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KAMATNI RIZIK ULAGANJA U OBVEZNICE - NEKONVENCIONALNE METODE MERENJA

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Rezime

Kamatni rizik obveznice najčešće se meri trajanjem i konveksnošću. Međutim, ove mere polaze od pretpostavke o ravnoj krivi prinosa i njenom paralelnom pomeranju. Za modeliranje realnijih slučajeva koriste se njihove modifikacije. Fisher-Weil-ovo trajanje služi za merenje osetljivosti na paralelno pomeranje neravne krive prinosa. Mere M-apsolutno i M-kvadrat pokazuju u kojoj meri je portfolio obveznica imunizovan na neparalelne promene krive prinosa, uzimajući u obzir dati vremenski horizont ulaganja. Neravna kriva prinosa može se aproksimirati i skupom odabranih ključnih kamatnih stopa, čija trajanja i konveksnosti mere osetljivost portfolija na promene ovih pojedinačnih kamatnih stopa.

Ključne reči: Fisher-Weil-ovo trajanje, kvazimodifikovano trajanje, M-apsolutno, M-kvadrat, trajanje ključnih kamatnih stopa

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INTEREST RATE RISK IN BOND INVESTMENT - UNCONVENTIONAL MEASUREMENT METHODS

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Summary

Interest rate risk of a bond is typically measured by means of duration and convexity. However, these measurements are based on the assumption of a flat yield curve and its parallel shifts. For the purpose of modelling more realistic cases, their modifications are used. Fisher-Weil duration is used to measure the sensitivity to parallel movements of a non-flat yield curve. M-absolute and M-square indicate to which extent a bond portfolio is immunized to non-parallel shifts of the yield curve, taking into account the given time horizon of the concerned investment. A non-flat yield curve can also be approximated by a set of selected key rates, whose duration and convexity measure the portfolio's sensitivity to the changes in specific interest rates.

Keywords: Fisher-Weil duration, quasi-modified duration, M-absolute, M-square, key rate duration

JEL: G11, G21

Uvod

Kamatni rizik predstavlja jedan od najznačajnijih rizika ulaganja u obveznice, koji se sastoji u tome da se cena obveznice menja u suprotnom smeru od promena tržišnih kamatnih stopa. Za njegovo merenje tradicionalno se koriste trajanje i konveksnost. Problem sa ovim pokazateljima je u tome što oni ne uvažavaju činjenicu da kriva prinosa nije ravna i da se često ne pomera paralelno.

U prvom delu rada objašnjavaju se osnovni pokazatelji kamatnog rizika, nakon čega se uvode njihove modifikacije koje uzimaju u obzir neravnu krivu prinosa. U drugom delu će biti predstavljeni pokazatelji kamatnog rizika koji mere disperziju novčanih tokova portfolija u odnosu na zadati vremenski horizont ulaganja. Pokazuje se da njih treba minimizovati u slučaju da se očekuju neparalelna pomeranja krive prinosa. U trećem delu se, polazeći od pretpostavke da se kriva prinosa može aproksimirati konačnim brojem tzv. ključnih kamatnih stopa, pokazuje kako se pomoću njih modelira njeno neparalelno pomeranje.

Klasične mere kamatnog rizika

Budući da je cena obveznice određena vremenom do dospeća, kuponskom stopom i prinosom do dospeća, cene obveznica sa različitim kuponskim stopama i različitim vremenima do dospeća će različito reagovati na identičnu promenu prinosa do dospeća. Trajanje obveznice je mera koja se koristi za poređenje osetljivosti različitih obveznica, kao i za računanje očekivane promene vrednosti portfolija obveznica, budući da bi računanje pojedinačnih promena bilo računski neefikasno. Najjednostavnija mera trajanja je **Macauley-evo trajanje** (D), koje se računa po sledećoj formuli:

$$D = \frac{1}{P} \sum_{t=1}^T t \frac{CF_t}{(1+y)^t}$$

gde je: y - prinos do dospeća, T - vreme do dospeća, P - cena, CF_t - novčani tok u trenutku t . Drugim rečima, Macauley-evo trajanje je ponderisano vreme do dospeća obveznice, gde se vreme do dospeća svakog novčanog toka ponderiše učešćem sadašnje vrednosti tog novčanog toka u ceni obveznice. Sledi da

je trajanje beskuponske obveznice jednako njenom vremenu do dospeća, budući da ona ima samo jedan novčani tok, koji pristiže na kraju njenog životnog veka. Za sve kuponske obveznice trajanje je nužno manje od vremena do dospeća, jer će ponder poslednjeg novčanog toka biti manji od 1, dok će se povećati ponderi ranijih perioda.

Trajanje obveznice, kao i volatilnost cene, zavisi od visine kuponske stope. Što je veća kuponska stopa, veći su novčani tokovi koji se isplaćuju pre roka dospeća, a time i njihova sadašnja vrednost (posebno ranijih tokova, jer imaju veći diskontni faktor), pa se trajanje obveznice smanjuje. Sa druge strane, trajanje se uglavnom povećava sa povećanjem roka dospeća, da bi kod kuponskih obveznica koje se prodaju uz veliki diskont u jednom trenutku počelo da pada. Kod svih kuponskih obveznica (i diskontnih i premijskih), sa povećanjem roka dospeća, trajanje se približava trajanju perpetuiteta sa datim prinosom do dospeća (za obveznicu bez dospeća (perpetuitet) može se dokazati da je njeno Macauley-evo trajanje jednako $(1+y)/y$). Međutim, ispostavlja se da na trajanje utiče i visina prinosa do dospeća - što je veći početni prinos, trajanje je niže. (Povećanje prinosa dovodi do smanjivanja svih diskontnih faktora, uz relativno veće smanjenje diskontnih faktora u kasnijim godinama, što dovodi do toga da se veći ponderi dodeljuju početnim novčanim tokovima.)

Ako Macauley-evo trajanje podelimo bruto prinosom do dospeća, dobijamo **modifikovano trajanje** (D_m), koje pokazuje procentualnu promenu cene ako se prinos promeni za jedan procentni poen:

$$\frac{dP}{P} = -\frac{1}{1+y} D dy = -D_m dy$$

Vidimo da se za aproksimaciju promene cene koristi prvi izvod. Iz definicije diferencijala znamo da on predstavlja proizvod prvog izvoda funkcije i infinitezimalno male promene njenog argumenta (to jest, $dP = P'_y dy, dy \rightarrow 0$), što znači da on predstavlja skoro savršenu aproksimaciju za vrlo male promene argumenta (u ovom slučaju prinosa do dospeća), a savršenu samo ako je funkcija linearna (kada je njen prvi izvod konstantan). Odnos cene i prinosa obveznice je najčešće strogo konveksan, što znači da ne možemo koristiti linearnu aproksimaciju za

Introduction

Interest rate risk is one of the most significant risks when it comes to investing in bonds, arising from the possibility of a bond price to shift in the opposite direction from the changes in market interest rates. Traditionally, to measure this risk we use duration and convexity. Yet, the problem with these indicators is that they do not take into account the fact that a yield curve often is not flat, and that it frequently does not record parallel shifts.

The first part of the paper explains the main indicators of interest rate risk, after which we introduce their modifications taking into consideration a non-flat yield curve. The second part will be focusing on interest rate risk indicators measuring the portfolio's cash flows dispersion in relation to the given time horizon of the concerned investment. It is demonstrated how they should be minimized in case that non-parallel shifts of the yield curve are expected. Starting from the assumption that the yield curve can be approximated by means of a finite number of the so-called key rates, the third part illustrates how to use them to model its non-parallel movements.

Classic Interest Rate Risk Measurements

Given that the bond price is determined by its maturity, coupon rate and yield to maturity, the prices of bonds with different coupon rates and different maturities will react differently to the identical change in yield to maturity. Bond duration is a measure used to compare the sensitivity of various bonds, and to calculate the expected changes in the bond portfolio's value, given that the calculation of individual changes would be inefficient. The simplest duration measure is **Macaulay duration** (D), calculated according to the following formula:

$$D = \frac{1}{P} \sum_{t=1}^T t \frac{CF_t}{(1+y)^t}$$

with: y - yield to maturity, T - time to maturity, P - price, CF_t - cash flow at the moment t . In other words, Macaulay duration is the weighted time until the bond's maturity, with time to maturity of each cash flow being weighted by the share

of that cash flow's present value in the bond's price. This implies that the duration of a zero coupon bond equals time to maturity, given that it only has one cash flow, maturing at the end of its life cycle. For all coupon bonds duration is necessarily shorter than time to maturity, because the weight of the last cash flow will be less than 1, and the weights of earlier periods will increase.

Duration of a bond, just like its price volatility, depends on the coupon rate level. The higher the coupon rate, the higher the cash flows disbursed before maturity, and thereby also their present value (especially of earlier cash flows, due to their higher discounting factor), hence the bond duration decreases. On the other hand, duration typically increases in parallel with maturity, only to start declining at one point in case of coupon bonds sold at a huge discount. In case of all coupon bonds (both discount and premium), as the maturity increases, duration approaches the duration of a perpetuity at the given yield to maturity (a bond with no maturity (i.e. perpetual bond) can be proven to have Macaulay duration which equals $(1+y)/y$). However, it turns out that the duration is also affected by the size of yield to maturity - the higher the initial yield, the lower the duration. (Increased yield leads to a reduction of all discount factors, with a relatively higher reduction of discount factors in later years, which results in bigger weights being awarded to initial cash flows.)

If we divide Macaulay duration by gross yield to maturity, we get **modified duration** (D_m), which indicates the percentage of the price change if the yield changes by one percentage point:

$$\frac{dP}{P} = -\frac{1}{1+y} D dy = -D_m dy$$

We can see that the first derivative is used to approximate the price change. Based on the definition of differentials, we know that it is the multiplication of the first derivative of the function and the infinitesimal change of its argument (i.e., $dP = P'_y dy$, $dy \rightarrow 0$), which means that it represents an almost perfect approximation for very small changes of the argument (in this case, yield to maturity), and a perfect approximation only if the function is linear (when its first derivative is constant).

veće promene prinosa. Budući da prvi izvod u geometrijskom smislu predstavlja tangentu na grafik funkcije, i znajući da je funkcija konveksna, zaključujemo da će aproksimacija prvim izvodom nužno dovesti do potcenjenosti cene u odnosu na njenu stvarnu vrednost. Ova potcenjenost će biti utoliko veća ukoliko kriva cena/prinos više odstupa od linearnog oblika (to jest, što je više zakrivljena / konveksna).

Kod većih promena prinosa dolazi do značajnijeg odstupanja projektovanih od stvarnih cena. Da bi se otklonio ovaj nedostatak, koristi se dodatna mera, koja je nazvana **konveksnost** jer je povezana sa zakrivljenošću krive cena-prinos.

Promenu cene možemo aproksimirati Tejlorovim polinomom drugog stepena:

$$\Delta P \approx -D_m P \Delta y + \frac{1}{2} C P (\Delta y)^2$$

gde $C = \frac{d^2 P}{dy^2}$ predstavlja konveksnost obveznice. Iz toga sledi da je konveksnost jednaka:

$$C = \frac{1}{P(1+y)^2} \sum_{t=1}^T \frac{t(t+1)CF_t}{(1+y)^t}$$

Vrednost konveksnosti sama po sebi nema nikakvo korisno značenje. Nju je potrebno dovesti u vezu sa kvadratom promene prinosa.

Smisao računanja trajanja i konveksnosti jeste u njihovom korišćenju za zaštitu portfolija od promena kamatnih stopa. Ova zaštita (hedžing) naziva se imunizacija, zato što se portfolio imunizuje tj. čini „otpornim“ na promenu kamatnih stopa. Suština imunizacije ogleda se u tome da se gubici na vrednosti imunizovanog portfolija nadoknade dobitkom u vrednosti zaštitnog portfolija (i obrnuto). Zaštitni portfolio se konstruiše tako da su njegovo trajanje i konveksnost jednaki onima imunizovanog portfolija, dok mu je vrednost suprotna (što znači da se klasičan portfolio imunizuje prodajom na kratko, a buduća obaveza kupovinom zaštitnih instrumenata).

Macaulay-evo i modifikovano trajanje implicitno podrazumevaju da je kriva prinosa ravna, zato što se svi novčani tokovi diskontuju istom stopom prinosa. Postoje druge mere trajanja koje su zasnovane na realnijim pretpostavkama. Jedno od njih je **Fisher-Weil-ovo trajanje**, koje se definiše pomoću spot stopa, na sledeći način (Urošević, Božović, 2009, str. 184):

$$D_{FW} = \frac{1}{P} \sum_{t=1}^T t CF_t e^{-s_t t}$$

gde su s_t spot stope pri kontinualnom ukamaćenju. Ono meri osetljivost cene obveznice na paralelno pomeranje spot krive. Ako stepen pomeranja spot krive označimo sa λ , nova cena iznosi:

$$P(\lambda) = \sum_{t=1}^T CF_t e^{-(s_t + \lambda)t}$$

a osetljivost na λ iznosi:

$$\frac{dP(0)}{d\lambda} = - \sum_{t=1}^T t CF_t e^{-s_t t}$$

tako da je

$$D_{FW} = - \frac{1}{P(0)} \frac{dP(0)}{d\lambda}$$

Slična mera je i **kvazimodifikovano trajanje**, koje koristi spot stope obračunate na godišnjem nivou:

$$D_Q = - \frac{1}{P(0)} \frac{dP(0)}{d\lambda}$$

gde je

$$P(\lambda) = \sum_{t=1}^T \frac{CF_t}{(1 + s_t + \lambda)^t}$$

$$\frac{dP(0)}{d\lambda} = - \sum_{t=1}^T t \frac{CF_t}{(1 + s_t)^{t+1}}$$

M-apsolutno i M-kvadrat

Svakoј terminskoj strukturi beskuponskih stopa prinosa (eng. *zero-coupon yields*) odgovara jedinstvena terminska struktura trenutnih forward stopa (eng. *instantaneous forward rates*). One su nam potrebne da bismo lakše uočili izvesne zakonitosti koje važe za pokazatelje kamatnog rizika koji se razmatraju u ovom odeljku.

Trenutna forward stopa za dospeće $t - f(t)$ - definiše se na sledeći način:

$$s_t t = \int_0^t f(s) ds$$

odnosno:

$$f(t) = \frac{\partial s_t}{\partial t} t + s_t$$

The relationship between the bond price and its yield is most frequently convex, which implies that we cannot use linear approximation for bigger changes in the yield. Given that the first derivative in geometrical terms represents a tangent on the function graph, and knowing that the function is convex, we may conclude that the first derivative approximation will necessarily lead to an underestimated price in relation to its real value. This underestimation will be all the bigger if the price/yield curve deviates more significantly from the linear form (i.e. the more curved/convex it is).

The bigger changes of yield result in more substantial deviations of projected prices from the real ones. To eliminate this drawback, an additional measurement is used, the so-called **convexity**, which is related to the slope of the price-yield curve.

The price change can be approximated by means of the second-degree Taylor polynomial:

$$\Delta P \approx -D_m P \Delta y + \frac{1}{2} CP (\Delta y)^2$$

with $C = \frac{d^2 P}{dy^2} \frac{1}{P}$ representing the bond's convexity. Therefore, the convexity equals:

$$C = \frac{1}{P(1+y)^2} \sum_{t=1}^T \frac{t(t+1)CF_t}{(1+y)^t}$$

The value of convexity itself does not have any useful meaning. It needs to be linked with the squared change in yield.

The point of calculating duration and convexity is in their usage to hedge the portfolio against interest rate changes. Such a hedge is referred to as immunization, because the portfolio is being immunized, i.e. made "immune" to interest rate changes. The essence of immunization is reflected in the losses based on the immunized portfolio's value being compensated by the gains in the hedged portfolio's value (and vice versa). The hedged portfolio is constructed in such a way as to make its duration and convexity equal to those of the immunized portfolio, whereas its value is the opposite (which means that a classic portfolio is immunized by means of short sales, whereas future obligations get immunized by purchasing hedge instruments).

Macauley and modified duration imply that the yield curve is flat, because all cash flows

are discounted by the same rate of return. There are other duration measurements based on more realistic assumptions. One of them is **Fisher-Weil duration**, defined by means of spot rates, in the following way (Urošević, Božović, 2009, p. 184):

$$D_{FW} = \frac{1}{P} \sum_{t=1}^T t CF_t e^{-s_t t}$$

with s_t being spot rates in case of continuous interest income. It measures the sensitivity of the bond price to the parallel shifts of the spot curve. If we mark the degree of spot curve movements with λ , the new price amounts to:

$$P(\lambda) = \sum_{t=1}^T CF_t e^{-(s_t + \lambda)t}$$

and its sensitivity to λ is:

$$\frac{dP(0)}{d\lambda} = - \sum_{t=1}^T t CF_t e^{-s_t t}$$

so that:

$$D_{FW} = - \frac{1}{P(0)} \frac{dP(0)}{d\lambda}$$

A similar measurement is the **quasi-modified duration**, using spot rates calculated at the annual level:

$$D_Q = - \frac{1}{P(0)} \frac{dP(0)}{d\lambda}$$

with:

$$P(\lambda) = \sum_{t=1}^T \frac{CF_t}{(1 + s_t + \lambda)^t}$$

$$\frac{dP(0)}{d\lambda} = - \sum_{t=1}^T t \frac{CF_t}{(1 + s_t)^{t+1}}$$

M-Absolute and M-Square

Each term structure of zero-coupon yields responds to a unique term structure of instantaneous forward rates. We need them for the purpose of detecting more easily certain rules in respect of interest rate risk indicators examined in this chapter.

The instantaneous forward rate for maturity $t - f(t)$ - is defined in the following way:

$$s_t t = \int_0^t f(s) ds$$

Samim tim, svakoj promeni terminske strukture besкупonskih stopa, Δs_t odgovara jedinstvena promena terminske strukture definisanoj preko trenutnih forward stopa, $\Delta f(t)$. Međutim, trenutne forward stope su volatilnije od besкупonskih stopa, budući da besкупonske stope predstavljaju neku vrstu proseka trenutnih forward stopa. Vidimo da su trenutne forward stope veće od besкупonskih stopa u slučaju kada je (besкупonska) kriva prinosa rastuća, i obrnuto.

U prethodnom odeljku je pokazano da svi pokazatelji trajanja (Macaulay-evo, modifikovano, Fisher-Weil-ovo i kvazimodifikovano) polaze od pretpostavke da se kriva prinosa pomera paralelno. Imunizacija korišćenjem trajanja će, stoga, implicirati da treba konstruisati zaštitni portfolio čije trajanje je jednako trajanju imunizovanog portfolija. Pri tome će portfolio menadžeru biti svejedno koje obveznice koristi za tu svrhu, u smislu da će npr. portfolio sastavljen od besкупonskih obveznica koje imaju ročnost po 2 i 10 godina biti podjednako prihvatljiv kao i portfolio sastavljen samo od besкупonskih obveznica sa dospećem od 6 godina.

Mere kamatnog rizika koje ovde definišemo zavise od vremenskog horizonta H na čijem kraju želimo da portfolio bude imunizovan. Prva takva mera naziva se **M-apsolutno** (eng. *M-absolute*), zbog toga što predstavlja ponderisanu apsolutnu vrednost razlike između vremena dospeća novčanih tokova i vremenskog horizonta ulaganja:

$$M^A = \frac{1}{P} \sum_{t=t_1}^{t_n} |t - H| \frac{CF_t}{e^{\int_0^t f(s) ds}}$$

Sledi da Fisher-Weil-ovo trajanje predstavlja specijalan slučaj M-apsolutnog kada je $H=0$. Vidimo da je M-apsolutno jednako nuli samo onda kada portfolio čini besкупonska obveznica čije je vreme dospeća jednako vremenskom horizontu ulaganja.

M-apsolutno je pokazatelj koji, za razliku od trajanja, treba minimizovati. Da bismo razumeli zašto, poći ćemo od toga da portfolio menadžer želi da minimizuje odstupanje vrednosti portfolija od njegove ciljane vrednosti u trenutku H , V_H . Drugim rečima, minimizuje se $\Delta V_H / V_H$, pri čemu odstupanje nastaje usled

promene kamatnih stopa u trenutku $t=0$.

Definišimo konstante K_1 , K_2 i K_3 na sledeći način:

$$K_1 \leq \Delta f(t) \text{ za svako } t$$

$$K_2 \geq \Delta f(t) \text{ za svako } t$$

$$K_3 = -\min\{K_1, -K_2\} = \max\{|K_1|, |K_2|\}$$

Ako pretpostavimo da je $CF_t \geq 0$ za svako t , dobija se (Nawalkha, Soto, Beliaeva, 2005, str. 105):

$$\frac{\Delta V_H}{V_H} \geq \frac{1}{V_0} \sum_{t=t_1}^{t_n} CF_t e^{-\int_0^t f(s) ds} (-K_2 |H - t|) = -K_2 M^A$$

K_3 zavisi od promena trenutnih forward stopa, te nije pod kontrolom portfolio menadžera. Ono na šta on može da utiče jeste M^A , koje se može smanjiti odabirom hartija čiji su novčani tokovi bliži vremenskom horizontu H .

Izjednačavajući trajanje portfolija sa vremenskim horizontom ulaganja, portfolio se štiti od malih i paralelnih pomeranja krive prinosa. Za razliku od toga, minimiziranje M-apsolutnog uglavnom neće u potpunosti zaštititi portfolio od takvih promena, ali će uzeti u obzir mogućnost većih i/ili neparalelnih promena krive prinosa. Zbog toga će izbor između ove dve mere zavisiti od toga kakve se promene krive prinosa očekuju.

Poređenje imunizacionih strategija zasnovanih na trajanju, odnosno na M-apsolutnom, obavili su Nawalkha i Chambers (2009) na podacima za period 1951-1986. Njihov rezultat je da je primena M-apsolutnog više nego dvostruko smanjila odstupanja od ciljane vrednosti portfolija u odnosu na strategiju imunizacije pomoću trajanja.

Druga mera koja zavisi od vremenskog horizonta je **M-kvadrat** (eng. *M-square*):

$$M^2 = \frac{1}{P} \sum_{t=t_1}^{t_n} (t - H)^2 \frac{CF_t}{e^{\int_0^t f(s) ds}}$$

Pošto u slučaju kontinualnog ukamaćenja važi:

$$C = \sum_t w_t t^2$$

sledi:

$$M^2 = \sum_t w_t (t^2 - 2tH + H^2) = C - 2DH + H^2$$

i.e.:

$$f(t) = \frac{\partial s_t}{\partial t} t + s_t$$

Therefore, each change in the term structure of zero-coupon yields, Δs_t , responds to a unique change in the term structure defined through instantaneous forward rates, $\Delta f(t)$. However, instantaneous forward rates are more volatile than zero-coupon yields, given that zero-coupon yields in a way represent the average of instantaneous forward rates. We observe that instantaneous forward rates are higher than zero-coupon yields when the (zero-coupon) yield curve has an upward slope, and vice versa.

The previous section illustrates that all duration indicators (Macaulay, modified, Fisher-Weil, and quasi-modified) start from the assumption that the yield curve has parallel shifts. Immunization by means of duration will, therefore, imply that one should construct a hedged portfolio whose duration equals the duration of the immunized portfolio. In the process, it does not matter which bonds the portfolio manager uses for the purpose, i.e. because the portfolio containing, for instance, zero-coupon bonds with 2- and 10-year maturity, will be equally acceptable as the portfolio containing only zero-coupon bonds with 6-year maturity.

The interest rate risk measures that we hereby define depend on the time horizon H , at the end of which we want the portfolio to be immunized. The first such measure is called **M-absolute**, because it represents the weighted absolute value of the difference between time to maturity of cash flows and time horizon of the concerned investment:

$$M^A = \frac{1}{P} \sum_{t=t_1}^{t_n} |t - H| \frac{CF_t}{e^{\int_0^t f(s) ds}}$$

It can be deduced that Fisher-Weil duration is a special case of M-absolute when $H=0$. We can see that M-absolute equals zero only when the portfolio contains a zero-coupon bond whose time to maturity is the same as the investment's time horizon.

M-absolute is the indicator which, as opposed to duration, should be minimized. In order to understand why, we will begin from the fact that the portfolio manager wishes to minimize the deviation of the portfolio's value

from its target value at the moment H , i.e. V_H . In other words, what is minimized is $\Delta V_H / V_H$, the deviation being caused by the changes in interest rates at the moment $t=0$.

Let us define the constants K_1 , K_2 and K_3 in the following manner:

$$K_1 \Delta f(t) \text{ for each } t$$

$$K_2 \Delta f(t) \text{ for each } t$$

$$K_3 = -\min\{K_1, -K_2\} = \max\{|K_1|, |K_2|\}$$

If we assume that $CF_t \geq 0$ for each t , we get the following (Nawalkha, Soto, Beliaeva, 2005, p. 105):

$$\frac{\Delta V_H}{V_H} \geq \frac{1}{V_0} \sum_{t=t_1}^{t_n} CF_t e^{-\int_0^t f(s) ds} (-K_3 |H - t|) = -K_3 M^A$$

K_3 depends on the changes in instantaneous forward rates, hence it is beyond the portfolio manager's control. What he can affect, though, is M^A , which may be reduced by selecting securities whose cash flows are closer to the time horizon H .

By setting the portfolio's duration to be equal to the investment's time horizon, we protect the portfolio from small and parallel shifts of the yield curve. As opposed to that, the minimization of M-absolute in most cases will not completely protect the portfolio from such changes, but it will take into account the possibility of bigger and/or non-parallel shifts of the yield curve. Thus, the choice between these two measures depends on the type of the expected changes in the yield curve.

A comparison of immunization strategies based on duration, i.e. M-absolute, was conducted by Nawalkha and Chambers (2009) focusing on the data for the period 1951-1986. According to their results, the application of M-absolute more than halved the deviations from the portfolio's target value, compared with the duration-based immunization strategy.

The second measure depending on the time horizon is **M-square**:

$$M^2 = \frac{1}{P} \sum_{t=t_1}^{t_n} (t - H)^2 \frac{CF_t}{e^{\int_0^t f(s) ds}}$$

Given that in case of continuous interest income we apply the following formula:

Vidimo da je M-kvadrat jednako nuli samo u slučaju beskuponske obveznice sa dospećem u trenutku $t=H$, kao i da konveksnost predstavlja specijalan slučaj M-kvadrata za $H=0$.

Nawalkha, Soto i Beliaeva tvrde (str. 106-107) da važi sledeća nejednakost:

$$\frac{\Delta V_H}{V_H} \geq -(D-H)\Delta f(H) - \frac{1}{2}K_4 M^2$$

Ukoliko je portfolio imunizovan u pogledu trajanja, $D=H$, iz čega sledi da donja granica odstupanja od ciljane vrednosti portfolija zavisi od konstante K_4 (koja odražava promenu terminske strukture) i mere M-kvadrat. Drugim rečima, portfolio menadžer će, za dato trajanje, želeći da minimizuje M-kvadrat portfolija.

U slučaju da je trajanje dato, M-kvadrat predstavlja linearnu transformaciju konveksnosti. Konveksnost je poželjna osobina obveznice, jer veća konveksnost znači veći dobitak u slučaju pada kamatnih stopa, odnosno manji gubitak u slučaju njihovog rasta. Sa druge strane, portfolio menadžer će hteti da minimizuje M-kvadrat portfolija. Pošto veća konveksnost nužno znači i veće M-kvadrat, ova protivrečnost je poznata kao **paradoks konveksnost/M-kvadrat** (eng. *convexity-M-square paradox*).

Ako se trenutne forward stope promene za $\Delta f(t)$ u trenutku $t=0$, prinos koji će se ostvariti do trenutka H biće jednak:

$$R(H) = \frac{V'_H - V_0}{V_0}$$

odnosno:

$$R(H) = \left(e^{\int_0^H f(s) ds} \sum_{t=t_1}^{t_n} \frac{CF_t}{e^{\int_0^t f(s) ds}} e^{\int_t^H \Delta f(s) ds} - V_0 \right) / V_0$$

Ukoliko terminska struktura ostane nepromenjena u periodu od $t=0$ do $t=H$:

$$R(H) = \frac{e^{\int_0^H f(s) ds} V_0 - V_0}{V_0} = e^{\int_0^H f(s) ds} - 1 = R_F(H)$$

što predstavlja nerizičnu stopu prinosa beskuponske obveznice sa dospećem H . Nakon aproksimacije $R(H)$ Tejlorovim polinomom drugog stepena dobija se sledeća jednakost (Nawalkha, Soto, Beliaeva, 2005, str. 109):

$$R(H) \approx R_F(H) + \gamma_1(D-H) + \gamma_2 M^2$$

$$\gamma_1 = -\Delta f(H)(1 + R_F(H))$$

$$\gamma_2 = -\frac{1}{2}(1 + R_F(H)) \left[\frac{\partial(\Delta f(H))}{\partial t} - (\Delta f(H))^2 \right]$$

Koeficijent γ_2 se dalje može razložiti na dva dela - efekat konveksnosti (CE) i efekat rizika (RE):

$$\gamma_2 = CE - RE$$

$$CE = \frac{1}{2}(1 + R_F(H))(\Delta f(H))^2$$

$$RE = \frac{1}{2}(1 + R_F(H)) \frac{\partial(\Delta f(H))}{\partial t}$$

Imunizacija portfolija podrazumeva da je $D=H$, pa vidimo da prinos imunizovanog portfolija zavisi od nerizične komponente i veličine M-kvadrat pomnožene odnosom pomenuta dva efekta. Efekat konveksnosti je uvek pozitivan, bez obzira da li dolazi do rasta ili pada kamatnih stopa. Sa druge strane, efekat rizika zavisi od toga da li dolazi do rasta ili do pada nagiba krive prinosa. Ako nagib poraste, prvi izvod promene je pozitivan, pa dolazi do smanjenja vrednosti koeficijenta γ_2 i time do smanjenja ukupnog prinosa. Obrnuto, prinos će se povećati ako dođe do pada nagiba. Suština paradoksa konveksnost/M-kvadrat je u tome što konveksnost uzima u obzir samo promenu nivoa krive prinosa tj. paralelne promene terminske strukture, dok neparalelne promene, koje uključuju i promenu nagiba, povećavaju apsolutnu vrednost efekta rizika. Zbog toga će portfolio menadžer težiti da smanji M-kvadrat portfolija ukoliko očekuje neparalelne promene terminske strukture.

Trajanje ključnih kamatnih stopa

Model trajanja ključnih kamatnih stopa je još jedan model koji uzima u obzir neparalelne promene terminske strukture. Prvi ga je uveo Ho 1992. godine. Ovaj model pretpostavlja da se promene terminske strukture mogu aproksimirati promenama konačnog broja odabranih, reprezentativnih kamatnih stopa. Od pojedinačnog istraživača zavisi koje će se kamatne stope koristiti. Na primer, Ho je predložio da se koristi 11 kamatnih stopa, sa

$$C = \sum_t w_t t^2$$

it implies that

$$M^2 = \sum_t w_t (t^2 - 2tH + H^2) = C - 2DH + H^2$$

We can see that M-square equals zero only in case of a zero-coupon bond maturing at the moment $t=H$, and that convexity represents a special case of M-square for $H=0$.

Nawalkha, Soto and Beliaeva (pp. 106-107) support the following inequality:

$$\frac{\Delta V_H}{V_H} \geq -(D - H)\Delta f(H) - \frac{1}{2}K_4M^2$$

If the portfolio is immunized in terms of duration, $D=H$, it implies that the bottom limit of the deviation from the portfolio's target value depends on the constant K_4 (reflecting the change in term structure) and the M-square measure. In other words, for the given duration, the portfolio manager would want to minimize the portfolio's M-square.

In case that the duration is given, M-square represents a linear transformation of convexity. Convexity is a desirable characteristic of a bond, because higher convexity implies higher profit in case of declining interest rates, i.e. smaller loss in case of their growth. On the other hand, the portfolio manager would strive to minimize the portfolio's M-square. Since higher convexity inevitably entails higher M-square, this contradiction is known as **convexity-M-square paradox**.

If instantaneous forward rates change by $\Delta f(t)$ at the moment $t=0$, the yield achieved until the time H will equal:

$$R(H) = \frac{V'_H - V_0}{V_0}$$

i.e.:

$$R(H) = \left(e^{\int_0^H f(s) ds} \sum_{t=t_1}^{t_n} \frac{CF_t}{e^{\int_0^t f(s) ds}} e^{\int_t^H \Delta f(s) ds} - V_0 \right) / V_0$$

If the term structure remains unchanged in the period from $t=0$ to $t=H$:

$$R(H) = \frac{e^{\int_0^H f(s) ds} V_0 - V_0}{V_0} = e^{\int_0^H f(s) ds} - 1 = R_F(H)$$

representing a risk-free rate of return of the

zero-coupon bond with maturity H . After the approximation $R(H)$ by means of the second-degree Taylor polynomial we get the following equation (Nawalkha, Soto, Beliaeva, 2005, p. 109):

$$R(H) \approx R_F(H) + \gamma_1(D - H) + \gamma_2M^2$$

$$\gamma_1 = -\Delta f(H)(1 + R_F(H))$$

$$\gamma_2 = -\frac{1}{2}(1 + R_F(H)) \left[\frac{\partial(\Delta f(H))}{\partial t} - (\Delta f(H))^2 \right]$$

The coefficient γ_2 can be further broken down to two segments: convexity effect (CE) and risk effect (RE):

$$\gamma_2 = CE - RE$$

$$CE = \frac{1}{2}(1 + R_F(H))(\Delta f(H))^2$$

$$RE = \frac{1}{2}(1 + R_F(H)) \frac{\partial(\Delta f(H))}{\partial t}$$

Immunization of the portfolio implies that $D=H$, hence we see that the immunized portfolio's yield depends on the risk-free component and the size of M-square multiplied by the ratio of the two mentioned effects. Convexity effect is always positive, regardless of whether the interest rates grow or decline. On the other hand, the risk effect depends on whether the slope of the yield curve is going up or down. If the slope increases, the first derivative of the change is positive, hence the value of the coefficient γ_2 decreases, and, in turn, the total yield drops. And vice versa, the yield increases if the slope goes down. The point of the convexity-M-square paradox is that convexity only takes into account the changes in the yield curve level, i.e. the parallel shifts in its term structure, whereas non-parallel shifts, including the changes in the slope, increase the absolute value of the risk effect. Therefore, the portfolio manager will strive to reduce the portfolio's M-square, if he expects non-parallel shifts in the term structure.

Key Rate Duration

The key rate duration model is another model taking into account non-parallel shifts in the term structure. It was first introduced by Ho in 1992. This model assumes that term

dospećima od 3 meseca, 1, 2, 3, 5, 7, 10, 15, 20, 25 i 30 godina.

Pretpostavimo da je odabrano N ključnih kamatnih stopa sa dospećima $t_1, t_2, \dots, t_N$. Radi jednostavnosti, pretpostavimo i da su dospeća novčanih tokova obveznice usklađena sa njima. U tom slučaju, promena kamatne stope za dospeće t_i dovede do promene vrednosti obveznice u srazmeri sa **trajanjem ključne kamatne stope** (eng. *key rate duration*) sa dospećem t_i :

$$\frac{\Delta P_i}{P} = -KRD(i) * \Delta y(t_i)$$

gde je trajanje ključne kamatne stope definisano kao relativna osetljivost obveznice na promenu kamatne stope $y(t_i)$:

$$KRD(i) = -\frac{1}{P} \frac{\partial P}{\partial y(t_i)}$$

Ukupna promena cene obveznice je jednaka zbiru N pojedinačnih efekata:

$$\frac{\Delta P}{P} = -\sum_{i=1}^N KRD(i) \Delta y(t_i)$$

Napomenimo da se promena kamatne stope sa dospećem $t \neq t_i$ dobija linearnom interpolacijom promena „susednih“ ključnih kamatnih stopa. Na primer, ako je $\Delta y(7)=0.5\%$ a $\Delta y(10)=0.8\%$, onda je $\Delta y(8)=0.67*0.5\%+0.33*0.8\%=0.6\%$.

Ukoliko je reč o većoj promeni terminske strukture, potrebno je uvesti i **konveksnosti ključnih kamatnih stopa** (eng. *key rate convexities*):

$$KRC(i, j) = KRC(j, i) = \frac{1}{P} \frac{\partial^2 P}{\partial y(t_i) \partial y(t_j)}$$

Promena vrednosti obveznice se onda aproksimira na sledeći način:

$$\frac{\Delta P}{P} = -\sum_{i=1}^N KRD(i) \Delta y(t_i) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N KRC(i, j) \Delta y(t_i) \Delta y(t_j)$$

Ako je reč o paralelnom pomeranju krive prinosa, dobijamo sledeću Tejlorovu aproksimaciju:

$$\frac{\Delta P}{P} = -D \Delta y + \frac{1}{2} C (\Delta y)^2$$

$$D = \sum_{i=1}^N KRD(i)$$

$$C = \sum_{i=1}^N \sum_{j=1}^N KRC(i, j)$$

Trajanje i konveksnost ključnih kamatnih stopa portfolija jednake su ponderisanom trajanju odnosno konveksnosti svih obveznica u portfoliju, gde su ponderi jednaki učešću svake obveznice u portfoliju:

$$KRD^P(i) = \sum_{j=1}^M w_j KRD_j(i)$$

$$KRC^P(i, j) = \sum_{k=1}^M w_k KRC_k(i, j)$$

U praksi su primećena tri ograničenja ovog modela (Nawalkha, Soto, Beliaeva, 2005, str. 281). Prva zamerka je da se ne uzimaju u obzir istorijska kretanja terminske strukture, zbog čega se izostavljaju značajne informacije o volatilnosti različitih segmenata krive prinosa. Pojedini autori su rešili ovaj problem tako što su u model uključili kovarijanse promena kamatnih stopa.

Druga zamerka se odnosi na proizvoljan izbor ključnih kamatnih stopa. U odsustvu jasnog kriterijuma, rešenje se može naći u tome da treba izabrati one kamatne stope koje najviše utiču na dati portfolio. Na primer, portfolio menadžer investicionog fonda tržišta novca će posmatrati kratkoročne kamatne stope.

Treći problem se ogleda u tome da je promena pojedinačne ključne kamatne stope malo verovatna u praksi, iako zajednička promena svih ključnih kamatnih stopa može verno predstaviti promenu terminske strukture. Naime, izolovana promena jedne (beskuponske) kamatne stope implicira neobičnu deformaciju krive trenutnih forward stopa. Ovaj problem se rešava posmatranjem forward stopa umesto beskuponske krive prinosa. Taj metod se naziva **pristup parcijalnih izvoda** (eng. *partial derivative approach*). On podrazumeva da se kriva trenutnih forward stopa podeli u više segmenata, a zatim se pretpostavlja da se sve forward stope u jednom segmentu pomeraju paralelno. *Parcijalno trajanje* koje odgovara svakom segmentu se onda definiše kao relativna osetljivost portfolija na promenu forward stope koja predstavlja taj segment. Razlika u odnosu

structure changes can be approximated by means of changes in a finite number of selected, representative interest rates. Every individual researcher chooses which interest rates to use. For instance, Ho suggested 11 interest rates, with respective maturities of 3 months, 1, 2, 3, 5, 7, 10, 15, 20, 25 and 30 years.

Let us assume that there were N selected interest rates, with maturities ranging from t_1 , $t_2, \dots$, to t_N . For the sake of simplification, let us assume that the maturities of the bond's cash flows are harmonized accordingly. In that case, the change in interest rate for maturity t_i will lead to the change in bond value proportionate to the **key rate duration** with maturity t_i :

$$\frac{\Delta P_i}{P} = -KRD(i) * \Delta y(t_i)$$

with key rate duration being defined as relative sensitivity of the bond to the changes in interest rate $y(t_i)$:

$$KRD(i) = -\frac{1}{P} \frac{\partial P}{\partial y(t_i)}$$

Total change in the bond price equals the summation of N individual effects:

$$\frac{\Delta P}{P} = -\sum_{i=1}^N KRD(i) \Delta y(t_i)$$

We should underline that the change in interest rate with maturity $t \neq t_i$ is calculated by linear interpolation of changes of its "neighboring" key rates. For instance, if $\Delta y(7)=0.5\%$ and $\Delta y(10)=0.8\%$, then $\Delta y(8)=0.67*0.5\%+0.33*0.8\%=0.6\%$.

In case of bigger changes in term structure, it is necessary to introduce **key rate convexities**:

$$KRC(i, j) = KRC(j, i) = \frac{1}{P} \frac{\partial^2 P}{\partial y(t_i) \partial y(t_j)}$$

A change in the bond value is then approximated in the following manner:

$$\frac{\Delta P}{P} = -\sum_{i=1}^N KRD(i) \Delta y(t_i) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N KRC(i, j) \Delta y(t_i) \Delta y(t_j)$$

In case of parallel shifts of the yield curve, we get the following Taylor approximation:

$$\begin{aligned} \frac{\Delta P}{P} &= -D \Delta y + \frac{1}{2} C (\Delta y)^2 \\ D &= \sum_{i=1}^N KRD(i) \\ C &= \sum_{i=1}^N \sum_{j=1}^N KRC(i, j) \end{aligned}$$

Duration and convexity of the portfolio's key rates equal the weighted duration, i.e. convexity of all bonds in the portfolio, with the weights equaling the share of each bond in the portfolio:

$$\begin{aligned} KRD^P(i) &= \sum_{j=1}^M w_j KRD_j(i) \\ KRC^P(i, j) &= \sum_{k=1}^M w_k KRC_k(i, j) \end{aligned}$$

There are three limitations to this model observed in practice (Nawalkha, Soto, Beliaeva, 2005, p. 281). The first objection is that it does not take into account the historical movements of the term structure, which is why significant information is omitted concerning the volatility of different yield curve segments. Certain authors solved this problem by integrating covariance of interest rates changes into the model.

The second objection refers to the arbitrary selection of key rates. Given the lack of a clear criterion, the solution may be the selection of those interest rates which most substantially affect the concerned portfolio. For instance, the portfolio manager of a money market investment fund will be focusing on short-term interest rates.

The third problem is reflected in the fact that a change of individual key rates is highly unlikely in practice, even though the common change of all key rates may truthfully represent the change in the term structure. Namely, an isolated change of a single (zero-coupon) interest rate implies an atypical deformation in the instantaneous forward rates curve. This problem is solved by observing forward rates instead of the zero-coupon yield curve. This method is called **partial derivative approach**. It implies that the instantaneous forward rates curve is divided into several segments, and the assumption is made that all forward rates within each segment record parallel shifts. *Partial duration* in respect of each segment

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na trajanje ključnih kamatnih stopa je u tome što sadašnja vrednost određenog novčanog toka zavisi od forward stopa za sve periode koji mu prethode, umesto samo od stope za taj period.

Zaključak

Jasno je da trajanje i konveksnost obveznice podrazumevaju značajno pojednostavljivanje oblika i dinamike krive prinosa. Prvi način da se ove pretpostavke ublaže jeste korišćenje Fisher-Weil-ovog umesto Macauley-evog trajanja, budući da ono meri paralelne promene neravne krive prinosa. Pošto se u praksi pokazalo da najveći deo (oko 70%) dinamike krive prinosa čine promene njenog nivoa, tj. paralelne promene, ovaj pokazatelj ne možemo odbaciti kao nekoristan.

Imunizacija portfolija se može vršiti i pomoću pokazatelja M-apsolutno i M-kvadrat, koji mere disperziju dospeća novčanih tokova portfolija oko vremenskog horizonta ulaganja. Uostalom, jedan od načina eliminisanja kamatnog rizika predstavlja tzv. strategija dedikacije. Iako je ona komplikovana i često nemoguća, prethodna dva pokazatelja idu njenim tragom, budući da mere odstupanje preduzete strategije imunizacije od teorijske strategije dedikacije, te upućuju na smanjenje tog odstupanja.

Treći metod podrazumeva da umesto krive prinosa posmatramo ograničen broj kamatnih stopa koje iz nekog razloga smatramo reprezentativnim, a da kamatne stope za sva ostala dospeća dobijemo njihovom linearnom interpolacijom. Ovaj metod je posebno od koristi portfolio menadžerima koji se koncentrišu na određeni segment krive prinosa.

is then defined as relative sensitivity of the portfolio to the changes in the forward rate representing that particular segment. The difference compared to key rate duration lies in the fact that the present value of a specific cash flow depends on forward rates of all preceding periods, instead of just on the rate for that period.

Conclusion

Evidently enough, duration and convexity of a bond substantially simplify the shape and dynamics of the yield curve. The first method to mitigate these assumptions is to use Fisher-Weil instead of Macaulay duration, given that it measures the parallel shifts in a non-flat yield curve. Since it turned out in practice that the biggest share (about 70%) of the yield curve dynamics is accounted for by the changes in its level, i.e. parallel shifts, this indicator cannot be dismissed as useless.

Portfolio immunization can also be performed by means of M-absolute and M-square indicators, measuring the dispersion of maturities of the portfolio's cash flows around the investment's time horizon. One of the ways to eliminate interest rate risk is the so-called dedication strategy. Although this strategy is complicated and often impossible, the two above mentioned indicators go along the same lines, given that they measure the deviation of the performed immunization strategy from the theoretical dedication strategy, therefore suggesting the reduction of this deviation.

The third method implies that, instead of the yield curve, we observe a finite number of interest rates which from some reason we deem representative, in which process we obtain interest rates for all other maturities through their linear interpolation. This method is particularly useful to portfolio managers focusing on a specific segment of the yield curve.