Specific Cost Ratio in a Port Modelling by M/E\(k\)/1 Queue

The notion of specific cost ratio involves different type of costs of a ship and a port modeled as a queuing system. Using the known general formula for the specific cost ratio of arbitrary port queuing system (denoted as \(R\)), here we derive the related expression for \(R\) of the M/E\(k\)/1 queue, where \(E_k\) \((k=2,3,...)\) is the Erlang-\(k\) probability distribution. This expression allows us to obtain a theoretical result which can be applied for determining the optimal values of shape parameter \(k\) of \(E_k\) under given constraints on other performances of the considered port queuing system. The related numerical and the graphical results are also presented. The obtained results would be a useful tool in future research in related subject areas.

**Keywords:** Port, M/E\(k\)/1 queue, Erlang-\(k\) distribution, Specific cost ratio, Traffic intensity.

1. **INTRODUCTION**

Recall that the total daily cost of a port queuing system with a certain number of berths (servers) is defined as a sum of total service cost per day, the total ship cost per day and the total marginal cost per day. Then the specific cost ratio, usually denoted as \(R\), is defined as a ratio of total daily cost of a port queuing system and the average daily cost of a ship in port. Notice that the optimal numbers of berths and the associated optimal intervals (the so-called the ranges of optimal server capacities; see, e.g., [1]) in the sense of minimization of \(R\) for certain single arrival and bulk arrival queuing systems were firstly studied by Noritake [2], Noritake and Kimura [3] and Radmilović et al. [1]. Furthermore, the total cost for different port systems is extensively studied by many authors in the last ten years ([4]-[14]). In particular, this concerns the problem of minimization of the specific cost ratio which is rarely investigated in the earlier literature on the subject. In addition, some queues are used to determine the utilization rate of the process [15], analytical calculation of throughput of the system [16] and to describe the loading processes at seaside link of seaport automobile terminal [17].

The remainder of the paper is organized as follows. In Section 2, using the general formula for the specific cost ratio of any port queuing system, here also denoted by \(R\), we derive the expression for \(R\) related to the M/E\(k\)/1 queue with one berth (i.e., with one server) and the infinite capacity, where \(E_k\) \((k=2,3,...)\) is the Erlang-\(k\) probability distribution, which can be used to model service times with a low coefficient of variation (less than one), but it can also arise naturally (see, e.g., [18]). For instance, if a job has to pass, stage by stage, through a series of \(k\) independent production stages, each stage takes an exponentially distributed time. Recall that the study of the M/E\(k\)/1 queue is similar to those of the well known M/G/1 queueing systems. For stability, it is assumed that the utilization factor \(\rho = \lambda/\mu\) is less than one, and it is equal to the related traffic intensity \(\theta\). On the other hand, M/E\(k\)/1 queueing model is just the special case of the M/G/1 queueing model. Then by the well known Pollaczek–Khintchine formula (see, [20]) - Subsection

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Correspondence to: Dr. Branislav Dragović, Professor Maritime Faculty of the University of Montenegro, Dobrota 36, 85330 Kotor, Montenegro

E-mail: branod@ac.me
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9.2, p. 348; cf. [12]), the average number of ships (customers) waiting in the queue, \( L_q \), is equal to

\[
L_q = \frac{\lambda^2 \sigma^2 + \rho^2}{2(1 - \rho)}.
\]

whence taking \( \rho = \theta \) we obtain

\[
L_q = \frac{\lambda^2 \sigma^2 + \theta^2}{2(1 - \theta)}.
\]

Then substituting (1) and \( \lambda/\mu = \rho \) into (3), we immediately obtain the following formula for the average number of ships (customers) waiting in queue, \( L_q(k) \), concerning the port queueing model \( M/E_k/1 \):

\[
L_q(k) = \frac{k + 1}{2k} \cdot \frac{\lambda^2}{\mu(\mu - \lambda)}.
\]

Finally, replacing \( \mu = \lambda/\theta \) into (4), we get

\[
L_q(k) = \frac{k + 1}{2k} \cdot \frac{\theta^2}{1 - \theta}.
\]

Note that for \( k = 1 \) the expression (5) reduces to the well-known formula for the average number of customers waiting in the queue \( M/M/1 \) with the traffic intensity \( \theta = \lambda/\mu \) [12].

Recall that the total cost of a port queueing system for considered period consists of a cost related to the berths and a cost related to the ships ([2-4]). Then the intensity concerning the port queueing model \( M/E_k/1 \) is the average service rate (i.e., \( 1/\mu \) for single-server port queueing system modeled as the Erlang-\( k \) distribution \( E_k \)).

In other words, let \( r > 0 \) and let \( \theta \in (0,1) \) be any fixed real numbers and let \( R(k,\theta) \) be defined by (8). Then \( \{R(k,\theta)\}_{k=2}^{\infty} \) is a decreasing sequence. In other words, it holds that

\[
R(2,\theta) > R(3,\theta) > \cdots > R(k,\theta) > R(k+1,\theta) > \cdots
\]

for all \( r > 0 \) and \( \theta \in (0,1) \).

**Proof of Proposition.** Since

\[
k \rho - 5\theta + 2\theta^2 + 2k\theta^2 = k(4 - 5\theta + 2\theta^2) + \theta(3 - 2\theta),
\]

taking this into the expression (8), immediately gives

\[
R(k,\theta) = r + \theta(4 - 5\theta + 2\theta^2) + \theta(3 - 2\theta).
\]

Notice that the first two terms on the right hand side of the above equality does not depend on \( k \) and the third term of this side decreases as \( k \) increases (since \( \theta(3 - 2\theta)/2(1 - \theta)^2 > 0 \) for all \( \theta \in (0,1) \)). This yields the assertion of Proposition.

In terms of port queueing systems, the above proposition can be reformulated as follows.

**Theorem.** Let \( M/E_k/1, k = 2,3,\ldots \) be the sequence of single-server port queueing systems which have the same values of \( \theta \) (the traffic intensity) and \( r = c_b/c_s \) (the quotient of the average daily cost of a berth, \( c_b \), and the average daily cost of a ship in port, \( c_s \)). Then the value \( R(k,\theta) \) of specific cost ratio of the queue \( M/E_k/1 \) decrease as \( k \) increases.

In particular, the above theorem yields that among all the port queueing systems \( M/E_k/1, K = 2,3,\ldots \) described
in this theorem, the maximal value of specific cost ratio is attained for the $M/E_k/1$ queue. Notice that the assertion of the above theorem is illustrated by some numerical and graphical results presented in the following section.

4. NUMERICAL RESULTS

Applying the expression (8), here we give some numerical and graphical results concerning the specific cost ratio for certain port queueing models described as a single-server queue $M/E_k/1$. More precisely, for any given (fixed) value $r > 0$ (the first term on the right hand side of (8)), we give some numerical and graphical results for the difference $R - r$ (which does not depend on $r$), where $R$ is given by (8), as a function of the shape parameter $k$ ($k = 2,3,4,\ldots,9$) and the traffic intensity $\theta$ ($0.25 \leq \theta \leq 0.85$). For these purposes, we write the expression (8) as a sum

$$ R(k, \theta) = r + f(k, \theta), $$

where

$$ f(k, \theta) = \frac{\theta(4k + 3\theta - 5k\theta - 2\theta^2 + 2k\theta^2)}{2k(1-\theta)^2}. \quad (9) $$

The graphic of the above function $f(k, \theta)$ in two variables $k \in \{2,3,4,\ldots,9\}$ and $\theta \in [0.2,0.8]$ is presented in Figure 1.

![Figure 1. The graphic of the function $f(k, \theta)$ for $k \in \{2,3,4,\ldots,9\}$ and $\theta \in [0.2,0.8]$](image1.png)

Using the expression (8), related numerical results are given in Tables 1 and 2.

| Table 1. The values of $f(k, \theta)$ for even values $k = 2,4,6,8$ and $\theta \in [0.25, 0.45, 0.65, 0.85]$ |
|-------------|-------------|-------------|-------------|
| $\theta$    | $k = 2$     | $k = 4$     | $k = 6$     |
| 0.25        | 0.7083      | 0.6736      | 0.6620      | 0.6563      |
| 0.45        | 1.9543      | 1.7786      | 1.7200      | 1.6908      |
| 0.65        | 5.6975      | 4.9645      | 4.7202      | 4.5981      |
| 0.85        | 33.0083     | 27.7903     | 26.0509     | 25.181      |

Figure 1, Table 1 and Table 2 show that for any considered fixed $k$ (and probably, also for all $k = 2,3,4,\ldots$) the function $\theta \rightarrow f(k, \theta)$ increases on the segment $[0.25,0.85]$ (and probably, also on the whole interval $(0,1)$). On the other hand, it is proved in Proposition of the previous section that for any fixed $\theta \in (0,1)$ the sequence $k \rightarrow f(k, \theta)$ decreases with respect to $k$ ($k = 2,3,4,\ldots$). Obviously, the same assertions are also true for the function $R(k, \theta) = r + f(k, \theta)$ defined by (8) which expresses the related values of specific cost ratio.

Finally, for $r = 0.4$ the graphic of the function $R(k, \theta)$/$R(k+1, \theta)$ in two variables $k \in \{2,3,4,\ldots,9\}$ and $\theta \in [0.25, 0.85]$ is presented in Figure 2. Related numerical results involving odd values $k = 3,5,7,9$ are given in Table 3.

| Table 2. The values of $f(k, \theta)$ for odd values $k = 3,5,7,9$ and $\theta \in [0.25, 0.45, 0.65, 0.85]$ |
|-------------|-------------|-------------|-------------|
| $\theta$    | $k = 3$     | $k = 5$     | $k = 7$     | $k = 9$     |
| 0.25        | 0.6852      | 0.6667      | 0.6587      | 0.6543      |
| 0.45        | 1.8372      | 1.7435      | 1.7033      | 1.6810      |
| 0.65        | 5.2089      | 4.8180      | 4.6504      | 4.5574      |
| 0.85        | 29.5296     | 26.7467     | 25.5540     | 24.8914     |

| Table 3. The values of $R(k, \theta)$/$R(k+1, \theta)$ with $r = 0.4$ for odd values $k = 3,5,7,9$ |
|-------------|-------------|-------------|-------------|
| $\theta$    | $k = 3$     | $k = 5$     | $k = 7$     | $k = 9$     |
| 0.25        | 1.0108      | 1.0044      | 1.0024      | 1.0015      |
| 0.45        | 1.0269      | 1.0111      | 1.0060      | 1.0038      |
| 0.65        | 1.0455      | 1.0191      | 1.0105      | 1.0066      |
| 0.85        | 1.0617      | 1.0263      | 1.0146      | 1.0093      |

![Figure 2. The graphic of the function $R(k, \theta)$/$R(k+1, \theta)$ with $r = 0.4$ for $k \in \{2,3,4,\ldots,9\}$ and $\theta \in [0.25, 0.85]$](image2.png)

5. CONCLUSION

In this paper, it is firstly derived the expression for the specific cost ratio in the dependence of shape parameter $k$ and the traffic intensity $\theta$ for the port queueing model described as a single-server queue $M/E_k/1$ with one berth. Applying this expression, some theoretical, numerical and graphical results for the total specific cost ratio involving different pairs of values $k$ and $\theta$ are presented. The obtained analytic expression for specific cost ratio also gives the possibilities of discussing and comparing the values of specific cost ratio of various $M/E_k/1$ queues with specific cost ratio of the port.
queueing models investigated in earlier authors’ papers. Moreover, the proposed analytical results would be useful for further development of this research field. In particular, this is closely related to the problem of minimization of specific cost ratio.

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МОДЕЛИРАЊЕ СПЕЦИФИЧНОГ ОДНОСА ТРОШКОВА У ЛУЦИ СА M/E_k/1 МОДЕЛОМ ТЕОРИЈЕ РЕДОВА ЧЕКАЊА

Б. Драговић, Р. Мештровић, Н. Зрнић, Д. Драгојевић

Појам специфичног односа трошкова односи се на разне типове трошкова брода и луке моделирани као систем реда чекања Користећи познату општу формулу за специфични однос трошкова (означен са R), изводимо одговарајући израз за R у односу на M/E_k/1 модел реда чекања, где је E_k (k = 2,3,...) Ерлангова k - распада вероватноће. Овај израз нам омогућава да добијемо теоријски резултат који се може приметити за одређивање оптималних вредности параметра облика k од E_k уз дата ограничења у односу на друге перформансе разматраног лучког модела теорије редова чекања. Такође су представљени одговарајући нумерички и графички резултати. Добијени резултати би могли бити корисни за будућа истраживања разматране проблематике.