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A METHOD FOR CHOOSING APPROPRIATE INVESTMENT PERIODS TO MAKE ARBITRAGE PROFIT AND EXPLAIN STOCK RETURNS

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Abstract

The current paper presents theoretical and experimental evidence to justify the need for paying attention to the investment horizon. Therefore, a criterion called the 'Safest Investment Horizon' (SIH) is utilized to select the appropriate investment horizon. To compute this quantity, a ratio called the 'Safest Investment Ratio' (SIR) is calculated, and the relationship between these criteria and arbitrage opportunities, along with methods for making an arbitrage profit through selecting an appropriate time horizon are discussed. Afterward, by applying this method for real-life data, the presence of arbitrage opportunities at different time horizons is confirmed. Furthermore, the effects of the time horizon on optimal portfolio composition are described. Finally, it is shown that these criteria outperform some of the conventional variables in CAPM, the 3-factor, and the 5-factor models for explaining stock returns and using SIH or SIR as a new variable increases the explanatory power of these models.

Keywords: investment horizon, arbitrage theorem, portfolio optimization, factor models

1. INTRODUCTION

Investment horizon is one of the main factors influencing investment decisions. For example, portfolio composition and asset allocation decisions depend on the investment horizon, and these change as the investment horizon becomes longer (Ferguson and Simaan, 1996). This is shown in various ways in a large number of studies. Gunthorpe and levy (1994) state that when returns are dependent and non-stationary over time, investment horizon affects portfolio composition; however, even if returns are independent and stationary, the weights of assets in a portfolio vary in a

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systematic way with the investment period. Marshal (1994) shows that investors will choose less risky single-period portfolios as their investment periods become shorter. Barberis (2000) argues that, compared to other investors, investors with longer investment horizons are more likely to allocate their wealth to stocks. Campbell and Viceira (2005) indicate that due to the changes in the risk to return ratio, the optimal portfolio varies with horizon. They call this concept 'structure of risk and return trade-off'. The study by Gil-Bazo (2006) demonstrates that if stock returns are predictable, the relationship between return and risk, and thus, the relative share of each risky asset in the portfolio, will change by altering the time horizon. Using the Fama and French method to create various portfolios, In and Gencay (2011) argue that as the investment horizon gets longer, the amount of investment in growth stocks changes significantly. In addition, Woodside-Oriakhi, Lucas, and Beasley (2013) show the importance of the investment horizon through modeling portfolio rebalancing by considering transaction costs and the investment horizon. In a nutshell, based on empirical results and typical recommendations of portfolio managers, Kunt (2017) suggests that the investment period affects the optimal allocated amount for investment in risky assets.

In addition, many other decisions, such as investment strategies, are influenced by the investment period. For instance, Dierkes, Erner, and Zeisberger (2010) examine the attractiveness of investment strategies over different investment periods, and demonstrate that strategy attractiveness depends on the investment horizon, i.e., strategies which are based on bonds are preferred by the short-term investors and stocks outperform in long term investments. This is similar to the results obtained by Butler and Domian (1991). On the other hand, Amadi and Amadi (2019) investigated the effects of the investment horizon on mutual funds investment strategies and found that small capital funds were interested in longer investment horizons.

Moreover, beta and a number of performance appraisal ratios depend on the investment horizon. In other words, as investment horizon lengthens, betas of high-beta stocks tend to increase, while those of low-beta stocks tend to decrease (Gunthorpe and levy, 1994). Levy (1972, 1981, and 1984), Levy and Samuelson (1992), and Levy and Duchin (2004) show that the investment horizon has a significant impact on performance measures. Likewise, Hodges, Taylor, and Yoder (1997) show that the Sharpe ratio increases at initial stages, and decreases as the investment horizon becomes longer, while the rankings of the portfolios, ranked based on the Sharpe ratio, change. Based on these results, using a static criterion such as the Sharpe ratio, without considering the investment horizon can lead to erroneous analysis. To explain this, Van Eaton, Douglas, and Conover (2001) argue that because of limitations on leverage and margin buy terms, using Sharpe ratio rankings as a basis for asset allocation for different time periods can result in error.

Despite this evidence, the majority of conventional financial theories fail to consider the investment horizon factor. This is because conventional approaches to describe the behavior of asset returns have long been dominated by a simple assumption positing that returns are independent and identically distributed (iid) random variables (Gilmore, 1976). This facilitating assumption has led to the neglect of the potential effects of the investment horizon on the distribution form of returns. Moreover, if this assumption is true, changing the time horizon of the investment will only change the accepted distribution parameters in finance, and it will not change the overall shape of the assumed distribution. For example, when assuming normal distribution, the mean coefficient of distribution is proportional to the time period of the investment, while its standard deviation is a factor of the time step square root according to the square-root-of-time rule. In any case, the return will have a normal distribution. However, a large number of studies have produced a substantial body of literature challenging the adequacy of the iid assumption (Gilmore, 1976), and by removing this assumption, the return distribution form will be fully dependent on the length of the investment period.

Contrary to these effects and despite the importance of determining the appropriate investment time horizon, little attention has so far been paid to how a suitable investment horizon can be determined. A few studies that have been carried out in this regard are mostly brief, lack deep and comprehensive empirical evidence, and do not consider different applications of selecting an appropriate horizon. For example, Simonsen, Johansen, and Jensen (2006) try to present a new criterion for choosing an optimal investment horizon and using this concept for performance appraisal. The initial idea of these authors was very similar to Roy's safety-first criterion, and since this criterion is the basis of the method used in the current study, the overall method of this paper is not unlike the work of these authors. However, in contrast to their study, the value of the threshold is selected based on economic theories, and in one case (risk-free rate as the threshold value), the value of this threshold is not considered constant, i.e. it changes according to the period under study. On the other hand, a large number of practical tests have been performed in this regard. Accordingly, in this study, an attempt is made to use a criterion for selecting the appropriate investment period, called the ‘Safest Investment Horizon’ (SIH) and the ‘Safest Investment Ratio’ (SIR). As noted earlier, the basic idea for these criteria is Roy's safety-first criterion (Roy, 1952), which is based on a threshold for the required rate of return. This period (SIH) is calculated for each asset in a way that the probability (or SIR) that the return of this period is higher than the threshold return is maximized. Then, the theoretical relationships between these criteria and the arbitrage theorem and portfolio optimization are analyzed. Then, its application for obtaining a profit from arbitrage opportunities, optimizing the portfolio, and explaining the stock returns behavior is examined based on real data.

The reminder of this paper is organized as follows: In Section 2, SIH and SIR (as an intermediate variable for calculating SIH) are explained mathematically and a pseudocode is provided for computing them. In Section 3, the relationship between SIH and arbitrage opportunity is analyzed. In Section 4, the importance of paying attention to the investment horizon in portfolio optimization is discussed. In Section 5, the results obtained using real data from the S&P-500 Index and stock returns with different views, including the explanatory power of SIH or SIR, are presented to test the extracted factor as an influential variable on stock returns. Finally, Section 6 presents the summary and concluding remarks.

2. SIH AND SIR

As noted in the introduction, the idea behind SIH and SIR is taken from Roy's safety-first criterion. Roy states that the best stock or portfolio is the one with minimum probability of producing a return below a threshold value, i.e., trying to minimize Probablity[Ln(P_(t+Δt)⁄P_t)<Treshold] (Goetzmann, Gruber, and Elton, 2014). In this expression, P_t and P_(t+Δt) signify prices at times t and t+Δt, respectively (adjusted for dividend and other kinds of payments).

As can be seen, this criterion takes the value of the investment horizon (Δt) as a given parameter and does not try to find the best value for it. Accordingly, in order to find the best period for investment, we have to find a value of Δt for which the probability that the return corresponding to this period is higher than a predefined threshold return is maximized. To do this, as the first step, it is necessary to calculate the ratio of the number of returns higher than the threshold return to the total number of returns for different investment periods Δt . This ratio is called the Safest Investment Ratio for the investment horizon Δt (SIR_Δt). Two values of zero return and risk-free interest rate were selected as thresholds in this paper to represent accounting and economic loss avoidance, respectively.

As mentioned earlier, in SIH, by taking a target return (threshold) into account, the optimal value of the investment time period is selected in a way that the probability that the asset return during this period is higher than the target return (SIR) is maximized. This can be expressed in mathematical terms as follows:

The value of the Δt that maximizes this statement will be SIH. Value of threshold_Δt can be anything, such as zero or risk-free rate (r_f), proportional to the length of the period. The probability statement in the bracket can be interpreted as SIR. This optimization problem can be solved analytically under predefined assumptions about the distribution of prices, such as log-normal, by using the first hitting time method. But as mentioned earlier, these kinds of assumptions do not usually hold in the real world. Therefore, in order to clarify the

```
Select the companies that have at least the minimum required trading days
for       First company to last company
|         Select return data for a company
|         for       Different investment horizons
|         |         Calculate percentage of returns more than respective threshold
|         |         if          The above percentage is equal to 100%
|         |         |           Save this percentage as SIR
|         |         |           Save this horizon as SIH
|         |         else if     The above percentage is less than 100%
|         |         |           Find the maximum percentage as SIR
|         |         |           Find the related horizon as SIH
|         |         end
|         end
end
```

Figure 1. Pseudocode to Calculate SIR and SIH

calculation of SIR and SIH, the pseudocode for these calculations is presented in Figure (1) as an iterative method.

$$Infimum\left\{\ t\ Max\left[Probablity\ \left(Ln\left(\frac{P_+}{P}\right) \geq Treshold\ \right)\right]\right\} \quad (1)$$

In this pseudocode, the first for loop is for different companies and the second for loop is for different investment horizons. By using if conditions, two different types of companies (with SIR=100% and SIR<100%) are separated, and in total, the minimum investment horizon that maximizes SIR is selected as SIH.

3. THE RELATIONSHIP BETWEEN SIH, SIR AND THE ARBITRAGE THEOREM

Obviously, the maximum value of probability in Statement (1) (or SIR) is one. If this value is obtained for a stock, arbitrage profit can be made by (short) selling or buying that stock. To explain this, following Hirsa and Neftci (2013), suppose that S_t is a vector representing N asset prices (S_i for i=1,2,…,N) in time t∈[0,∞):

$$\boldsymbol{S_t} = \begin{bmatrix} S_1(t) \\ S_2(t \\ \vdots \\ S_N(t) \end{bmatrix} \quad (2)$$

Each of these assets have payoff d_{ij} in mutually exclusive states of the world j=1,2,…,K and time t, which are grouped in matrix D_t below:

$$\boldsymbol{D_t} = \begin{bmatrix} d_{11} & \dots & d_{1K} \\ \vdots & \vdots & \vdots \\ d_{N1} & \dots & d_{NK} \end{bmatrix} \quad (3)$$

If the first asset (S_1 (t)) in matrix S_t is a riskless asset, e.g., treasury bills, regardless of the realized state of the world, its payoff will be a fixed value of 〖1+r〗 $_f$ Δt in the first row of D_t. In addition, for simplicity, suppose that the amount of risk-free borrowing and lending is equal to one. Thus, D_t will be given by:

$$\begin{bmatrix} (1 + r_f\Delta t) & \dots & (1 + r_f\Delta t) \\ \vdots & \vdots & \vdots \\ S_{N1}(t + \Delta t) & \dots & S_{NK}(t + \Delta t) \end{bmatrix} \quad (4)$$

Now, based on the arbitrage theorem, given S_t and D_t provided in (2) and (3), if positive constants ψ_j>0 for all j=1,2,…,K can be found such that asset prices satisfy:

$$\begin{bmatrix} 1 \\ \vdots \\ S_N(t) \end{bmatrix} = \begin{bmatrix} (1 + r_f\Delta t) & \dots & (1 + r_f\Delta t) \\ \vdots & \vdots & \vdots \\ S_1(t + \Delta t) & \dots & S_K(t + \Delta t) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_K \end{bmatrix} \quad (5)$$

or

$$\boldsymbol{S_t = D_t \psi} \quad (6)$$

where ψ=[(ψ_1 … ψ_K)]', then there are no arbitrage opportunities (and vice versa). Using the arbitrage theorem, and multiplying D_t by the S_t vector, we obtain:

$$\begin{cases} 1 = (1 + r_f\Delta t)\psi_1 + \dots + (1 + r_f\Delta t)\psi_K \\ \vdots \\ S_N(t) = S_{N1}(t + \Delta t)\psi_1 + \dots + S_{NK}(t + \Delta t)\psi_K \end{cases} \quad (7)$$

Let's define the gross returns as R_{ij} (t+Δt)=(S_{ij} (t+Δt))⁄(S_i (t)) and write the above equation using these new symbols:

$$\begin{cases} 1 = (1 + r_f\Delta t)\psi_1 + \dots + (1 + r_f\Delta t)\psi_K \\ \vdots \\ 1 = R_{N1}(t + \Delta t)\psi_1 + \dots + R_{NK}(t + \Delta t)\psi_K \end{cases} \quad (8)$$

Subtracting the first equation from the other N-1 equations gives:

$$\sum_{j=1}^{K}\left[R_{ij}(t+\Delta t)-\left(1+r_f\Delta t\right)\right]\psi_j=0 \quad , \quad i=2,3,\dots,N \quad (9)$$

Hence, ψ_j>0 for all j=1,2,…,K. Because of the no-arbitrage condition, the above equation will be satisfied if and only if:

$$R_{i1}(t+\Delta t)<\cdots<\left(1+r_f\Delta t\right)<\cdots<R_{iK}(t+\Delta t) \quad (10)$$

In other words, if there is a period of time Δt in which for one or more risky assets S_i we have:

$$\left(1+r_f\Delta t\right)<R_{i1}(t+\Delta t)<\cdots<R_{iK}(t+\Delta t) \quad (11)$$

or

$$R_{i1}(t+\Delta t)<\cdots<R_{iK}(t+\Delta t)<\left(1+r_f\Delta t\right) \quad (12)$$

respectively, it is possible to arbitrage by buying or (short) selling that (those) asset(s). In addition, based on the definition provided for the safest investment horizon, this Δt will be equal to SIH, while the maximum value of SIR is equal to one. In other word, if it is possible to find SIH for asset i with the corresponding SIR equal to 100%, the return of that asset in all states of the world for that time period will be more than the risk-free rate of that period, and it falls under the condition described in Equation (11). This shows that it is possible to find arbitrage opportunities by selecting appropriate investment horizons so that SIR=100%.

4. INVESTMENT HORIZON AND PORTFOLIO OPTIMIZATION

When short sales are allowed and there is a riskless lending and borrowing rate, the standard derivation of the efficient set becomes possible by maximizing the Sharpes ratio (1994) as the objective function of weights, subject to a constraint to equalize the sum of the relative weights of the investment in each asset with one:

$$\underset{\{\boldsymbol{w}\}}{Max}\ Sharp\ ratio=\frac{R_P-R_f}{\sigma_P}$$

$$s.t: \quad (13)$$

$$\sum_{i=1}^{N}w_i=1$$

where R_P and σ_P are the expected return and the standard deviation of the portfolio return, respectively, while w=[w_1 w_2…w_N]' signifies the weight of the asset (w_i for i=1,2,…,N). Under the assumption of serially uncorrelated returns, Sharpe (1998) offers the square-root-of-time rule as:

$$Sharpe\ ratio_{\Delta t}=\sqrt{\Delta t}.Sharpe\ ratio_{1} \quad (14)$$

where Sharpe $ratio_{\Delta t}$ and Sharpe $ratio_1$ denote the Sharpe ratios for investment horizons equal to Δt and 1, respectively. Therefore, based on the serially uncorrelated returns assumption, the Sharpe ratio is a monotonic function of the horizon, and there is no need to consider the investment horizon for portfolio optimization. However, Levy (1972) shows that the Sharpe ratio depends on investment horizons. As mentioned earlier, Hodges, Taylor, and Yoder (1997) show that the Sharpe ratio has a relationship with the investment horizon with various slopes. Furthermore, portfolio rankings based on the Sharpe ratio change, and the Sharpe ratio depends on the investment horizon.

To explain the effects of the investment horizon on portfolio optimization and the asset allocation problem, following Gil-Bazo (2006), the final wealth of the investor from time t with an investment horizon of Δt (or $W_{t,t+\Delta t}$) will be:

$$W_{t,t+\Delta t} = (1 - \boldsymbol{w}'I)e^{r_f \Delta t} + \sum_{i=1}^{N} w_i e^{r_f \Delta t + r_{i,t+\Delta t}} \quad (15)$$

where, $r_{i,t+\Delta t}$ denotes the cumulative continuous excess return of asset i=1,2,…,N between t and t+Δt, while I is a vector of ones. Based on the above formula for final wealth of the investor, this variable depends on the investment horizon (Δt). Therefore, Δt will be one of the main factors for any utility function that depends on the investor's final wealth and is used to calculate and maximize the problem of portfolio optimization. This effect can be analyzed from two perspectives. Firstly, based on the coefficient of the r_f variable in Equation (15), the investment horizon affects investor wealth by selecting different periods of risk-free investing, and for many interest rate time structures (e.g., increasing, decreasing, or hump structure), changing the investment horizon will change the risk-free return. The second effect is through the factor r_(i,t+Δt). Based on the results of studies mentioned in introduction section and the empirical evidence that will be addressed in the following sections, the resulting return is not a monotonic function of the length of the investment period.

The result of these two factors is that the final wealth would be a function of w and Δt ($W_{t,t+\Delta t}$=W(w,Δt)), and the optimal portfolio composition must be determined based on these two factors. If U[W(w,Δt)] denotes the utility function of the investor, the optimization problem will become:

$$\underset{\{\boldsymbol{w},\Delta t\}}{Max}\ U[W(\boldsymbol{w},\Delta t)]$$

$$s.t: \qquad (16)$$

$$\sum_{i=1}^{N} w_i = 1$$

The main differences between Equations (13) and (16) are:

1. The objective function in Equation (16) cannot be the Sharpe ratio based on previous explanations for this ratio and its inadequacy for different time horizons. Therefore, for different utility functions, the objective function will vary; and

2. Optimization in (16) is carried out using the two variables of w and Δt; however, in (13), the only variable under consideration is w.

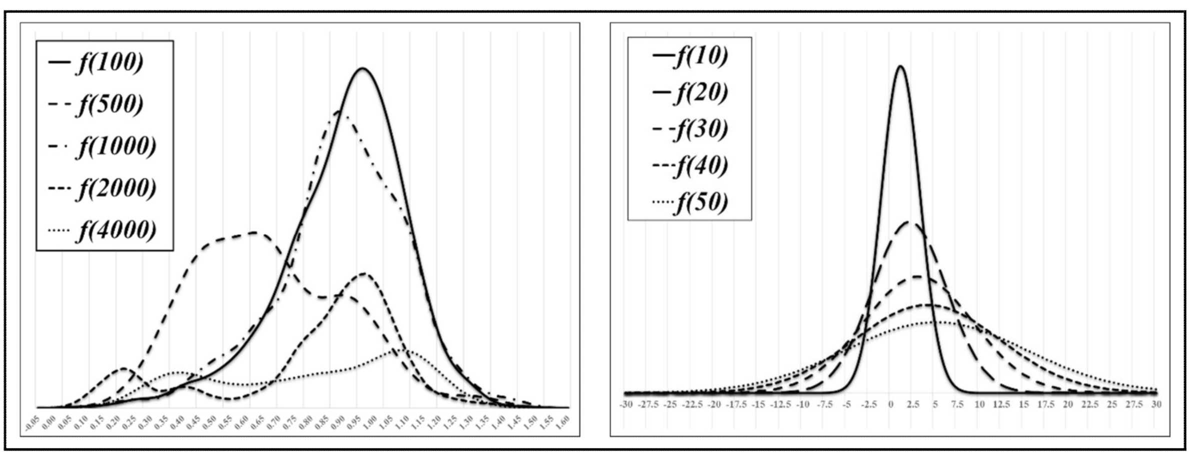
Based on these differences, it is possible to optimize the problem in Equation (13) using common optimization methods, such as quadratic programing and Lagrange methods. However, the problem in (16) is based on two variables, and it is required to use different multivariable optimization methods or stochastic methods, such as first hitting time with fixed and variable boundaries for threshold values of zero or risk-free rate, respectively.

5. DATA ANALYSIS

This section contains different subsections as follows. Firstly, the effects of the time period on distribution of returns are examined. Then, the calculations of SIH and SIR are explained, and the results are analyzed. Afterward, the effects of the investment period on portfolio optimization results are presented. Finally, the explanatory power of SIH and SIR is tested. To realize these objectives, two categories of data are used. In the first part, the S&P-500 Index data are used. This set of data will be used as an example to justify the need to examine the time horizon by comparing the return distribution in different horizons, as well as to explain the concepts related to SIH and SIR. The second set of data includes stock returns that are in the S&P-500 Index composition. Using this set of data, the results of SIH and SIR calculations are presented and analyzed based on arbitrage theorem, and their application is tested in two parts: the investment horizon's effects on portfolio optimization and the explanatory power of SIH or SIR. For both of these datasets, the daily data from 2000 to 2019 (20 years) are considered.

5.1. Changing Return's PDF by Increasing the Investment Horizon

As noted in the introduction, because of the inaccuracy of the assumption of iid about the return data, the shape of the probability density function (PDF) of the return data can be changed by varying the investment period. To illustrate this, the following figure shows the probability density function of normal simulated returns based on an iid distribution (e.g., the geometric Brownian motion (GBM) process) and real S&P-500 stock index returns over several different time horizons after smoothing by Kernel density function. As can be seen, the general shape of the PDF of the simulated price (the right-hand side diagram) has not changed with the variation in the time horizon (10, 20, 30, 40, and 50 days), and in all these cases, the logarithmic returns of the simulation prices have a normal distribution (with different parameters). However, the actual returns distribution of the S&P Index for periods of 100, 500, 1000, 2000, and 4000 days (the left-hand side diagram) does not have this property, and when changing the time horizon, the general shape of the distribution completely changes as well. In other words, for some periods, the direction of the distribution skewness changes, and,



Notes: This figure shows the smoothed probability distribution function of simulated and real historical returns of S&P-500 on different time horizons. The figure on the right is for the simulated returns based on GBM for the 10, 20, 30, 40 and 50-day investment horizons, while the figure on the left is for the actual S&P-500 Index for the period of 2000 to 2019 for the 100, 500, 1000, 2000 and 4000-day investment horizons.

Figure 2. Investment Horizon and Shape of the Return Distribution

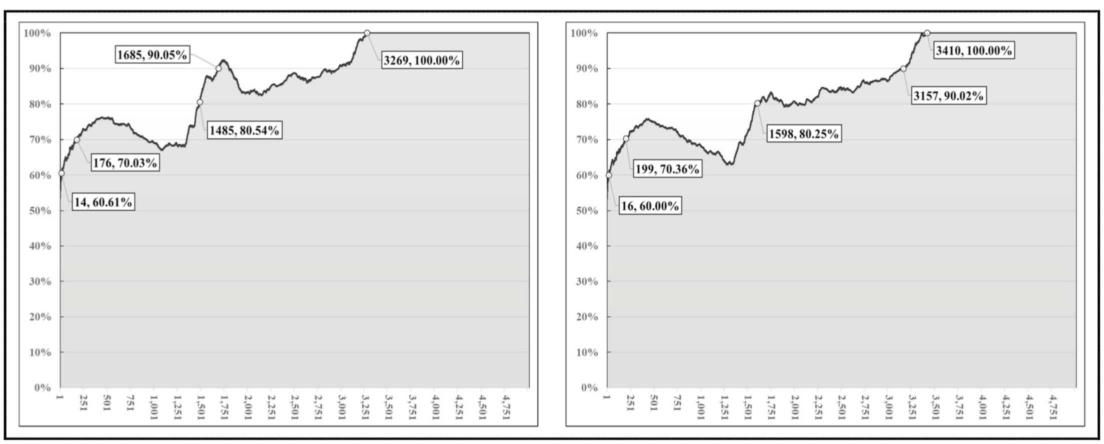
sometimes, even the distribution becomes multimodal and fat tail. It should be noted that the selection of the mentioned time periods is only for the purpose of explaining the notions, and the selection of other time periods will lead to similar results.

These results show the importance of the investment horizon as a factor for investment decisions. Moreover, these results are consistent with the findings of the study carried out by Levy and Duchin (2004), which fitted different distributions for various investment horizons and found that, in different horizons, some distributions are fitted to the data better than other distributions. Therefore, it is possible to find the best return distribution (based on different factors, such as risk or reward) by changing the investment horizon. This will be presented in later sections.

5.2. SIR and SIH Calculation for the S&P-500 Index

In this section, in order to better explain the concept, SIR_Δt is calculated for the S&P 500 Index, and the results are shown in Figure (3).

As shown in the above figure, if an investment horizon of 3,269 days (3,410 days) is selected (about 14 years), there is 100% certainty (SIR=100%) that S&P return will be higher than zero (risk-free rate), and these investment horizons are safe based on different thresholds. On the other hand, if a 14- day (16-day) investment horizon is selected, only in 60% of cases, having a return more than zero (risk-free rate) is certain. Moreover, to increase the safety of the investment with probabilities of 70%, 80%, and 90%, the best investment horizons are 176 days, 1,485 days, and 1,685 days for the zero threshold, and 199 days, 1,598 days, and 3,157 days for the risk-free rate threshold, respectively. Another noteworthy point in this figure is that the value of SIR does not increase monotonically as the duration of the period increases. For example, based on the zero-value threshold diagram, for the time period of 1,685 days, a gain of about 90% is possible; however, if the length of the investment horizon



Notes: This figure shows SIR values for the S&P-500 Index return from the beginning of 2000 to the end of 2019 using two different values for the thresholds (zero value in the left-hand side diagram and risk-free rate in the right-hand side diagram) based on different investment horizons (Δt) on the horizontal axis. The values in the boxes are investment horizon lengths in trading days and the percentage value of SIR.

Figure 3. SIR in Different Investment Horizons for the S&P-500 Index

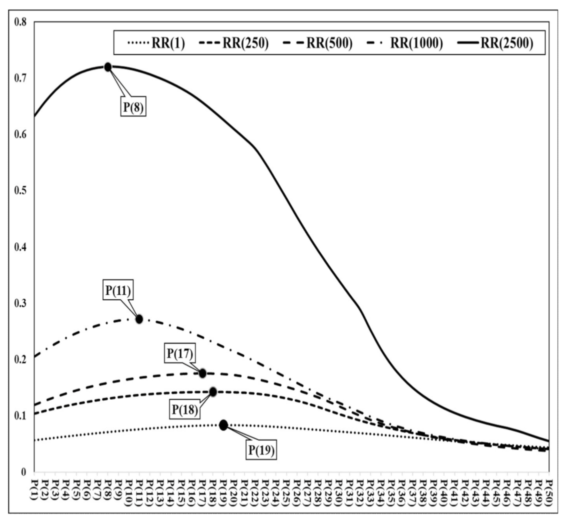
increases, after a few days, SIR starts decreasing for a while. In addition, according to Figure (3), for S&P 500, based on a zero-value threshold, the SIH was 3,269 days, while based on the risk-free rate selected as the threshold, the SIH was 3,410 days, maximizing the above probability to the value of one, or a 100% certainty. Regarding S&P 500, it can be said that there are arbitrage opportunities by trading this index composition or index-based securities if the investment horizon is equal to 3,410 days based on considerations discussed in Section 2.

5.3. The Impact of the Investment Horizon on Portfolio Optimization

Based on considerations discussed in Section 3, due to the differences in the Sharpe ratio for different time periods, the results of portfolio optimization will depend on the time horizon as well. In order to show this, based on stock returns described before, the Markovitz efficient frontier is calculated for different investment horizons (1 day, 250 days, 500 days, 1000 days, and 2500 days in order to test different periods from one day to 10 years), and the reward to risk ratio is computed for 50 points on each of these efficient frontiers. The results are presented in Figure (4).

As shown in the figure, the highest reward to risk ratio (which corresponds to the second order utility function, which is a function of reward and risk) varies for different investment horizons. For example, with the 1-day investment period, portfolio 19 is the best portfolio; however, by increasing this period, portfolios 18, 17, 11, and 8 are the best. More details about these best portfolios are presented in Table (1).

Based on the information presented in this table and Figure (4), it is easy to see that the optimized portfolio composition varies for different investment horizons. For example, when the investment horizon is one day, the best portfolio consists of 24 stocks, and the highest weight is 21 percent. By increasing the investment horizon to 250, 500, and 1000 days (equal to 1, 2, and 4 years, respectively), the number of stocks in each portfolio decreases. Therefore, it is possible to diversify the portfolio with a smaller number of stocks. In the 250-day period, we can see the highest value for the standard deviation of weights (0.0711), and more than 37 percent of the investor's wealth in one stock. Again, for the period of 2500 days, we see an increase in the number of stocks (21 different stocks).



Notes: In this figure, the reward to risk ratio (RR) of 50 points ($P(1), P(2), \ldots, P(50)$) on efficient frontier, calculated for different investment horizons (1 day, 250 days, 500 days, 1000 days, and 2500 days), is plotted. Each line represents the reward to risk ratio for a specific investment horizon. To make the different horizons comparable, the mean and standard deviation of returns are standardized with Δt and $\sqrt{\Delta t}$, respectively. The number of portfolios with the highest reward to risk ratio is marked with a label.

Figure 4. Reward to Risk Ratios for Different Investment Horizons

Table 1. Selected Information about Optimized Portfolios

Investment Horizon (Day)	1	250	500	1000	2500
Number of Stocks Selected	24	17	15	11	21
Maximum Weight of Stocks	21%	37%	28%	28%	25%
Standard Deviation of Weights	.0446	.0711	.0595	0.0597	.0548

Notes: In this table, selected information is presented about the best portfolio for different horizons based on the reward to risk ratio. This information includes the number of stocks in each portfolio (among 455 stocks), the maximum weight of the stocks in each portfolio, and the standard deviation of the weights (after eliminating stocks with weights equal to zero for all portfolios).

5.4. Running the Model on Stock Returns

In this section, total return data for stocks in the S&P-500 Index composition for the period of 2000 to 2019 (20 years) are used if they include data for at least 1000 trading days in this period (455 stocks for 5,031 trading days). As the first step of analysis, SIH and its correspondent probability are calculated based on two threshold values (zero and risk-free rate). Main results are reported in Table (2).

As can be seen, based on the two threshold values, more than 90 percent of companies have a SIH with the probability of 100% that shows presence of arbitrage opportunities in some of these companies if the investment horizon is at least 371 days. In other word, arbitrage opportunities only exist for the period of more than a year and a half (assuming 250 trading days per year). For the risk-free rate threshold, this result is consistent with the study carried out by Dierkes, Erner and Zeisberger (2010), indicating the attractiveness of investing in bonds for short periods and investment in stocks for long periods. For the other companies, the minimum value of SIR is about 43%, indicating that if the true value of the investment horizon is selected based on the proposed method, the chance of obtaining a return more than the threshold return is more than 43 percent.

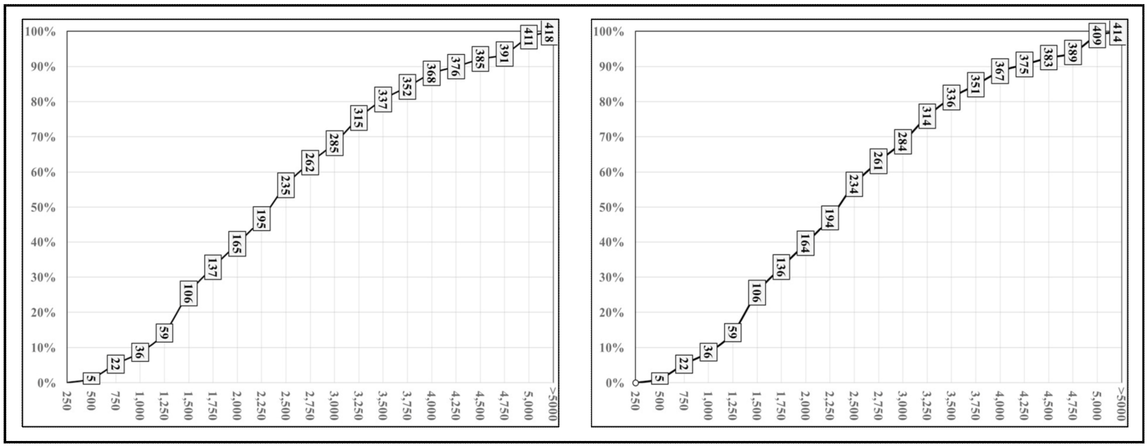
In order to clarify this point, Figure (5) shows the number and percentage of companies with a SIR of 100% based on different values of SIH on the x axis for different threshold values.

Based on this figure, for two threshold values, only 5 companies have a SIH less than 500 trading days (or about two years). In addition, based on different slopes in these curves, if investors increase their investment horizons, the frequency of companies increases with variable intensity. For example, if the investment horizon is 1,250

Table 2. Selected Results for Calculated SIR and SIH Values for Different Companies

Threshold value	Number of Companies with $SIR = 100\%$	Percentage of Companies with $SIR = 100\%$	Percentage of Companies with $SIR < 100\%$	Minimum value of SIR	Minimum value of SIH
$Treshold = 0$	418	91.87	8.13	43.89	371
$Treshold = r_f$	414	90.99	9.01	43.43	371

Notes: This table contains the values of SIR and SIH calculated for 455 companies from 2000 to 2019 by using two values of zero and risk-free rate as threshold values. To better explain the results, these values are reported separately for companies with 100% of SIR and other companies.



Notes: This figure shows the percentage frequency of companies with a SIR equal to 100% on the y-axis with the number of these companies in boxes on the curve for different values of SIH on the x-axis. The left-hand side diagram uses a zero value for the threshold, while the right-hand side diagram is based on the risk-free rate for the threshold.

Figure 5. Frequency of Companies with a SIR of 100% Based on Different Values of SIH

days (5 years), 59 companies have returns more than the threshold value for certain, and this value approximately doubles (106 companies, i.e., about a quarter of the companies) with the investment period of 1500 trading days (6 years). For half of the companies, it is necessary for the investors to wait about 10 years (2,500 trading days) to have a safe investment. At the final portions of these charts, the slope gradually decreases, and then increasing the investment horizon has a little effect on having a safe investment.

5.5. The Explanatory Power of SIH and SIR

To test the explanatory power of SIH as a factor that affects stock return, the capital asset pricing model (CAPM, Sharpe, 1964), the three-factor model (Fama and French, 1993), and the five-factor model (Fama and French, 2015) are used, and the SIH and SIR factors are added to these models, followed by comparing the results before and after introducing this factor. In order to do this, the regression method of Fama and Macbeth (1973) is ussed based on the return data of companies, discussed in the previous section, for the same period. Results are presented in Table (3) in three different panels (panel a: CAPM, panel b: the 3-factor model, and panel c: the 5-factor model). In each panel, the first column presents the estimation results for the original model (without adding the SIH and SIR variables), and the other four columns present the results of each model when the SIH variable in the form of a logarithmic scale, i.e., Ln(SIH), or the SIR variable are added. It should be noted that using the logarithm of SIH is to make it scale-free. On the other hand, the SIH and SIR factors are calculated with two threshold values, i.e., risk-free rate and zero, tested separately in different columns. Moreover, the F statistics and the adjusted R-squared are calculated and presented for each regression.

As the results in the above table show, based on values of the F statistics, only the CAPM regression is not statistically significant, while the coefficient of SIH in

Table 3. Results for the Explanatory Power of SIH and SIR

Panel a					
Name	*CAPM*	$Threshold: R_f$		$Threshold: Zero$	
		$Ln(SIH)$	SIR	$Ln(SIH)$	SIR
Constant	0.0004***	0.0009***	-0.0018***	0.0012***	-0.0018***
$R_m - R_f$	-0.0035	0.0000	0.0004	0.0034	0.0000
New Variable	---	-0.0001**	0.0022***	-0.0001***	0.0023***
F-statistic	0.41	3.50**	67.37***	8.31***	63.11***
Adj R^2	-0.0013	0.0113	0.2326	0.0323	0.2209
Panel b					
Name	*3-Factor*	$Threshold: R_f$		$Threshold: Zero$	
		$Ln(SIH)$	SIR	$Ln(SIH)$	SIR
Constant	0.0004***	0.0008***	-0.0018***	0.0012***	-0.0018***
$R_m - R_f$	0.0006	0.0042	0.0058	0.0075	0.0054
HML	-0.0150***	-0.0147***	-0.0126***	-0.0144***	-0.0125***
SMB	-0.0017	-0.0019	-0.0043	-0.0018	-0.0045
New Variable	---	-0.0001**	0.0022***	-0.0001***	0.0022***
F-statistic	7.25***	6.99***	39.83***	9.41***	37.34***
Adj R^2	0.0411	0.0519	0.2618	0.0714	0.2492
Panel c					
Name	*5-Factor*	$Threshold: R_f$		$Threshold: Zero$	
		$Ln(SIH)$	SIR	$Ln(SIH)$	SIR
Constant	0.0006***	0.0008***	-0.0011***	0.0011***	-0.0011***
$R_m - R_f$	-0.0133	-0.0113	-0.0080	-0.0089	-0.0081
HML	-0.0321***	-0.0316***	-0.0261***	-0.0309***	-0.0264***
SMB	-0.0003	-0.0005	-0.0026	-0.0005	-0.0027
RMW	0.0348***	0.0339***	0.0263***	0.0327***	0.0272***
CMA	-0.0552***	-0.0546***	-0.0445***	-0.0536***	-0.0452***
New Variable	---	0.0000	0.0017***	-0.0001**	0.0017***
F-statistic	33.09***	27.99***	48.15***	28.95***	46.97***
Adj R^2	0.2681	0.2700	0.3924	0.2769	0.3863

Notes: In this table, the results for three models, i.e., CAPM, the 3-Factor, and the 5-Factor models are presented in panels a, b and c, respectively. In each panel, the natural logarithm of SIH, i.e., $Ln(SIH)$*, or SIR are added to the relevant model. The values of SIH and SIR are calculated based on different threshold values (risk-free rate and zero). The models with the additional factors are estimated for all companies in the sample (455 companies). The values of the F statistics and adjusted* R^2 *are presented at the bottom of each panel. In all estimated regressions, classic assumptions of regression and robustness are tested, and relevant remedies are applied. The figures marked with *, **, and *** indicate significant coefficients in 90%, 95%, and 99% confidence intervals, respectively.*

the 5-factor model, estimated for all companies based on the risk-free rate as the threshold value (in panel c), is not statistically significant. However, in all other parts, the coefficients of SIH and SIR are significant at least in the 95% and 99% confidence intervals, respectively. Nonetheless, the SMB factor is not significant in regressions. This means that the explanatory powers of SIH and SIR are more than that of the SMB, which is one of the main variables in the 3 and 5-factor models. The same is true for the CAPM model based on the results of estimating the coefficients of the variable R_m-R_f, all of which are not statistically significant. The variable coefficients of SIH are negative in all estimated models (except for the one in panel c), and coefficients of SIR are positive in all regressions, which is perfectly consistent with the concept of SIH and SIR because with higher values of SIH (lower values of SIR), the company under analysis reaches the safest state over a longer period (with more probability), and investors are less (more) interested in this company; as a result, the estimated coefficient is expected to be negative (positive). In all regressions, by adding the SIH and SIR variables to the original models, the explanatory power of the models is increased based on the values of adjusted R-squared.

5. CONCLUSIONS

While the importance of the investment horizon in financial decisions has been confirmed in several studies, due to the assumptions of classical financial models (iid returns), this influential variable has been ignored in mainstream financial models. However, this assumption has been challenged and rejected in several studies.

Accordingly, in this study, the effects of the time horizon were studied and analyzed, and a practical approach for selecting the appropriate investment period was tested. For this purpose, using the idea of the Safety-First Criterion (Roy, 1952), a criterion called the safest investment horizon (SIR) was introduced as the ratio of the number of times the return on assets is greater than the threshold return value. Then, the shortest horizon in which this ratio is maximized was introduced as the best investment horizon, i.e., the safest investment horizon (SIH). It is worth mentioning that two values of zero return and risk-free interest rate were selected as threshold values in the current study. However, any other values for the threshold variable can also be tested and analyzed.

It has also been theoretically shown that if the optimal investment period is accompanied by an SIR value of 100% (maximum possible value) based on the risk-free rate as the threshold value, it can be said that by choosing this period as the investment period, arbitrage opportunities can be achieved. Therefore, the proposed criterion can be considered as a novel way to discover arbitrage opportunities in the market. On the other hand, by theoretically analyzing the effects of the time period on optimal portfolio selection, a new optimization problem was proposed to select the optimal portfolio, and, at the same time, determine the optimal time horizon in portfolio selection.

In addition to the abovementioned theoretical analyses, the proposed method was implemented on real data and tested in a number of different ways. For this purpose, two separate datasets were employed. The first dataset is related to the S&P-500 Index information, while the second dataset is related to the stock return information of the companies in this index, both for the period of 2000 to 2019. In the first step, the effects of altering the time horizon on the overall distribution shape of returns were presented, which is inconsistent with the assumption of iid of returns, and indicates the necessity to pay attention to the investment horizon. In addition, the results of implementing SIR and SIH criteria on the S&P-500 Index indicate the possibility of achieving a 100% confidence by choosing a 14-year investment period for this index. Also, by shortening this period, the probability of returns higher than the threshold returns decreases. For instance, with a period of 176 days or 199 days with 70% confidence, surplus returns can be obtained at zero and risk-free returns, respectively. Moreover, based on stock returns, most of the companies have an SIR equal to 100%, so it is possible to find arbitrage opportunities by choosing the right investment horizon. Another result presented in this study involves the effects of the time horizon on optimal portfolio composition, which indicates the importance of paying attention to this variable in portfolio optimization and asset allocation. In order to further investigate the capabilities of SIH, the explanatory power of this variable was investigated using CAPM, the three-factor, and the five-factor models proposed by Fama and French. Based on the results of the estimates, not only is the SIH variable in these models significant (while some of the main variables of these models are not statistically significant), but also adding this variable to the original models leads to an increase in the explanatory power of the models.

In general, the results of this study are consistent with previous studies, indicating the importance of paying attention to the investment time horizon. Therefore, the SIH criterion is useful for (1) selecting the optimal time horizon, (2) identifying arbitrage opportunities in various periods, and (3) improving the explanatory power of existing models. It is also necessary to use this criterion to determine the optimal portfolio.

References

Amadi, Felicia & Amadi, Confidence. (2019). Investment Horizon and the Choice of Mutual Fund. International Journal of Business and Management.

МЕТОДА ЗА ОДАБИР ОДГОВАРАЈУЋИХ ПЕРИОДА УЛАГАЊА ЗА ОСТВАРИВАЊЕ ПРОФИТА ОД АРБИТРАЖЕ И ОБЈАШЊЕЊЕ ПОЈМА “STOCK RETURNS“

Moslem Peymany

Извод

Овај рад представља теоријске и експерименталне доказе који оправдавају потребу обраћања пажње на хоризонт улагања. Због тога се за одабир одговарајућег хоризонта улагања користи критеријум под називом „Хоризонт најсигурнијег улагања“ (СИХ). Да би се израчунала ова количина, израчунава се коефицијент који се назива 'најсигурнији однос улагања' (СИР) и разматра се однос између ових критеријума и могућности арбитраже, заједно са методама за остваривање профита од арбитраже кроз одабир одговарајућег временског хоризонта. Након тога, применом ове методе за податке из стварног живота, потврђује се присуство могућности арбитраже у различитим временским хоризонтима. Надаље, описани су ефекти временског хоризонта на оптималну композицију портфеља. Коначно, показано је да ови критеријуми надмашују неке од конвенционалних варијабли у ЦАПМ, моделима са 3 фактора и 5 фактора за објашњење приноса акција и коришћење СИХ или СИР као нове варијабле повећава снагу објашњења ових модела.

Кључне речи: хоризонт улагања, теорема арбитраже, оптимизација портфолија, факторски модели

Barberis, N. (2000), Investing for the Long Run when Returns are Predictable, Journal of Finance, Vol. 55, No. 1 (February), pp. 225–64.

Butler, Kirt C; Domian, Dale L. (1991). Risk, Diversification, and the Investment Horizon, Journal of Portfolio Management; New York Vol. 17 (3): 41.

Campbell, John Y. and Viceira, Luis M. (2005), The Term Structure of the Risk-Return Trade-off, Financial Analysts Journal, 61(1), 34-44.

Dierkes, M., Erner, C., & Zeisberger, S. (2010). Investment horizon and the attractiveness of investment strategies: A behavioral approach. Journal of Banking & Finance, 34(5), 1032-1046.

Fama, E. F., & French, K. R. (1993). Common risk factors in the returns on stocks and bonds. Journal of Financial Economics, 33 (1): 3-56.

Fama, E. F., and K. R. French. (2015). A five-factor asset pricing model. Journal of Financial Economics, 116(1): 1-22.

Fama, Eugene F.; MacBeth, James D. (1973). Risk, Return, and Equilibrium: Empirical Tests. Journal of Political Economy. 81 (3): 607–636.

Ferguson, Robert and Simaan, Yusif. (1996). Portfolio Composition and the Investment Horizon, Journal of Portfolio Management, Vol. 22, No. 4.

Gil-Bazo, J. (2006). Investment horizon effects. Journal of Business Finance & Accounting, 33(1-2), 179-202.

Gilmore, C.G. (1996), Detecting Linear and Nonlinear Dependence in Stock Returns: New Methods Derived from Chaos Theory. Journal of Business Finance & Accounting, 23: 1357-1377.

Goetzmann, W. N., Brown, S. J., Gruber, M. J., & Elton, E. J. (2014). Modern portfolio theory and investment analysis. John Wiley & Sons, 237.

Gunthorpe, D., & Levy, H. (1994). Portfolio composition and the investment horizon. Financial Analysts Journal, 50(1), 51-56.

Hirsa, A., & Neftci, S. N. (2013). An introduction to the mathematics of financial derivatives (2nd Edition). Academic press.

Hodges, C. W., Taylor, W. R., & Yoder, J. A. (1997). Stocks, bonds, the Sharpe ratio, and the investment horizon. Financial Analysts Journal, 53(6), 74-80.

In, F., Kim, S., & Gençay, R. (2011). Investment horizon effect on asset allocation between value and growth strategies. Economic Modelling, 28(4), 1489-1497.

Knut K. Aase. (2017). The investment horizon problem: a possible resolution, Stochastics, 89:1, 115-141.

Levy H. (1972), Portfolio Performance and the Investment Horizon, Management Science, 18, 645-653.

Levy H. (1981), The CAPM and the Investment Horizon, Journal of Portfolio Management, 7, 32-40.

Levy H. (1984), Measuring Risk and Performance Over Alternative Investment Horizons, Financial Analyst Journal, 40, 61-68.

Levy H. and Duchin R. (2004). Asset Return Distributions and the Investment Horizon, The Journal of Portfolio Management, 30 (3): 47-62.

Levy H., and Paul A. Samuelson (1992), The Capital Asset Pricing Model with Diverse Holding Periods, Management Science, 38, 1529-1542.

Marshall, J.F. (1994), The Role of the Investment Horizon in Optimal Portfolio Sequencing (An Intuitive Demonstration in Discrete Time). Financial Review, 29: 557-576.

Roy, Arthur D. (1952). Safety First and

the Holding of Assets. Econometrica. 20 (July): 431–450.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. The journal of finance, 19(3), 425-442.

Sharpe, W.F. (1994), The Sharpe Ratio, The Journal of Portfolio Management, 49, 49-58.

Simonsen, I., Johansen, A., & Jensen, M. H. (2006). Investment horizons: A time-dependent measure of asset performance. In Practical Fruits of Econophysics (pp. 246-251). Springer, Tokyo.

Van Eaton, R. Douglas, and James A. Conover. (2001). Stock and Bond Sharpe Ratios and Long-Horizon Asset Allocation, The Journal of Investing, 10 (1): 35-42.

Woodside-Oriakhi, M. Lucas, C. and Beasley, J.E. (2013). Portfolio rebalancing with an investment horizon and transaction costs, Omega, Volume 41 (2): 406-20.