originalni naučni rad

UDK 336.781.5 ; 330.133.2:336.763.3

KONSTRUISANJE KRIVE PRINOSA OBVEZNICE

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Originalna verzija teksta je na engleskom jeziku

Rezime

U najširem smislu, kriva prinosa ukazuje na gledište tržišta o evoluciji kamatnih stopa tokom vremena. Međutim, s obzirom na to da je trošak zaduživanja blisko povezan sa kreditnom sposobnošću (sposobnost otplate), različite krive prinosa primenjivaće se na različite valute, sektore tržišta ili čak na pojedinačne emitente. Pošto zaduživanje države ukazuje na nivoe kamatnih stopa raspoloživih za druge učesnike na tržištu u određenoj zemlji, i uzimajući u obzir da emitovanje obveznica i dalje ostaje dominantna forma suverenog duga, ovaj materijal opisuje konstruisanje krive prinosa korišćenjem obveznica. Dat je odnos između prinosa zero-kupona, paritetnog prinosa i prinosa do dospeća i opisano je njihovo korišćenje u utvrđivanju diskontnih faktora krive. Diskutuje se i njihovo korišćenje u izvođenju terminskih stopa i cene povezanih derivatnih instrumenata.

Ključne reči: kriva prinosa, cene obveznica, zero-kupon obveznice, konstrukcija krive prinosa

JEL: G12, G13, E43

Rad primljen: 24.04.2014.

Odobren za štampu: 14.05.2014.

UDC 336.781.5 ; 330.133.2:336.763.3

original scientific paper

BOND YIELD CURVE CONSTRUCTION

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Summary

In the broadest sense, yield curve indicates the market's view of the evolution of interest rates over time. However, given that cost of borrowing it closely linked to creditworthiness (ability to repay), different yield curves will apply to different currencies, market sectors, or even individual issuers. As government borrowing is indicative of interest rate levels available to other market players in a particular country, and considering that bond issuance still remains the dominant form of sovereign debt, this paper describes yield curve construction using bonds. The relationship between zero-coupon yield, par yield and yield to maturity is given and their usage in determining curve discount factors is described. Their usage in deriving forward rates and pricing related derivative instruments is also discussed.

Keywords: yield curve, bond prices, zero-coupon bonds, curve construction

JEL: G12, G13, E43

The original version of the text is in English

Paper received: 24.04.2014

Approved for publishing: 14.05.2015

Uvod

Kod određivanja svih izuzev najprostijih finansijskih instrumenatad neophodno je napraviti brojne pretpostavke, od kojih je jedna ročna struktura terminskih kamatnih stopa, jer ona utiče na valorizovanje budućih tokova gotovine i diskontovanje na sadašnju vrednost da bi se eliminisale bilo koje mogućnosti za arbitražu. Uprkos primene kompleksnih kvantitativnih tehnika i modela, buduća evolucija bilo koje promenljive, time i kamatnih stopa, može se procenjivati sa većim ili manjim uspehom. Zato su, kada se konstruiše kriva prinosa, obveznice sa zero-kuponom tako atraktivne, jer ti instrumenti, kako im naziv implicira, ne plaćaju među kupone. Time, zbog samo dva toka gotovine (inicijalno investiranje i otplata kod dospeća), međuročna struktura postaje irelevantna i rizik reinvestiranja je uklonjen. Jasno, pošto se ne prima prihod do dospeća obveznice, investitor očekuje da se njena cena odredi po diskontu u odnosu na nominalnu vrednost (izraženo u odnosu na 100 jedinica konkretne valute), čime se prinos zero-kupona na obveznicu izračunava iz odnosa između inicijalne i nominalne cene. Važno je istaći da, mada nema reinvestiranja između kupovine i dospeća, stopa koja se postigne investiranjem u obveznicu sa zero-kuponom ipak ima strukturu i nije izračunata kao prosta kamata. Razlog za korišćenje ovog pristupa je da bi se omogućilo poređenje između prinosa impliciranog kod obveznica sa zero-kuponom i drugih dužničkih instrumenata na tržištu (kamatonostnih obveznica i naročito obveznica sa paritetnom cenom).

S obzirom na to da se, na većini tržišta, trguje sa vrlo malo obveznica sa zero-kuponom, većina od kojih je sa vrlo kratkim dospećem, pojam prinosa po zero-kuponu postaje uglavnom teoretski, jer to nije vidljiva promenljiva. Ipak, s obzirom na jednostavnost i korisnost ovog koncepta za poređenje ne samo različitih obveznica kojima se trguje na tržištu, već i za vrednovanje drugih, složenijih finansijskih instrumenata, prinosi po obveznicama sa zero-kuponom tipično se izvode iz tržišnih podataka i koriste za konstruisanje krivih prinosa na obveznice. U daljem tekstu ovog materijala, najpre se daju neke osnovne informacije u vezi sa istorijskim događajima u pogledu modeliranja krive prinosa i neka pitanja u vezi sa njegovim korišćenjem. Posle toga sledi izvođenje prinosa po zero-kuponu za različite ročnosti i njihovu primenu na izračunavanje terminskih stopa. Najzad, diskutuju se neka pitanja u vezi sa konstruisanjem krive prinosa da bi se pružile smernice za praktično korišćenje tih tehnika.

Istorijski razvoj

Kriva prinosa po zero-kuponu konstruisana neposredno iz tržišnih podataka ima mnoge primene u praksi kao i u istraživanju. U prošlosti, Tanggaard i Jakobson (1988) uspešno su konstruisali krivu obveznica i primenili je na dansko tržište obveznica, a kasnije, Bhar i Chiarella (1996) primenili su sličan pristup na australijanske obveznice trezora. Međutim, vrednost takvog rada je najočigledniji u njegovim kasnijim primenama na utvrđivanje cene instrumenata zavisno od ročne strukture kamatne stope. Konkretno, modeli koji pokušavaju da obuhvate različite karakteristike proizvoda koji nastaju na tržištima derivata znatno se oslanjaju na evoluciju cele krive prinosa (a ne na rad sa terminalnim vrednostima kamatnih stopa). Takvi modeli mogu se šire kvalifikovati kao ravnotežni modeli i nearbitražni modeli.

Ravnotežni modeli opisuju ponašanje ekonomskih promenljivih u smislu trenutne kratkoročne kamatne stope r, koja se ne vidi u realnom svetu i tako nema odnos sa stvarno vidljivim prinosima. Dalje, pošto je inicijalna ročna struktura krive prinosa autput modela, a ne njegov imput, teško je ustanoviti vezu između tržišnih cena promenjivih u modelu. Primeri takvih modela su modeli koje su razvili Rendleman i Barterr (1980), Vasicek (1977), Cox, Ingersoll i Ross (1981) i Longstaff i Schwarz (1992).

Mnogo korisnija klasa modela su oni koji se tačno uklapaju u današnju ročnu strukturu, poznati kao 'nearbitražni modeli'. Pošto ti modeli uzimaju današnju ročnu strukturu kao imput i definišu proces svoje evolucije tokom vremena, znatno više su korišćeni u praksi. Pošto tri pristupa modeliranju mogu da se koriste - korišćenje cena obveznica, terminskih stopa ili kratkoročnih stopa - očigledno je zašto

Introduction

When pricing all but the simplest financial instruments numerous assumptions need to be made, one of which is the forward interest rate term structure, as it affects the valuation of future cashflows and discounting to the present value in order to eliminate any arbitrage opportunities. Despite complex quantitative techniques and models employed, the future evolution of any variable, thus interest rates, can only be estimated with a greater or a lesser success. This is why, when constructing a yield curve, zero-coupon bonds are so attractive, for these instruments, as their name implies, pay no interim coupon. Thus with only two cashflows (initial investment and repayment at maturity), the intermediate term structure becomes irrelevant and the risk of reinvestment is removed. Clearly, as no income is received until bond maturity, the investor expects it to be priced at discount to its notional value (expressed in relation to 100 units of a specific currency), whereby the zero-coupon yield on the bond is calculated from the relationship between the initial and nominal price. It is important to note that, even though no reinvestment takes place between the purchase and maturity, the rate achieved by investing in a zero-coupon bond is still decompounded, rather than calculated as simple interest. The reason for using this approach is to enable comparison between yields implied by zero-coupon bonds and other debt instruments in the market (coupon bearing bonds, and in particular those priced at par).

Given that, in most markets, very few zero-coupon bonds are traded, most of which have very short maturities, the notion of zero-coupon yield becomes rather theoretical, as it is not an observable variable. Nonetheless, given the simplicity and usefulness of this concept for comparing not only different market-traded bonds, but also valuing other, more complex, financial instruments, zero-coupon bond yields are typically derived from market data and used to construct bond yield curves. In the remainder of this paper, firstly some background information regarding the historical developments with respect to yield curve modelling and some issues related to its use is given. It is followed by derivation of zero-coupon yields for different maturities and their application in calculating forward rates. Finally, some issues related to yield curve construction are discussed, in order to provide directions for practical use of these techniques.

Background

Zero-coupon yield curve constructed directly from market data has many applications in practice as well as research. In the past, Tanggaard and Jakobsen (1988) have successfully constructed bond curve and applied it to Danish bond market, and later, Bhar and Chiarella (1996) applied similar approach to Australian treasury bonds. However, the value of such work is most evident in its subsequent application to pricing instruments depended on interest rate term structure. In particular, models attempting to accommodate different features of the products emerging in the derivatives markets rely heavily on the evolution of entire yield curve (rather than working with terminal values of interest rates). Such models can be broadly classified as equilibrium models and no-arbitrage models.

Equilibrium models describe behaviours of economic variables in terms of the instantaneous short-term rate *r*, which is not observable in the real world and thus has no relation to the actual observable yields. Moreover, as the initial yield curve term structure is the output of the model, rather than input to it, it is difficult to establish the link between market prices and model variables. Examples of such models are those developed by Rendleman and Bartter (1980), Vasicek (1977), Cox, Ingersoll and Ross (1981), Brennan and Schwartz (1982), and Longstaff and Schwartz (1992).

A much more useful class of models are those that fit today's term structure exactly, thus known as 'no-arbitrage models'. As these models take today's term structure as input and define the process of its evolution over time, they are much more useful in practice. Given that three modelling approaches can be used-using bond prices, forward rates, or short rate as input-is it evident why bond yields are essential in pricing. Although models based on bond prices and forward rates typically have to be solved analytically, no-arbitrage approach su prinosi na obveznice bitni kod određivanja cene. Mada modeli zasnovani na cenama obveznica i terminskim stopama tipično moraju da se rešavaju analitički, nearbitražni pristup je našao široku primenu u praksi, sa modelima koje su razvili Ho i Lee (1986), Hull i White (1990), Heath-Jarrow-Morton (1992), Black, Derman i Toy (1990) i Black i Karasinski (1991) kao najistaknutiji primeri.

Pošto većina analitičkih modela zahteva inicijalnu ročnu strukturu terminske stope kao jedan od imputa, cene obveznica sa zero-kuponom tipično se koriste za izvođenje terminskih prinosa. Međutim, kao što je napomenuto napred, malobrojnost (ili čak odsustvo) takvih instrumenata na tržištu implicira da je potrebna dalja obrada podataka obveznica za naredno korišćenjel u modeliranju. Ostavljajući tu temu po strani za trenutak, pošto će ponovo biti reč o njoj u daljoj diskusiji, naredna sekcija opisuje izvođenje prinosa po zero-kuponu iz tržišnih cena kuponskih obveznica.

Odnos između prinosa po zero-kuponima i tržišnih cena kuponskih obveznica

Idealno, kada se pokuša izvođenje prinosa po zero-kuponima iz obveznica kojima se trguje na tržištu, treba koristiti instrumente iste ročnosti sa paritetnom cenom (cena jednaka otplatnoj vrednosti). Prednost takvog pristupa proizilazi iz činjenice da, uprkos pretpostavci nepostojanja arbitraže na tržištu, gde bi likvidnost ubrzo uklonila bilo kakve povoljne investicione mogućnosti koje proističu iz neispravne procene vrednosti raznih ugovora, poređenje obveznica koje plaćaju različite kupone izuzetno je teško. Na primer, kod date dve 10-godišnje obveznice, koje plaćaju 4% i 8% godišnji kupon respektivno, prva će jasno imati nižu cenu od druge. Međutim, manji kupon će implicirati da će veći deo toka gotovine biti plativ na kasnije datume (tj. duracija obveznice biće duža), u poređenju sa obveznicom od 8%. Tako, u normalnim tržišnim uslovima, gde su dugoročne kamatne stope više, kasniji tokovi gotovine na obveznicu od 4% vredeće komparativno više, pomerajući ravnotežu neznatno u njenu korist. Pošto ovo vrednovanje zavisi od mnogo pretpostavki, mnogo je pogodnije koristiti paritetne obveznice kod vrednovanja.

Međutim, pošto nije verovatno da će se na tržištu u bilo koje vreme trgovati paritetnim obveznicama svih potrebnih ročnosti, bitno je uspostaviti odnos između stopa zero-kupon, paritetnih i neparitetnih obveznica da bi se kretalo od jedne do druge, prema potrebi. Ovde je takođe bitno istaći da takvi odnosi nisu važeći samo između ugovora na istom tržištu, jer će različiti emitenti (korisnici kredita), zbog raznolikosti svojih kreditnih sposobnosti, biti u mogućnosti da se zadužuju po različitim stopama. Tako, kod izbora ugovora za analizu, njih mora da su emitovali na pr. država ili korporacije jednakog stendinga. U takvom slučaju, biće moguće izgraditi krivu prinosa iz koje mogu da se izvedu paritetna kao i zero-kupon kriva.

Izvođenje prinosa po zero-kuponu iz prinosa kuponskih obveznica - butstrapovanje

Mada se izraz ''butstrapovanje'' koristi kod konstruisanja ''standardnih'' kriva prinosa korišćenjem novca, fjučersa i svopova, gde on opisuje izračunavanje diskontnih faktora za sva svop plaćanja rešavajući tačke rastućih dospeća iterativno, kod krivih za obveznice on implicira unekoliko različit proces. Ovde se ''butstrapovanje'' odnosi na proces u kome se obveznice kojima se trguje na tržištu (paritetne ili neparitetne) kombinuju sa fiktivnim zaduživanjem da bi se eliminisala kuponska međuplaćanja do dospeća. Primenom tog metoda, sva osim inicijalnog i finalnog toka gotovine eliminisana su, izjednačavajuči tako dinamiku plaćanja sa onim postignutim zero-kuponom obveznicom iste ročnosti.

Drugim rečima, pri datoj 1-godišnjoj kamatnoj stopi i, 2-godišnja obveznica A, po ceni od X i koja plaća godišnji kupon x imala bi sledeću strukturu plaćanja:

Inicijalni odliv:	$-X$
Godina 1:	x
Godina 2:	$100 + x$

Jasno, da bi se izjednačili tokovi gotovine između obveznice A i ekvivalentne zero-kupon obveznice Z, kupon x plativ u Godini 1 moraće has found widespread usage in practice, with models developed by Ho and Lee (1986), Hull and White (1990), and Heath-Jarrow-Morton (1992), Black, Derman and Toy (1990), and Black and Karasinski (1991) as the most prominent examples.

Given that most analytical models require initial forward rate term structure as one of the inputs, zero-coupon bond prices are typically used to derive forward yields. However, as noted above, paucity (or even absence) of such instruments in the market implies that further processing of bond data is required for subsequent use in the modeling. Leaving this topic aside for the moment, as it will be revisited in a subsequent discussion, the next section describes derivation of zero-coupon yields from coupon bearing bond market prices.

The relationship between zero-coupon yields and coupon bearing bond market prices

Ideally, when attempting to derive zero-coupon yields from market-traded bonds, instruments of equal maturity priced at par (price equal to redemption value) should be used. The advantage of such approach stems from the fact that, despite the assumption of no-arbitrage in the market, whereby liquidity would soon remove any advantageous investment opportunities arising from mispricing different contracts, comparisons of bonds paying different coupons are notoriously difficult. For example, given two 10-year bonds, paying 4% and 8% annual coupon respectively, the first would clearly be priced lower than the latter. However, the low coupon would imply that most of the bond cashflows will be payable at later dates (i.e. bond duration will be longer), compared to the 8% bond. Thus, in normal market conditions, whereby long-term interest rates are higher, the later cashflows on 4% bond will be worth comparatively more, thus tipping the balance slightly in its favor. As this valuation is subject to many assumptions, it is more convenient to use par bonds in valuations.

However, as it is unlikely that par bonds of all required maturities will be traded in the market at any point in time, it is essential to establish the relationship between zero-coupon, par and non-par bond rates in order to move from one to the other, as required. Here, it is also important to note that such relationships are only valid between contracts in the same market, as different issuers (borrowers) will, due to the differential in their creditworthiness, be able to borrow at different rates. Thus, when choosing contracts for analysis, they all must be issued by e.g. government, or corporates of equal standing. In such case, it will be possible to build a yield curve, from which a par curve as well as zero coupon one can be derived.

Deriving zero-coupon yields from coupon bearing bond yields-Bootstrapping

Although the term 'bootstrapping' is used in construction of 'standard' yield curves using cash, futures and swaps, where it describes calculation of discount factors for all swap payments by solving points of increasing maturities iteratively, in bond curves it implies a somewhat different process. Here 'bootstrapping' refers to the process by which market-traded bonds (either par or non-par) are combined with fictitious borrowing in order to eliminate interim coupon payments until maturity. By applying this method, all but the initial and the final cashflows are eliminated, thus equating the payment schedule to that of a zero-coupon bond of the same maturity.

In other words, given current 1-year interest rate i, a 2-year bond A, priced at X and paying annual coupon x would have the following payment structure:

Initial outlay:	$-X$
Year 1:	x
Year 2:	$100 + x$

Clearly, in order to equate the cashflows between the bond A and an equivalent zero-coupon bond Z, the coupon x payable at Year 1 will have to be eliminated by borrowing the amount b, which will, at current rate i, grow to x that will be repaid at Year 1. In other words, the cashflow structure will become:

da se eliminiše zaduživanjem za iznos *b*, koji će, po tekućoj stopi *i*, porasti na *x* koji će se otplatiti u Godini 1. Drugim rečima, struktura toka gotovine postaće:

Godina	Obveznica	Zaduženje	Ukupno
0	-X	$b = x/(1+i)$	$-X + b$
1	x	$-x$	0
2	$100 + x$		$100 + x$

Otuda, eliminisanjem međuplaćanja, 2-godišnji prinos po zero-kuponu izračunava se kao:

$$z = \left(\frac{100 + x}{X - x/(1+i)}\right)^{1/2}$$

Sličan proces može se primeniti za obveznice dužih ročnosti kojima se trguje na tržištima, gde bi fiktivno zaduživanje negiralo svako kuponsko plaćanje da bi ostali samo inicijalni i finalni tokovi gotovine, iz kojih se može izvesti prinos zero-kupona za tu ročnost.

Izvođenje paritetnih prinosa iz prinosa zero-kupona

Pri datoj navedenoj diskusiji u pogledu odsustva obveznica sa zero-kuponom na tržištu, objašnjenje kako ih koristiti za izvođenje vrednosti koje su bolje vidljive na tržištu može izgledati suvišno. Međutim, pošto prinosi po zero-kuponima služe kao vredan pokazatelj za druge prinose na tržištu, jer oni precizno daju odnos poznatog toka gotovine na konkretan budući datum sa njegovom sadašnjom vrednošću, oni se mogu koristiti za poređenje alternativnih investicionih opcija kao i za proveru koliko je dobro utvrđena cena konkretne obveznice na tržištu. Još važnije, izračunavanje sadašnje vrednosti svih gotovinskih tokova obveznice diskontovanjem zero-kuponske stope za odgovarajuće periode rezultiralo bi cenom, iz koje, korišćenjem jednačine za određivanje cene obveznice, može da se izračuna prinos do dospeća obveznice. Pored toga, kada se struktura obveznice prilagodi paritetnoj ceni korišćenjem diskontnih faktora izvedenih iz stopa zero-kupona, rezultirajući kupon biće jednak paritetnom prinosu obveznice. Tako, uspostavlja se odnos između tri mere prinosa.

Izračunavanje prinosa do dospeća korišćenjem prinosa zero-kupona

Kako je napred napomenuto, prinos do dospeća izračunava se izjednačavanjem cene obveznice dobijene korišćenjem zero-kupon stopa koje odgovaraju pojedinačnim tokovima gotovine (kuponi i otplata) sa cenom izvedenom pod pretpostavkom jedinstvenog prinosa kada se diskontuju svi gotovinski tokovi obveznice. Tako, mora da važi sledeća jednakost:

$$\sum_{n=1}^{N} \frac{C}{(1+z_n)^n} + \frac{R}{(1+z_N)^N} = \sum_{n=1}^{N} \frac{C}{(1+y)^n} + \frac{R}{(1+y)^N}$$

Gde *C* pokazuje kuponska plaćanja, *R* iznos otplate, *N* je broj kupona plativih do dospeća, z_n je zero-stopa za kuponski period *n*, a *y* je prinos obveznice do dospeća.

Treba napomenuti da, radi pojednostavljenja, u gornjem izrazu, pretpostavlja se da se obveznica poseduje od početka ugovora do dospeća, tj. ne valorizuje se na sekundarnom tržištu (nema transfera ugovora). Ako bi to bio slučaj, bilo bi verovatno da bi, u vreme kupovine/prodaje, prodavac obveznice bi bio kompenzovan za deo vrednosti kupona koji je prispeo od poslednjeg plaćanja. Sledstveno, prispeli deo kupona bi morao da se inkorporiše u gornji izraz. Dalje, svi porezi, provizije ili bilo koji drugi troškovi takođe se zanemaruju.

Izračunavanje paritetnog prinosa iz diskontnih faktora zero-kupona

Kako je napomenuto ranije, obveznica koja dobija paritetnu cenu korišćenjem zero-kupon stopa za diskontovanje svih gotovinskih tokova imaće kupon jednak svom paritetnom prinosu. Tako, za obveznicu od N-godina koja plaća godišnje kupone, sledeći odnos između paritetnog prinosa *i* i diskontnih faktora zero-kupona (označeno kao df_k, gde je $k = 1, \ldots, N$) mora da važi:

$$(i \times \mathrm{df}_1) + (i \times \mathrm{df}_2) + \ldots + (i \times \mathrm{df}_N) + (1 \times \mathrm{df}_N) = 1$$

Otuda,

$$i \times (\mathrm{df}_1 + \mathrm{df}_2 + \ldots + \mathrm{df}_N) = 1 - \mathrm{df}_N$$

Year	Bond	Borrowing	Total
0	-*X*	$b = x/(1+i)$	-*X* + *b*
1	*x*	-*x*	0
2	100 + *x*		100 + *x*

Consequently, the by eliminating the interim payments, the 2-year zero-coupon yield is calculated as:

$$z = \left(\frac{100 + x}{X - x/(1+i)}\right)^{1/2}$$

Similar process can be applied for market traded bonds of longer maturities, whereby fictitious borrowing would negate each coupon payment to leave only the initial and the final cashflows, from which a zero-coupon yield for that maturity can be derived.

Deriving par yields from zero-coupon yields

Given the above discussion regarding the absence of zero coupon bonds in the market, the explanation on how to use them to derive values that are more readily observable in the market may seem redundant. However, given that zero-coupon yields serve as a valuable anchor for other yields in the market, for they precisely relate a known cahsflow at a specific future date with its present value, they can be used for comparing alternative investment options as well as checking how well a specific bond is priced in the market. More importantly, calculating present value of all bond cashflows by discounting at the zero-coupon rate for the corresponding periods would result in a price, from which, using the bond pricing equation, bond's *yield to maturity* can be calculated. In addition, when the bond structure is adjusted to price at par using discount factors derived from zero-coupon rates, its resulting coupon will be equal to the *bond par yield*. Thus, the relationship between the three yield measures is established.

Calculating yield to maturity using zero-coupon yields

As noted above, yield to maturity is calculated by equating the bond price obtained using the zero-coupon rates corresponding to individual cashflows (coupons and the redemption) with the price derived by assuming a single yield when discounting all bond cashflows. Thus, the following equality must hold:

$$\sum_{n=1}^{N} \frac{C}{(1+z_n)^n} + \frac{R}{(1+z_N)^N} = \sum_{n=1}^{N} \frac{C}{(1+y)^n} + \frac{R}{(1+y)^N}$$

where C indicates coupon payments, R redemption amount, N is the number of coupons payable until maturity, z_n is the zero-rate for the coupon period n, and y is the bond yield to maturity.

It should be noted that, for simplification, in the above expression, it is assumed that the bond is held from the contract initiation to maturity, i.e. it is not valued in the secondary market (there is no transfer of contract). If that were the case, it would be likely that, at the time of purchase/sale, some coupon would have accrued since the last payment and the bond seller would have to be compensated for it. Consequently, accrued coupon would have to be incorporated into the above expression. Moreover, all taxes, fees or any other charges are also ignored.

Calculating par yield from zero-coupon discount factors

As noted earlier, a bond that prices at par using zero-coupon rates to discount all the cashflows will have coupon equal to its par yield. Thus, for a N-year bond paying annual coupons, the following relationship between its par yield i and the zero-coupon discount factors (denoted as df_k, where $k = 1, \ldots, N$) must hold:

$$(i \times \mathrm{df}_1) + (i \times \mathrm{df}_2) + \ldots + (i \times \mathrm{df}_N) + (1 \times \mathrm{df}_N) = 1$$

Hence,

$$i \times (\mathrm{df}_1 + \mathrm{df}_2 + \ldots + \mathrm{df}_N) = 1 - \mathrm{df}_N$$

which gives the relationship between zero-coupon and par yield:

$$i_N = \frac{1 - \mathrm{df}_N}{\sum_{k=1}^{N} \mathrm{df}_k}$$

where i_N is a N-year par yield and df_k is zero-što daje odnos između zero-kupona i paritetnog prinosa:

$$i_N = \frac{1 - \mathrm{df}_N}{\sum_{k=1}^{N} \mathrm{df}_k}$$

gde i_N predstavlja N-godišnji paritetni prinos a df_k je zero-kupon diskontni faktor za period od k godina, izveden iz zero-stopa z_k kao $1/(1+z_k)^k$

Izvođenje forvord-forvord prinosa iz prinosa po zero-kuponu

Izvođenje bilo koje forvord-forvord stope zasniva se na ključnom konceptu da, ako je iznos glavnice uzet na kredit za, na pr. m godina i odmah deponovan za prvih k godina, neto rezultat u inicijalnom periodu je nula, stvarajući na taj način forvord-forvord zaduženje koje počinje k godina od sada i ističe na kraju godine m. Da bi se ove dve strategije izjednačile, forvord-forvord stopa treba da je jednaka:

$$\text{Forvord-forvord stopa} = \left[\frac{(1+z_m)^m}{(1+z_k)^k}\right]^{\frac{1}{m-k}} - 1$$

Gornji izraz podjednako važi za stope zero-kupona kao i za kratkoročne instrumente kao što su FRA, čija se cena zapravo utvrđuje na osnovu ovog principa (mada korišćenjem tačnih konvencija o brojanju dana da bi se odrazila njihova kraća dospeća).

Diskusija

Napred prezentovane jednačine služe izračunavanju prinosa u specifičnim vremenskim tačkama, tj. u vreme gotovinskih tokova obveznica (kupona i iznosa otplate). Međutim, izraz 'kriva prinosa' implicira glatku funkciju koja pruža vrednost prinosa u bilo koje vreme. Modeliranje krive (nezavisno od instrumenata korišćenih za izvođenje osnovnih tačaka podataka izvedenih sa tržišta) predstavlja kompleksno pitanje, o kome se široko debatuje u akademskom svetu. Jasno, praktičari na tržištu želeli bi glatku, pouzdanu krivu iz koje se mogu izvući sve tržišne cene korišćene za njenu konstrukciju. Dalje, s obzirom na to da, zbog propisa i računovodstva, većina institucija moraju da utvrde cene svih svojih instrumenata iz jedine krive za datu valutu, konstruisanje krive prinosa koja bi ispunila sve navedene kriterijume težak je zadatak. Tipično, podešavanje krive ima za cilj postizanje optimalnog kompromisa između glatkosti i minimalnih grešaka kod izvlačenja tržišnih podataka. Tokom godina, korišćeno je nekoliko pristupa. Mada Bhar i Chiarella (1996) nude odličan istorijski pregled u svom radu, neki aspekti razvoja modeliranja krive daće se ovde. Kada se modelira kriva, moraju da se daju neke pretpostavke njene funkcionalne forme. Na primer, McCulloch (1975) i Chambers, Carleton i Waldman (1984) aproksimirali su funkciju sadašnje vrednosti polinomom, izjednačavajući izvođenje ročne strukture kamatne stope sa procenom parametara polinoma. Međutim, Shea (1985) je pokazao da takav pristup ne modelira dugoročnu strukturu sa dovoljno tačnosti. To je gledište koje su ponudili Vasicek i Fong (1982), tvrdeći da oblik koji proizvode polinomski 'spline-ovi' obično ne odgovara obliku ročne strukture prinosa koja se vidi na tržištu, što je takođe pokazano u slučaju eksponencijalnih 'spline-ova' kod Shea (1984). Skorije, pri datom povećanom broju derivatnih instrumenata čije vrednovanje (kako je ukazano nizom modela prikazanim u ovoj studiji) i isplate zavise od razvoja krive, pažnja je usmerena na postizanje krivih terminske stope koje rezltiraju iz tržišno impliciranih stopa zero-kupona. Neki primeri radova koji imaju za cilj maksimizovanje glatkosti jesu Adams i Van Deventer (1994), Konno i Takase (1995) i Kim, Moon i Lee (1997). Međutim, pošto se veliki deo kvantitativnog istraživanja događa u institucijama koje aktivno učestvuju na tržištu, one teže da usvoje vlastite pristupe konstrukciji krive prinosa koji se retko objavljuju. Tako, radovi prezentovani ovde samo su indikacija trenutnog smera razvoja u ovoj oblasti.

Zaključci

Mada odnosi dati napred izgledaju jednostavni i u teoriji treba da proizvode konzistentne vrednosti različitih mera prinosa, u praksi to nije uvek slučaj. Ipak, imati alate za poređenje različitih instrumenata kojima se coupon discount factor for the period of k years, derived from zero-rates z_k as $1/(1+z_k)^k$

Deriving forward-forward yields from zero-coupon yields

Deriving any forward-forward rate rests on the key concept that, if a principal amount is borrowed for, e.g., m years and immediately deposited for the first k, the net result in the initial period is zero, thus creating a forward-forward borrowing that starts k years from now and expires at the end of year m. In order to make these two strategies equivalent, forward-forward rate should be equal to:

$$\text{Forward-forward rate} = \left[\frac{(1+z_m)^m}{(1+z_k)^k}\right]^{\frac{1}{m-k}} - 1$$

The above expression is equally valid for zero-coupon rates as is for shorter dated instruments, such as FRAs, which are, indeed, priced based on this principle (albeit using exact daycount conventions to reflect their shorter maturities).

Discussion

The equations presented above serve to calculate yields at specific points in time, i.e. at the time of the bond cashflows (coupons and redemption amount). However, a term 'yield curve' implies a smooth function that can provide yield value for any point in time. Modelling yield curves (irrespective of the instruments used to derive the basic data points derived from the market) is a complex issue, widely debated in the academic field. Clearly, market practitioners would like a smooth, well behaved curve from which all market prices used to construct it can be recovered. Moreover, given that, for regulatory and accounting purposes, most institutions are required to price all their instruments off a single curve for a given currency, constructing a yield curve that would meet all the above criteria is a tall order. Typically, curve fitting aims to achieve optimal compromise between smoothness and minimal errors when recovering market data. Over the years, several approaches have been used. Although Bhar and Chiarella (1996) offer excellent historical overview in their paper, some aspects of the curve modelling evolution will still be highlighted here. When modelling a curve, some assumptions of its functional form must be made. For example, both McCulloch (1975) and Charmbers, Carleton and Waldman (1984) approximated the present value function by a polynomial, thus making derivation of the interest rate term structure equivalent to estimating the parameters of the polynomial. However, Shea (1985) showed that such approach fails to model the long-dated term structure with sufficient accuracy. This is the view offered by Vasicek and Fong (1982), who argue that shape produced by polynomial splines generally does not correspond to that of the yield term structure observed in the market, which was also shown to be the case for exponential splines by Shea (1984). More recently, given the increased number of derivative instruments whose valuation (as indicated by the range of models given in the background of this study) and payouts are contingent on curve evolution, the focus has been on achieving smooth forward rate curves that result from market-implied zero-coupon rates. Some examples of works that aim to maximize smoothness are Adams and Van Deventer (1994), Konno and Takase (1995) and Kim, Moon and Lee (1997). However, as a large portion of quantitative research takes place in institutions that actively participate in the market, they tend to adopt proprietary yield curve construction approaches that are rarely published. Thus, the works presented here are only an indication of the direction developments in this field are likely to be taking currently.

Conclusions

Although the relationships given above seem simple, and should in theory produce consistent values of different yield measures, in practice this is not always the case. Nonetheless, having the tools for comparison of different market-traded instruments is always beneficial for investors. They, however, must use their judgment to decide whether bonds priced above their theoretical value are actually overpriced, as there may be other practical considerations that influence both the market price and the theoretical valuation. Moreover, when valuing bonds, it is their 'dirty' price (one that includes trguje na tržištima uvek je korisno za investitore. Oni, međutim, moraju da koriste svoje rasuđivanje kod odlučivanja da li su obveznice sa cenom iznad njihove teoretske vrednosti stvarno precenjene, jer može biti drugih praktičnih razloga koji utiču na tržišne cene i teoretsko vrednovanje. Dalje, kada se vrednuju obveznice, treba da se koristi njihova 'prljava cena' (cena koja uključuje narasli kupon), što se radi pojednostavljenja zanemaruje u navedenom izračunu. Pored toga, pri datoj malobrojnosti (ili čak odsustvu) instrummenata sa zero-kuponom na tržištu, njihovo vrednovanje je uveliko teoretsko i često dalje komplikovano teškoćom nalaženja pogodnih ugovora u istom sektoru sa svim ročnostima potrebnim za 'butstraping'. Otuda, aproksimacija uvedena u izračunavanje zero-stopa (kao i konsgtruisanje krive) dalje će smanjivati pouzdanost bilo kakvog zaključivanja iz njihove primene u poređenju ugovora kojima se trguje na tržištu. Kao opšte pravilo, pozitivna kriva prinosa imaće pozitivniju odgovarajuću zero-krivu i obrnuto. Pored toga, forvord-forvord kriva (konstruisana izvođenjem terminskih stopa iz odgovarajućih diskontnih faktora zero-kupona) biće još više naglašena, tipično sa nekom konveksnošću u svom obliku. To se mora uzeti u obzir kada se odlučuje o proceni vrednosti bilo kog instrumenta koristeći konkretnu krivu.

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the accrued coupon) that should be used, which was, for simplification, ignored in the above calculation. In addition, given the paucity (or even absence) of zero-coupon instruments in the market, their valuation is largely theoretical and is often further complicated by the difficulty of finding suitable contracts in the same sector with all the maturities required for bootstrapping. Hence, approximations introduced in zero-rate calculations (as well as curve construction) will further reduce the reliability of any conclusions made by their application in comparing market-traded contracts. As a general rule, a positive par yield curve will have a more positive corresponding zero-curve, and vice versa. In addition, forward-forward curve (constructed by deriving forward rates from corresponding zero-coupon discount factors) will be even more exaggerated, typically with some convexity in its shape. This must be taken into account when deciding to price any instruments using a particular curve.

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