THE INVERSE PROBLEM OF RECOVERING AN UNSTEADY LINEAR LOAD FOR AN ELASTIC ROD OF FINITE LENGTH

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The main purpose of the paper is to obtain solutions for new non-stationary inverse problems for elastic rods. The objective of this study is to develop and implement new methods, approaches and algorithms for solving non-stationary inverse problems of rod mechanics. The direct non-stationary problem for an elastic rod consists in determining elastic displacements, which satisfies a given equation of non-stationary oscillations in partial derivatives and some given initial and boundary conditions. The solution of inverse retrospective problems with a completely unknown space-time law of load distribution is based on the method of influence functions. With its application, the inverse retrospective problem is reduced to solving a system of integral equations of the Volterra type of the first kind in time with respect to the sought external axial load of the elastic rod. To solve it, the method of mechanical quadratures is used in combination with the Tikhonov regularisation method.

Key words: inverse problem, elastic rod, influence function, Fourier series, integral transformations, integral equations, Tikhonov regularisation, quadrature formulas

INTRODUCTION

An elastic homogeneous isotropic rod of finite length is considered, the left end of which is rigidly fixed, and the right end of the rod is free. At the initial moment of time, a distributed unsteady load begins to act on the rod, the dependence of which on time and the distribution law along the coordinate are unknown and must be determined in the process of solving the inverse problem. It is assumed that the displacements are known in some vicinity of the free end of the rod. In practice, this information can come from sensors for measuring longitudinal displacements installed in several sections in the vicinity of the free end of the rod.

To construct a method for solving the inverse problem, it is first necessary to obtain solutions to the direct problem, in which the axial load is known and it is required to determine the unsteady displacements for the elastic rod. The methodology for solving the direct problem is based on the principle of superposition, in which displacements and contact stresses are related by means of integral operators with respect to the spatial variable and time [1-3]. The cores of the latter are the so-called influence functions. These functions represent fundamental solutions to systems of differential equations of motion for the beam under consideration [4-6]. Their construction is a separate task. The influence functions are found using the Laplace transform in time and expansions in Fourier series in the system of eigenfunctions [7, 8].

In inverse problems, the right-hand side in the equations of unsteady oscillations is not specified, but there is some information about the displacements at the points of the sensor's installation. Using the method of influence functions, the inverse problem is reduced to solving a system of integral equations of the Volterra type of the first kind in time with respect to the expansion coefficients of the required load in a Fourier series in terms of the system of eigenfunctions [9-11]. To solve integral equations, the mechanical quadrature method is used in combination with the Tikhonov regularisation algorithm [12, 13]. The possibilities of using the proposed identification method in the presence of measurement noise are investigated.

It should be noted that along with many studies aimed at solving theoretically and practically important direct problems for thin-walled structural elements, for example, works [14-16], including, taking into account the effect of temperature, material anisotropy, multilayer structure of elements, there are practically no papers devoted to solving non-stationary inverse problems [17-19]. This determines the relevance of this paper.

METHODS FOR SOLVING DIRECT AND INVERSE NON-STATIONARY PROBLEM

The mathematical formulation of the direct problem includes the equation of motion for a homogeneous rod of constant cross-section, boundary conditions, and zero initial conditions in dimensionless form [20-22] (Eqs. 1-3):

$$\frac{\partial^2 u\,(x,\tau)}{\partial \tau^2} = \frac{\partial^2 u\,(x,\tau)}{\partial x^2} + p\,(x,\tau) \tag{1}$$

$$u(x,\tau)\big|_{x=0} = 0, \ \frac{\partial u(x,\tau)}{\partial x}\bigg|_{x=l} = 0 \tag{2}$$

$$u(x,\tau)\big|_{\tau=0} = 0, \ \frac{\partial u(x,\tau)}{\partial \tau}\bigg|_{\tau=0} = 0 \tag{3}$$
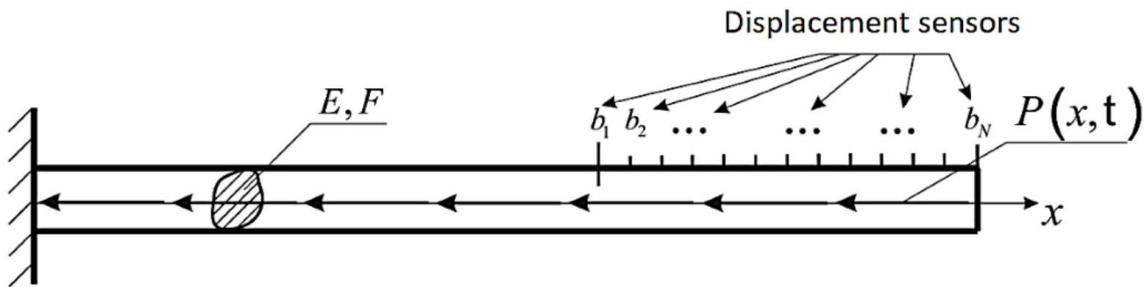
Figure 1: Sensor installation diagram

In the inverse problem, it is assumed that the displacements $u(x,\tau)$ in certain vicinity $x \epsilon [b_1, b_N]$ of the free end of the rod are known. In practice, this information can come from displacement sensors installed in the rod sections $x = b_1, b_2, ..., b_N$ (Fig. 1). It is required, according to the data obtained from the displacement sensors, installed at points $b_1, ..., b_N$, to restore the axial load $p(x,\tau)$.

Thus, the mathematical formulation of the inverse problem consists of equation (1) with an unknown right-hand side $p(x,\tau)$, boundary and initial conditions (3), as well as additional conditions (Eq. 4):

$$u(b_n, \tau) = U_n(\tau), n = \overline{1, N} \quad (4)$$

where $U_n(\tau)$ – known functions of time. In practice, they represent the displacement values from the sensors.

To solve the direct and inverse problems, we construct the influence function $G(x,\tau)$ [23-25]. It is a solution of problem (1), with the replacement of the load $p(x,\tau)$ by a single load $\delta(x-\xi)\delta(\tau)$, where $\delta(x-\xi)\delta(\tau)$ – the Dirac delta function [23] (Eqs. 5-7):

$$\frac{\partial^2 G(x,\xi,\tau)}{\partial \tau^2} = \frac{\partial^2 G(x,\xi,\tau)}{\partial x^2} + \delta(x - \xi)\delta(\tau) \quad (5)$$

$$G(x,\xi,\tau)|_{x=0} = 0, \left.\frac{\partial G(x,\xi,\tau)}{\partial x}\right|_{x=l} = 0 \quad (6)$$

$$G(x,\xi,\tau)|_{\tau=0} = 0, \left.\frac{\partial G(x,\xi,\tau)}{\partial \tau}\right|_{\tau=0} = 0 \quad (7)$$

To calculate the influence functions, we apply to problem (5) the Laplace transform in time, (the index "L" is the Laplace transform, s is the parameter of the Laplace transform $G^L = G^L(x,\xi,s)$) (Eqs. 8-9):

$$s^2 G^L = \frac{\partial^2 G^L}{\partial x^2} + \delta(x - \xi) \quad (8)$$

$$G^L|_{x=0} = 0, \left.\frac{\partial G^L}{\partial x}\right|_{x=l} = 0 \quad (9)$$

Let us find the eigenfunctions and eigenvalues of the homogeneous problem (1). For this we apply the Fourier variable separation method. Substitute $u(x,\tau) = X(x)T(\tau)$ in, then (Eqs. 10-13):

$$\frac{1}{T(\tau)}\frac{\partial^2 T(\tau)}{\partial \tau^2} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\lambda^2 \quad (10)$$

$$\frac{\partial^2 X(x)}{\partial x^2} + \lambda^2 X(x) = 0 \quad (11)$$

$$X(0) = 0, X'(1) = 0 \quad (12)$$

$$\lambda_n = \frac{2n-1}{2}\pi, X_n(x) = \sin \lambda_n x \quad (13)$$

Thus, $X_n(x) = \sin \lambda_n x$ – eigenfunctions, and $\lambda_n = \frac{2n-1}{2}\pi$ –eigenvalues of homogeneous problem (1). To solve (5-7), we apply the method of incomplete separation of variables. We represent the required function $G(x,\tau)$ and the function $\delta(x-\xi)\delta(\tau)$ in the Fourier series in eigenfunctions (Eqs. 14-15):

$$G(x,\xi,\tau) = \sum_{n=1}^{\infty} G_n(\xi,\tau) \sin \lambda_n x, \dot\iota$$

$$(x - \xi)\delta(\tau) = \delta(\tau) \sum_{n=1}^{\infty} \delta_n(\xi) \sin \lambda_n x \quad (14)$$

$$\delta_n(\xi) = 2 \int_0^1 \delta(x - \xi) \sin \lambda_n x dx = 2 \sin \lambda_n \xi \quad (15)$$

Substituting (14-15) into (5-7), we arrive at the problem in terms of the coefficients of the series (14-15):

$$\ddot{G}_n(\xi,\tau) = -\lambda_n^2 G_n(\xi,\tau) + 2\delta(\tau) \sin \lambda_n \xi \quad (16)$$

$$G_n(\xi,0) = 0, \dot{G}_n(\xi,0) = 0 \quad (17)$$

Applying the integral Laplace transform with respect to time to (Eqs. 16-17), we arrive at the equation for the Laplace images of the coefficients of the expansion series of the influence function (s is the transformation parameter) (Eq. 18):

$$s^2 G_n^L(\xi,s) = -\lambda_n^2 G_n^L(\xi,s) + 2\sin \lambda_n \xi \quad (18)$$

by solving which we find (Eq. 19):

$$s^2 G_n^L(\xi,s) = -\lambda_n^2 G_n^L(\xi,s) + 2\sin \lambda_n$$

$$\xi, G_n^L(\xi,s) = 2\frac{\sin \lambda_n \xi}{s^2 + \lambda_n^2} \quad (19)$$

Performing the inverse transformation, we obtain the originals of the desired coefficients (Eq. 20):

$$G_n(\xi,\tau) = 2\frac{\sin \lambda_n \xi \sin \lambda_n \tau}{\lambda_n} \quad (20)$$

Then the original of the influence function will have the form (Eq. 21):

$$G(x,\xi,\tau) = 2\sum_{n=1}^{\infty} \frac{\sin \lambda_n \xi \sin \lambda_n \tau \sin \lambda_n x}{\lambda_n} \quad (21)$$

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SOLUTION OF THE DIRECT AND INVERSE NON-STATIONARY PROBLEM

Knowing the influence function $G(x,\xi,\tau)$ and based on the principle of superposition, the solution to problem (1) can be represented as (Eq. 22):

$$u(x,\tau) = \int_0^1 \int_0^\tau G(x,\xi,\tau-t)p(\xi,t)dtd\xi \qquad (22)$$

Expand the external axial load in a Fourier series (Eq. 23):

$$p(x,\tau) = \sum_{n=1}^\infty p_n(\tau)\sin\lambda_n x$$

$$p_n(\tau) = 2\int_0^1 p(x,\tau)\sin\lambda_n x dx \qquad (23)$$

Substituting series (21) and (23) into (22) and taking into account the orthogonality of the system of eigenfunctions, we arrive at an integral representation similar to (22) for the displacements (Eq. 24):

$$u(x,\tau) = \sum_{n=1}^\infty I_n(\tau)\frac{\sin(\lambda_n x)}{\lambda_n}, I_n$$

$$(\tau) = \int_0^\tau \sin[\lambda_n(\tau-t)]p_n(t)dt \qquad (24)$$

For an approximate determination of the displacements of the elastic rod $u(x,\tau)$ we use the formula for average rectangles. We split the segment of integration $[0,\tau]$ into M parts with a uniform step $h=\tau/M$. In equation (23), we restrict ourselves to the first N terms of the series. The integrals in (23) are replaced by approximate quadrature formulas of the method of mean rectangles, then (Eq. 25):

$$u_h(x,\tau) \approx h\sum_{n=1}^N \frac{\sin\lambda_n x}{\lambda_n}\sum_{m=1}^M \widetilde{G}_n(\tau-t_m)p_n(t_m)$$

$$t_m = h\frac{2m-1}{2} \qquad (25)$$

Examples of solving the direct problem with an estimate for the convergence of formula (25) are given in [26-28]. The inverse problem is to determine the coefficients $p_n(\tau)$ of series (23). Suppose N sensors are installed on a certain rod segment, which measure the values of rod displacements $U_1(\tau)=u(b_1,\tau), U2(\tau)=u(b2,\tau),...,U_N(\tau)=u(b_N,\tau)$ depending on time τ (Fig. 1), where

$$b_n = \frac{b_1-b_N}{2} + \frac{b_1-b_N}{2}\cos\left(\frac{2n-1}{2N}\pi\right)$$

b1 – coordinate of the first sensor on the rod, b_N– coordinate of the last sensor. Restricting ourselves to the first N terms, from (24) we obtain N integral representations (Eq. 26) which form a system of algebraic equations for the Volterra integral operators $I_n(\tau)$, $n=\overline{1,N}$:

$$U_k(\tau) = \sum_{n=1}^N I_n(\tau)a_{kn}, a_{kn} = \frac{\sin b_k \lambda_n}{\lambda_n}, k=1,..,N \quad (26)$$

System (26) is written in vector-matrix form (Eq. 27):

$$U = AI, A = (a_{kn})_{N\times N}, U = [U_k(\tau)]_{N\times 1}, I = [I_n(\tau)]_{N\times 1} \quad (27)$$

Solving this system, we obtain the vector I (Eq. 28):

$$I = U^* \qquad (28)$$

where (Eq. 29):

$$U^* = A^{-1}U = [U_n^*(\tau)]_{N\times 1} \qquad (29)$$

Vector-matrix equality (28) is equivalent to N independent Volterra integral equations of the first kind with respect to the sought coefficients of series (23):

$$u_{nm}^* \approx h\sum_{k=1}^m G_{W_{nmk}}p_{nk}, m=1,...,M \qquad (30)$$

It is known that for G(0)=0, (Eq. 30) are incorrect in the sense of J. Hadamard [29, 30]. Therefore, to solve problem (Eq. 30), we apply the regularisation method by A.N. Tikhonov [31, 32].

NUMERICAL SOLUTION OF THE VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND

To solve equations (30), we will use the formula for mean rectangles. Let us fix some finite time T. Divide the integration time interval $[0, T]$ into M equal parts with a uniform step $h=T/M$. For each time moment $\tau_m=hm$ we replace equation (30) with a numerical analogue using the method of mean rectangles (Eqs. 31-33):

$$u_{nm}^* \approx h\sum_{k=1}^m G_{W_{nmk}}p_{nk}, \quad m=1,\ldots,M \qquad (31)$$

$$u_{nm}^* = u_{nm}^*(\tau_m), G_{U_{nmk}} = G_{U_n}(\tau_m-t_k) \qquad (32)$$

$$p_{nk} = p_n(t_k), t_k = h\frac{2k-1}{2} \qquad (33)$$

As a result, we arrive at a system of linear algebraic equations with respect to p_{nk}, which are the values of the sought coefficients $p_n(\tau)$ at times t_k, k=1,...,M (Eq. 34):

$$G_n P_n = U_n^* \qquad (34)$$

where (Eqs. 35-36):

$$G_n = (G_{U_{nmk}})_{M\times M} = \begin{pmatrix} G_{U_{n11}} & 0 & 0 & ... & 0 \\ G_{U_{n21}} & G_{U_{n22}} & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ G_{U_{nM1}} & G_{U_{nM2}} & G_{U_{nM3}} & ... & G_{U_{nMM}} \end{pmatrix} (35)$$

$$P = (p_{nk})_{M\times 1}, U_n^* = \left(\frac{u_{nm}^*}{h}\right)_{M\times 1} \qquad (36)$$

Due to the ill-posedness of problem (30), the matrix G_n is ill-conditioned; therefore, we solve the system of equations (34) using the regularisation method by A.N. Tikhonov [31, 32]. Tikhonov's regularization method is an algorithm that allows finding an approximate solution to ill-posed operator problems of the form $A_x = u$. Tikhonov's method is perhaps the most popular in solving problems with approximately given information. It also essentially uses a priori information about the exact solution of the problem. In this case, (34) is replaced by the problem of finding the minimum of the Tikhonov functional (Eq. 37):

$$\Omega_\alpha(\tau) = |G_n\tau - U_n^*|^2 + \alpha|\tau|^2 \qquad (37)$$

It can be shown [33-35] that the problem of minimising the Tikhonov functional is reduced to solving another system of algebraic equations (38):

$$(G_n^T G_n + \alpha E)\widetilde{P}_n = G_n^T U_n^* \qquad (38)$$

where α – small positive regularisation parameter, which is selected in some optimal way, $\tilde{P}_n$ – quasi-solution vector of the equation of the system (34).

EXAMPLES OF SOLVING THE INVERSE PROBLEM

Consider several problems with different assigned right-hand sides for equation (24) (Fig. 2). The rod material is steel with the following dimensional parameters: ρ=7850kg/m^3, E=2·10^{11} Pa, b_1=0.1l – coordinate of the first sensor, b_N=0.9l coordinate of the last sensor. The length of the rod is 1 m.

The corresponding dimensionless parameters are:

c=5047.55, b_1=0.1, b_N=0.9, l=1. The external axial load given for finding the displacements of the rod, shown in Figure 2a, equals – $P(x,\tau)$=10^{-3}$(e^{-\tau}sin(\pi x)+e^{-\tau}sin(2\pi x))$, in Figure 2b, equals – $P(x,\tau)$=$e^{-\tau}(-(3x-1.5)^2+2.5)$.

Here, the dashed line is the reconstructed external axial load, the solid line is the external axial load specified to find the displacements of the rod. To simulate experimental data on displacement at the points of sensors installation, we add to the displacements found a column vector composed of small random numbers and analyse the behaviour of the solution. As seen from Fig. 3, when adding a column vector, the solution to the inverse problem remains practically unchanged. Here, the specified external axial load is a solid line, the restored external axial load is dashed.

CONCLUSIONS

A new inverse problem of recovering a non-stationary linear load for an elastic rod of finite length is solved. A method has been developed for constructing a non-stationary influence function for an elastic rod, which is used
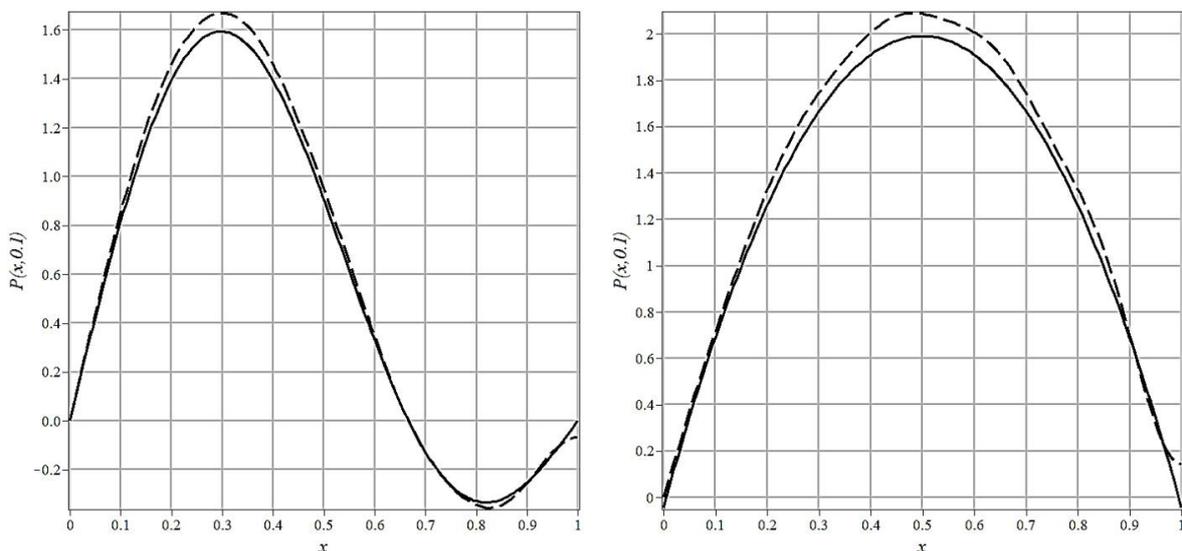

Figure 2: Comparison of the solution to the inverse problem with the external axial load specified for the solution of the direct problem
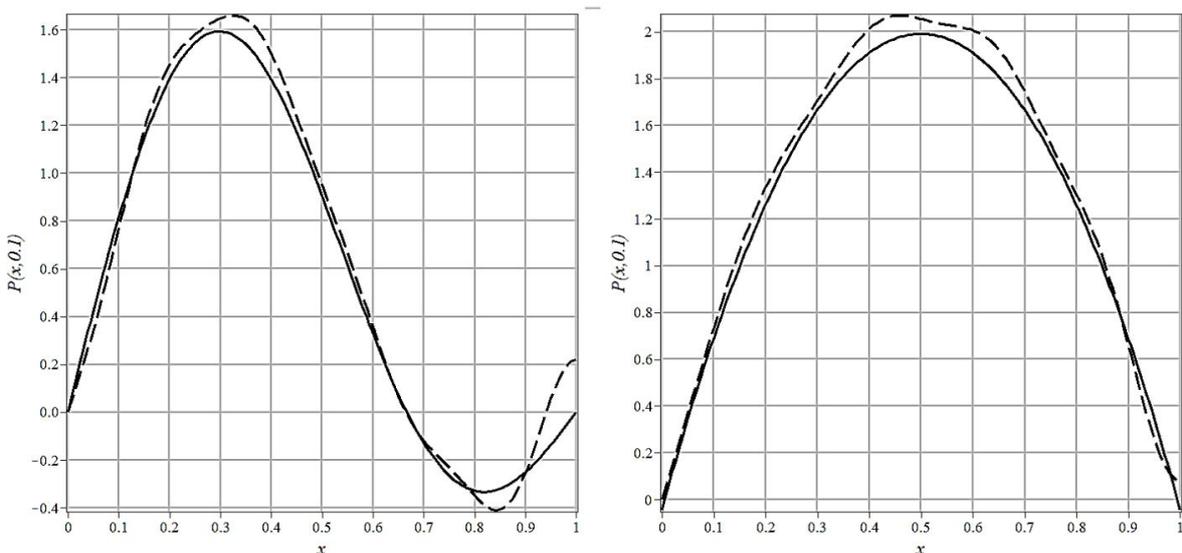

Figure 3: Comparison of the solution of the inverse problem with the external axial load specified for the solution of the direct problem, taking into account noise

to solve the direct and inverse problems. This function is a fundamental solution to the differential equation of motion of the investigated rod. The influence function is found using the Laplace transform in time and expansions in Fourier series in the system of eigenfunctions. In order to obtain a solution to the inverse retrospective problem, the direct problem of determining the displacements of an elastic rod was solved. The methodology for solving the direct problem is based on the principle of superposition, in which displacements and linear load are connected by an integral operator with respect to the spatial variable and time. Its core is the influence function.

In the inverse problem, the linear load is not known and must be identified. In this case, the movements of the rod at the points of installation of the sensors are assumed to be known. The construction of the solution to the reverse retrospective problem is based on the method of influence functions. With its application, the inverse problem is reduced to solving a system of Volterra integral equations of the first kind for the expansion coefficients of the distributed load in a Fourier series. To solve the system of integral equations, the mechanical quadrature method is used in combination with the Tikhonov regularisation method.

The proposed formulation and method for solving the nonstationary inverse problem can serve as the basis for the creation of complexes for monitoring structures in real time. They will allow to monitor and timely prevent the occurrence and development of damage directly during operation, monitor various structural transformations, restore the space-time laws of external loads acting on the structure. In connection with the rapid development of computers, automation and robotics, the tasks of this class are at the forefront of modern science.

ACKNOWLEDGEMENTS

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