], [6, 277-286], [7], [9, p. 158], or [15, 141-148]).

In Fig. 7 three of the six types are depicted. Due to Scheffers [11], the curve c with $K = 1$, $0 < a < 1$ and $b = 0$ (type 1) shows up at the development of an elliptic cylinder when bounded by a circular section. At type 2 with $K = 1$ and $(a,b) = (1,0)$ the meridian c is a half-circle centered on the x-axis. The meridian c for $K = -1$ and $(a,b) = (1,1)$ of type 3 has the arc-length parametrization

$$x = \sqrt{1 - e^{-2s}} - \operatorname{ar}\cosh e^{s}, \ y = e^{-s}, \ s > 0.$$

This defines a *tractrix* c, for which the contacting segment PT at the point $P \in c$ with T on the x-axis satisfies $\overline{PM_1} \cdot \overline{PM_2} = \overline{PT}^2 = 1$.

Theorem. *Let $\mathcal{F}_0$ be the family of meridians of surfaces of revolution with constant Gaussian curvature $K \ne 0$. Suppose the curve $c_0 \in \mathcal{F}_0$ bounds together with the corresponding axis a_0 (= x-axis) the development Φ_0 of a cylindrical patch with generators orthogonal to a_0.*

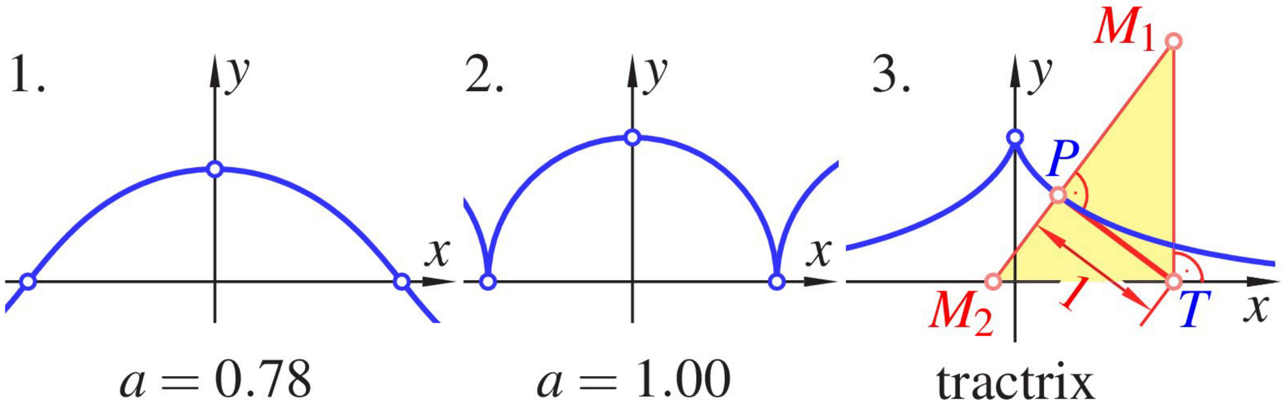

Figure 7. Curves of the family $\mathcal{F}_0$ of meridians of surfaces of revolution with constant Gaussian curvature $K \neq 0$

If at a cylindrically bent pose Φ of Φ_0 the corresponding boundary curve c is located in a plane ε then c is again a member of the family $\mathcal{F}_0$ and even with the same curvature K. The axis of c is the meet of ε and the plane of the orthogonal section through the bent counterpart a of the original axis a_0.

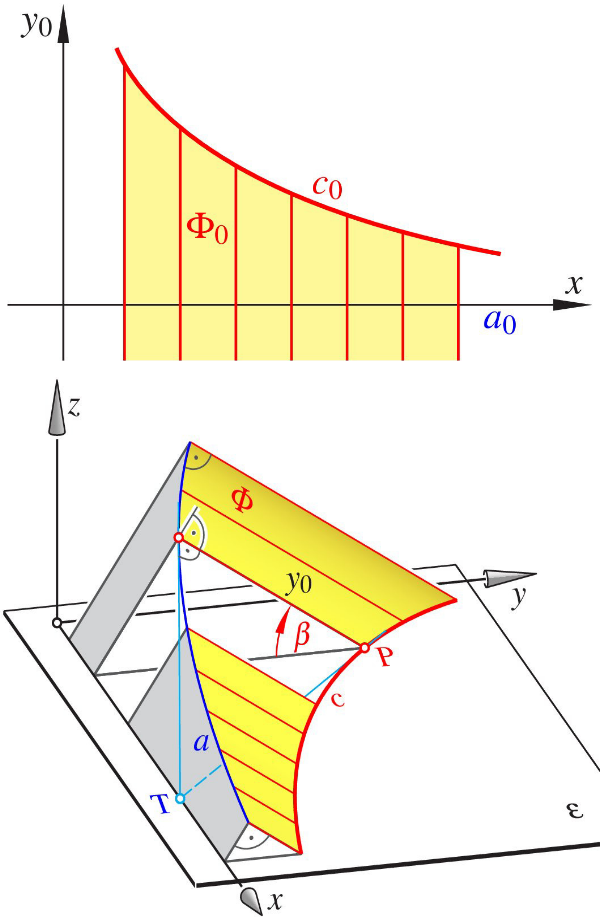

Figure 8. Development Φ_0 and bent pose Φ

Proof. The bending from the flat initial pose Φ_0 to the cylindrical shape Φ is an isometry. Therefore the arc-length s of c_0 serves also as arc-length of $c \subset \varepsilon$. If at the bent pose Φ the line of intersection between ε and the plane of the bent cross section a is used as x-axis then the ordinate y_0 of any point of c_0 and the y-coordinate of the corresponding point of c satisfy

$$y_0(s) = y(s)\cos\beta\,, \text{ with } \beta = \text{const.} \tag{3}$$

being the angle of inclination of the generators of Φ w.r.t. the plane ε (Fig. 8). We have $\beta < \frac{\pi}{2}$ since otherwise c_0 would be a line.

The ordinate $y_0^{(s)}$ of the given boundary curve c_0 satisfies (1). Consequently, the planar section c of Φ satisfies the same equation $y'' + K_y = 0$. This means in particular that the Gaussian curvature K of the corresponding surfaces of revolution is preserved.

If we plug (3) into the general solutions $y_0 = y_0^{(s)}$, as listed in (2), the coefficients a_0, b_0 are replaced with

$$a = \frac{a_0}{\cos\beta} \geq a_0 \text{ and } b = \frac{b_0}{\cos\beta} \geq b_0\,. \tag{4}$$

This proves the Theorem.

We can perform a *continuous* bending from Φ_0 to Φ by varying the angle β of inclination in an intervall $0 \leq \beta \leq \beta_1$. The condition $|y'| \leq 1$, by (1), implies an upper limit β_1 for β: The angle β cannot be greater than the angle in the initial flat pose between the generators and the boundary c_0.

Here are a few examples of the Theorem:

- In Fig. 7, type 2, the curve c is a circular arc. Hence, the cylinder Φ is elliptic and c_0 a curve of type 1. This confirms Scheffers' result in [11].
- If c_0 lies on a tractrix (type 3) then c is congruent to another portion of the same tractrix. This follows from (4) for $a=b$, but it can also be concluded from the fact that the distance $\overline{PT}$ along the tangent from the point $P \in c_0$ to the intersection T with the x-axis (see Fig. 8) remains unchanged during the bending from $\mathbf{\Phi_0}$ to Φ.
- The statement in the Theorem above includes also Wunderlich's particular case (see Fig. 5) with c_0 of type 2. However, we still need to prove the planarity of the boundary curve c at the box depicted in Fig. 5, bottom. This can be done as follows:

Proof. Physical models of the box demonstrate that its shape is uniquely defined by its development. We show that there is a solution where the four bounding surfaces are cylinders.

The spatial counterpart c of the circular arc c_0 in Fig. 5, top, is common for two cylinders. At each point P of c the geodesic curvature of c is the same for both cylinders since it is equal to the curvature of c_0 in the development at the corresponding point. Therefore at P the tangent planes to the two cylinders must be symmetric w.r.t. the osculating plane of c at P.

Also the generators of the two cylinders at P must be symmetric w.r.t. this osculating plane since in both tangent planes the generators include the same angle with the tangent to c, which is the meet of the two planes. The direction of the generators does not change when P traverses c. Hence, also the osculating plane as well as the binormal of c cannot change its direction. But then the torsion of c must be zero, and c is planar.

Remark: An analytic proof of a slightly generalized theorem can be found in [14, Theorem 3].

When we reflect the second cylinder through c in the plane of c, we obtain an extension of first cylinder beyond its original boundary c. (In rigid origami this is called reflection operation (see, e.g., [10, p. 187]). For each pair of generators, which meet on c, the plane of c is an exterior angle-bisecting plane.

3. FOLDING WITH UNKNOWN RULINGS

Figure 9 shows a development Φ_0 with a boundary c_0, which is a C^1-curve composed from two straight line segments and two semicircles of equal lengths. Let A_0 and C_0 be two opposite points of transition between semicircles and straight line segments for bisecting the boundary. Now the spatial form Φ is obtained by gluing together, from A_0 on, the semicircle of one part with the straight segment of the other, and vice versa (Fig. 10, top to bottom). The question is, how to obtain a mathematical model of the resulting body?

In contrast to Example 1, the crucial point is now that the ruling is unknown, and local conditions are not sufficient for modelling the bent shape. The constraint is of global nature: the boundary c_0 of Φ_0 must finally give a two-fold covered closed curve c.

In [8] a general and effective appoximating method is presented in order to compute the spatial form from given planar shapes. Our approach in this particular case is different. The inspection of a physical model (Fig. 10) reveals:

- The corresponding spatial body with the boundary Φ is convex and uniquely defined.
- The helix-like curve c is a proper edge of Φ; the resulting solid is the *convex hull* of c.
- The spatial body has an axis a of symmetry which connects the spatial position M of the center M_0 with the remaining transition point $B = D$ on c.
- The semicircular disks are bent to cones with apices A and C. Hence, the boundary Φ is a C^1-composition of two cones and a torse between.

We traverse c from A to C and subdivide it at the transition point $B = D$ into the two parts c_1 and c_2. Due to the observations at the physical model, we can state:

(a) The rotation about the axis a of symmetry through 180° maps Φ onto itself and interchanges c_1 and c_2. Hence, a is orthogonal to the tangent t_B at the transition point B. The apices A and C are symmetric w.r.t. a. The generator g_M of Φ through the central point M is cylindric and has a tangent plane orthogonal to a.

(b) Because of the straight segments of the development c_0, the developable surface on the left hand side of c_1 belongs to the *rectifying torse* of c_1. With respect to the right hand side, c_1 is a geodesic circle of Φ.

(c) Since at A_0 the semicircle is tangent to the adjacent straight segment, the surface Φ has cone singularities with the intrinsic curvature π at the points A and C. Therefore, at point A the surface can be approximated by a right cone with the apex angle 60°. The initial tangent t_A to c_1 is a generator of this cone. Consequently, the osculating plane of c_1 at A coincides with the cone's tangent plane along t_A, and the rectifying plane at A passes through the cone's axis.

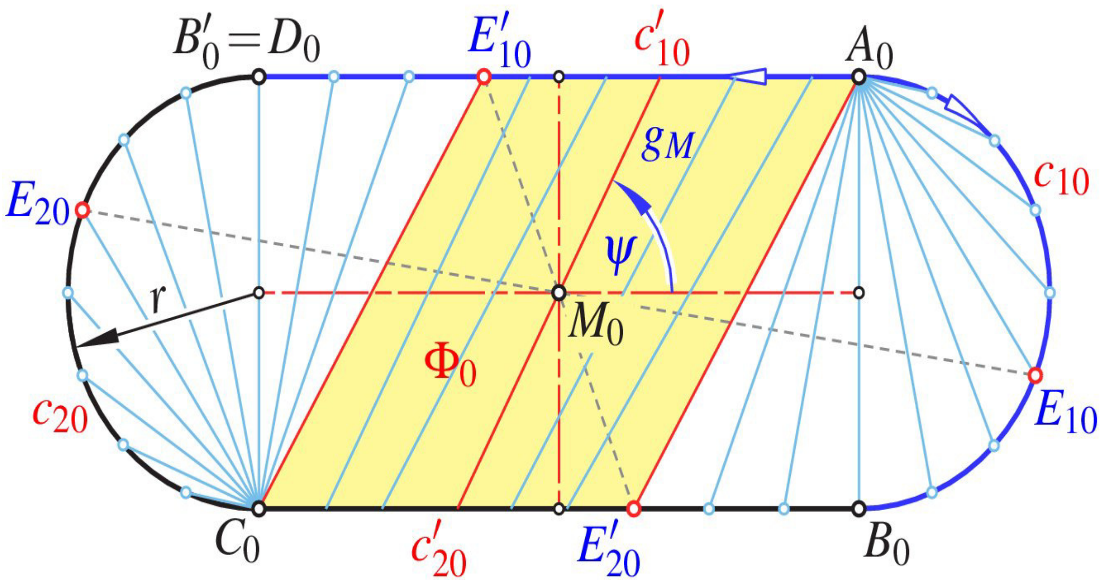

Figure 9. Development of the second example

(d) Φ belongs to the connecting torse of c_1 and c_2. If g is a generator of Φ whose counterpart in the development meets both straight segments of the boundary c_0 then g meets c_1 and c_2 at points with parallel and equally oriented tangent vectors.

(e) At the point $E_2 \in c_2$ of transition between the cone with apex A and the continuing torse (see Fig. 9) the tangent to c_2 must be parallel to the initial tangent t_A to c_1 at A. The symmetric point $E_1 \in c_1$ has a tangent parallel to the final tangent t_c of c_2. By virtue of item (d), the *tangent indicatrices* of the subarcs $AE_1$of c_1 and E_2C of c_2 must coincide.

Figure 10. The transition from the development to the spatial form (photos: G. Glaeser)

It turned out that a good approximation arises when we specify the conciding subarcs of the tangent indicatrices as a circular arc. Then the corresponding space curves $AE_1 \subset c_1$ and $E_2C \subset c_2$ are of *constant slope*, and their common rectifying torse is a cylinder.

Conversely, if the middle part of Φ is supposed as a cylinder the arcs AE_1 and E_2C, being geodesics, are curves of constant slope w.r.t. generators g of this cylinder. In the development (Fig. 9) all generators g_0, which meet the two straight segments of c_0, are of equal lengths. There is a *translation* along g which maps the arc $AE_1 \subset c_1$ onto the arc $E_2C \subset c_2$. The half-rotation about the axis a of symmetry maps E_2C back to E_1A. Hence, there must be a *half-rotation* with an axis a_1 parallel to a which exchanges A with E_1 while the arc AE_1 is mapped onto itself.

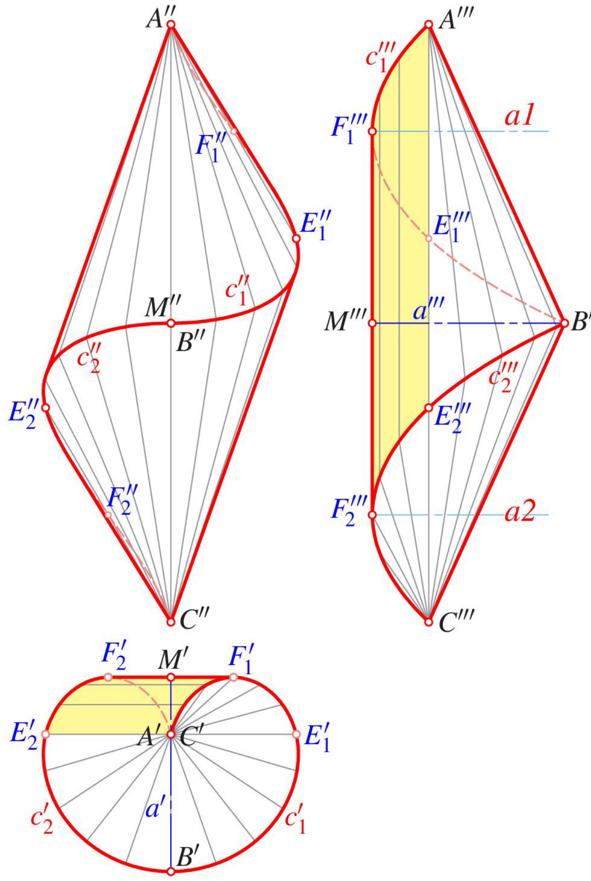

Figure 11. Principal views of the approximation; the cylindrical patch is shaded

This means, the slope curve AE_1 has an axis a_1 of symmetry which meets c_1 at a point F_1. The axis a_1 must be orthogonal to the tangent plane of the cylinder at F_1 in order to guarantee that the complete arc AE_1 is a smooth slope curve. Since E_1 and C are the images of A under reflections in parallel axes a_1 and a, respectively, the points A, C, E_1, and E_2 lie in a plane orthogonal to a (note the side view in Fig. 11).

The approximation depicted in Figs. 11 and 12 yields the following numerical results: The slope angle of c w.r.t. the cylinder is approximately $\psi = 54.53°$. The 'width' $\overline{MB}$ of the solid, in terms of the semicircles' radius r, is $1.18r$, the 'height' $\overline{AC} = 3.635r$, and the angle $ABC = 130.67°$.

These data correspond quite well to the physical model. However, in [14] it is noted that there is still a tiny contradiction inherent in the model with the cylindrical patch. So, at the exact model, the torsal part between the two cones of Φ must deviate slightly from a cylinder.

We can fold the sheet shown in Fig. 9 in two ways, since we can choose the depicted side either in the interior of the solid or in the exterior. At the first choice the curve c has negative torsion (see photos in Fig. 10). Otherwise its torsion is positive (see Fig. 12).

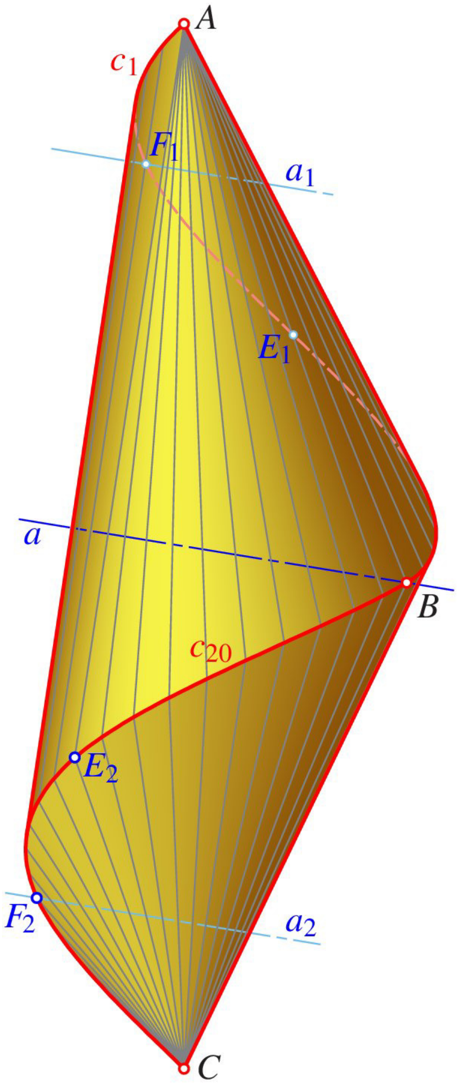

Figure 12. Approximation with a cylindrical patch

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О ПРОРАЧУНУ САВИЈАЊА КОД ПОЛИЕДАРСКИХ СТРУКТУРА

Х. Штахел

Поступак одређивања развијања полиедра у мрежу или развојне површине се назива развијање и даје јединствени резултат, поред постављања различитих компонената у раван. Обрнути поступак који се назива савијање много је сложенији. У случају полиедара оно доводи до система алгебарских једначина.

Дато развијање може да одговара неколицини или бесконачном броју разноликих полиедара. Исто важи и за глатке површине. У раду су дата два примера таквих савијања. У оба случаја просторна реализација је везана за тела за која су потребни

математички модели. У првом примеру су приказани цилиндри са кривим наборима. У таквом случају кривине се могу тачно описати. У другом примеру чак је и понашање обухваћене развојне површине непознато. Добијени модел овде представља само апроксимацију.