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LANCHESTER'S DIFFERENTIAL EQUATIONS AS OPERATIONAL COMMAND DECISION MAKING TOOLS

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Abstract

This paper investigates the application of Lanchester's equations as a scientific method and tool for examining the functioning of the armed forces as complex organizational systems in combat. It is important to assess the reliability of the knowledge obtained by this method, about the facts of the operational environment and the effectiveness of the use of forces, in order to support the process of planning and making optimal decisions, in conditions of uncertainty and risk, which are inherent in warfare. According to this hypothesis, a mathematical model was developed based on the well-known Lanchester's equations, which defined the quadratic and linear law of combat between two opponents with a heterogeneous force structure (air force and army). The created model enables a correct simplified analysis in the decision-making process. Real war and combat operations are very complex and require the use of complex simulators, whose methodological background is often unknown to decision makers, which is why reliable approximate simulation and modeling methods are necessary and desirable.

Keywords: differential equations, warfare modeling, attrition process of forces

1. INTRODUCTION

The operational environment in modern warfare is far more complex and time-sensitive for the decision-making process than ever before in history. If this process is based only on experience, advanced operational capabilities and new operational knowledge in the process of planning operations or developing the operational design of new military doctrines will already be obsolete. On the other hand, if combat postulates are not confirmed by experience but only implemented in Military Doctrine, it leads to failure or disaster. Therefore, the optimal war plan must be based on a correct

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decision-making process that takes into account mutual combat capabilities and the correct way of using all forces that enables victory or defeat (Doctrine, 2000) in modern warfare. This is not possible without the synergy of experience and scientific methods, and it is the only way to define a proper operational concept in Military Doctrine (Doctrine, 2003). To do this, strategic and operational military management must be able to use combat modeling and simulation in the planning process. This claim was proven by the planning process of Operation Desert Storm (Survey I, 1993) where standard doctrinal principles were abandoned in favor of solutions obtained through modeling and simulation.

The aim of the research in this paper was to consider application of Lanchester's (1916) equations as a scientific method and tool for examining the functioning of the armed forces as complex organizational systems in combat, a method for quick assessment of the possibility of victory in battle, as well as the optimal use of combat forces in accordance with the principles of military doctrine at the operational-strategic level of war. The method used is based on the process of loss of forces over time, caused by the way of use, the passage of time, the strength of the forces at the beginning of the fight and the effective rate of attrition (combat capabilities). The actual application of the Lanchester's equations can be relatively simple, taking into account a few parameters, or very complex. Lanchester's equations can be used to evaluate and verify doctrinal principles in the conditions of modern combat, making it suitable for rapid strategic assessment. In the second part of the paper, the theoretical foundations of the used method and a brief historical overview of its beginning and evolution are given. The third part presents the modeling and mathematical description of the problem. This process was carried out with Lanchester's equations, applying the linear and quadratic laws of combat in accordance with operational scenarios and defined combat action flows. The fourth section presents the experimental results. This process was carried out by experimenting on the model performed according to the specific scenario, in accordance with the way of engagement of different forces and combat effectiveness and efficiency. The fifth section considers results analysis and discussion about their correlations to warfare experience and Lanchester's combat Law.

2. THEORETICAL BACKGROUND

Originally, Lanchester's (1916) work presented two mathematical laws of combat that explained two historical modes of combat. The first is linear and is immanent to ancient and medieval battles. The basic idea of Frederick Lanchester is a fight between two opponents whose forces are homogeneous. The second is the modern struggle, which characterizes the war history of the second half of XIX century. Although these equations represent the beginnings of a special branch of applied mathematics "operational research", their utility value is still current as well as their significance for the development of modern combat simulations. The further development of his originally idea is more complex and represents a more realistic fight between heterogeneous forces, where there is a mutual influence of the state of the forces, methods of engagement and combat capabilities (Caldwell et al., 2000).

Lanchester's (1916) linear law is characterized by the use of melee weapons, on foot or on horseback and possibly dueling archers, catapults and similar ancient weapons. The forces attrition was caused by the duel of two individual opponents. Maneuver has certain significance, especially when it causes the disintegration of the order of battle of the forces. The concentration of troops was not so important compared to modern warfare. It is assumed that the opponents, regardless of the number of forces, basically have the same or similar fighting capability, and the fight is an individual duel that can be expressed as a rate of attrition. For example: one adversary with the strength of forces (Y) can cause losses to the other party whose number is (X) with the rate of attrition (β). Or in another case, the side with power strength (X) can impose losses on the other side with power strength (Y) at the rate of attrition (α). This means that if two opponents have different numbers of forces at the beginning of the battle (X_0) and (Y_0), the total losses at the end will be proportional to the number of troops engaged. The combat result will not be affected by tactics or operational skill in terms of the concentration of force created by the maneuver. Lanchester did not provide any equations for ancient warfare, but as Taylor (1980) states, it is clear from his work that he is referring to the combat attrition of forces for which the current ratio of losses inflicted is independent of the number of combatants. Analytical expression for this process is (Caldwell et al., 2000):

$$\frac{dX}{dt} = -\alpha \quad \text{and} \quad \frac{dY}{dt} = -\beta, \qquad \text{for} \quad X(t) > 0 \;\; \Lambda \;\; Y(t) > 0 \tag{1}$$

This expression implies that force (X) is superior compared to force (Y) only, and only if: (α>β). However, if we assume the same attrition rate for both opponents, the one with the larger number of troops has the advantage in battle (Kress, 2020). Both sides will weaken until one suffers unacceptable losses and surrenders or is completely destroyed. According to this, the rate of change of (Y) with respect to (X) is a constant and has oblique as slope. By eliminating time, the state of forces is related by the equation (Caldwell et al., 2000):

$$\beta * (X_0 - X_i) = \alpha * \left(Y_0 - Y_j\right) \tag{2}$$

Relation (2) can also be expressed as (Washburn, 2000):

$$X_i = X_0 - \frac{\alpha}{\beta} * \left(Y_0 - Y_j\right), \;\; \text{if} \;\; Y_j = 0$$

$$\text{and} \qquad \beta * X_0 \geq \alpha Y_0, \qquad \text{or} \tag{3}$$

$$Y_j = Y_0 - \frac{\beta}{\alpha} * (X_0 - X_i), \;\; \text{if} \;\; X_i = 0$$

$$\text{and} \qquad \beta * X_0 \leq \alpha Y_0 \tag{4}$$

A special form of the Lanchester's linear law of combat is represented by operations characterized by the law of probability without precise shooting, such as bombing with artillery or air support of an area that is evenly occupied by the opposing armed force.

Lanchester's square law, on the other hand, assumes the decisive influence of force concentration. Even in the case where both sides have the same attrition rate or one is slightly better, the advantage in force numbers has a decisive influence (MacKay, 2002). This implies: the winner is on the side with a better concentration of forces at the right time and in the right place or perhaps has larger army divisions or air wings and squadrons. It also means that a numerically inferior force must possess much greater combat capability to attain equality of power (Lanchester, 1916). Osipov (1915), who is considered to be Lanchester contemporary, stated this conclusion independently of Lanchester in his work. After analyzing 38 known battles, he came to the conclusion that in 55% of cases the theory of the victory of numerically superior forces was proven, in 37% the weaker side won, and in 8% the result was undecided. This led him to the conclusion that, in addition to the numerical strength of the opponent, there are other factors that affect the speed of inflicting losses, and ultimately the victory. The influence of various parameters such as: maneuver, tactical decisions, logistics, firing process, operational situation factors (weather, geography, etc.) should be taken into account. These factors affect the rate of attrition and the law of battle of the belligerents.

Lanchester's square law of combat, where homogeneous forces of two adversaries fight against each other (for example fighter jets) can be expressed by a system of differential equations as a transformation of the system from one state to another. This method is extremely complex as evidenced by the example of air combat between two groups of opponents (Petric, 1974). Fortunately, in practical use of Lanchester's square law there is no need to describe each state of the system in detail. If there is no change in time for the attrition rates (α) and (β), then the differential equations can be expressed as a system of ordinary differential equations (Washburn, 2000):

$$\frac{dX_i}{dt} = -\beta * Y_j \quad \text{and} \quad \frac{dY_j}{dt} = -\alpha * X_i, \tag{5}$$

$$\text{for} \quad X(t) > 0 \;\; \Lambda \;\; Y(t) > 0$$

The solution to the problem is more precise if the number of opposing forces is large enough to avoid the problem of random events inherent in small force problems. It is very easy to see that the number of remaining aircraft will decrease faster (attrition rate) if the enemy's combat capabilities are greater or if the enemy possesses greater strength (Petric, 1974). This is Lanchester's basic quadratic law model and the algebraic form of the equations for any moment of combat (Petric, 1974):

$$X_i = X_0 \cdot \cosh\sqrt{\alpha \cdot \beta \cdot t} - Y_0\sqrt{\beta/\alpha} \cdot \sinh\sqrt{\alpha \cdot \beta \cdot t} \tag{6}$$

and

$$Y_j = Y_0 \cdot \cosh\sqrt{\alpha \cdot \beta \cdot t} - X_0\sqrt{\alpha/\beta} \cdot \sinh\sqrt{\alpha \cdot \beta \cdot t} \tag{7}$$

It is important to note the appropriate application of this model in the analysis of the influence of various factors on the success of the battle (Petric, 1974) such as: sudden strike, tempo of force building (mobilization, reinforcement, etc.). This method is also suitable to perceive combat dynamic from the tactical to the strategic level. It is very useful as a simple combat model because it can be easily solved using a specific explicit form of the analytic function. These differential equations are basic for the application of the slightly more complex Ditchman's (1962) law of mixed combat, which enables the simulation of the combat dynamics of qualitatively different or heterogeneous opponents. For example: the warfare of two opponents in guerilla and conventional combat. This problem could be solved by a combination of quadratic and linear laws (Handbook, 1979).

During World War II, an adapted Lanchester's model was used in differential game theory to solve ammunition distribution problems in short and long-term logistics support missions. The research of this problem showed a practical quality for solving a strategic problem in the distribution of resources, in addition to the analysis of the historical struggle and the importance of the concentration of force efforts. Many works have been published on historical battles such as the Ardennes campaign (Fricker, 1997), The battle of the Iwo Jima (Engel, 1954) and Kursk (Thomas 2004), artillery and air support; strategy optimization in relation to weapon range, enemy attrition rate and operational costs (Isaacs, 1965); problem solving for air operations with regard to combat resources due to the distribution of combat sorties in air support operations, offensive and defensive counterair operations (Berkovitz & Dresher, 1959), SEAD operation (Timothy et al., 2002), and air combat model engagement and attrition processes (Patrick, 1998).

A shortcoming of the basic Lanchester's model methodology is the consideration of a constant attrition rate of forces. For this purpose, combat modeling with partial differential equations was developed (Protopopescu et al., 1990) which can even calculate the contribution of Intelligence (Coulson, 2019). Given these facts, it is understandable why the basic model is not suitable for modeling a real war combat. The above confirms the historically proven structure of forces in battle, which is always heterogeneous (infantry, artillery, etc.). The basic Lanchester's model considers only a homogeneous power structure. As Taylor (1983) said „for small-scale operations it may be possible to reasonably represent force interactions and attendant attrition rates with a few differential equations, but for large-scale operations of conventional armed forces the same approach might well involve hundreds (and possibly even thousands) of differential equations tied together through battlefield operations ". Based on these arguments, given the complexity of the methodology for the practical solution of this problem, Taylor (1983) pointed out that only a few useful analytical models have been developed. Furthermore, he asserts three main approaches in simulating an attrition-based combat model:

- Monte-Carlo simulation,
- Aggregated Force-Fire Power Score approach and
- Detailed Lanchester type model

For modeling large-scale combat operations, such as a strategic land-sea-air operation or campaign, more suitable are Aggregated Force and Detailed Lanchester type model. Monte-Carlo simulation is more suitable for small-scale combat model (bellow battalion force level). Regardless of the stochastic and deterministic nature of these methods, many authors consider both models quite similar in terms of results, but the deterministic model is more practical to use (Taylor, 1983). A modern approach to the methodology of solving Lanchester type models could be found in "Enrichment of Lanchester type models", by Caldwell et al. (2000) work. This methodology is based on developed procedures for the numerical solution of Lanchester type models. The simplest of the numerical methods according to Washburn (2000) is Euler's method, which he emphasized works very efficiently in both cases. A significant contribution to the development of this methodology is given by the works: Kress (2020), Caldwell et al. (2000) and Washburn (2000). An interesting war model created by Seung (2013) is based on a multi-weapon extension. However, the basic Lanchester's model is still very interesting for practical application in a simple strategic simulation as an auxiliary method for a practical and quick assessment of the outcome of the battle and the development process of the battle. There is a very interesting paper by Hsiao and Guu (2004), which proves this claim. This is the battle model of two component forces such as Air Force and Army in which the air force acts as close fire support to the ground forces.

Table 1. The meaning of symbols

Forces by type	Army Force		Air Force	
Belligerent parties	*Blue*	Red	Blue	Red
The strength of forces	*X*	Y	Z	Q
Coefficients of attrition	α_1	β_1	α_2	β_2

The meaning of symbols is as follows:

- Symbols (X) and (Y), as well as symbols (Z) and (Q), represent different type of forces, e.g. army and air component for two opponents in combat, e.g. Blue and Red;
- Symbols (α_1/ β_1) are the force attrition coefficients, with which ground forces (X/Y) cause losses to opposite ground forces (Y/X);
- Symbols (α_2/ β_2) are the force attrition coefficient, with which air forces of Blue or Red (Z/Q) cause losses to opposite ground forces Red and Blue (Y/X).

Combat process is presented in algebraic form (Hsiao & Guu, 2004):

$$X_i = X_0 \cdot \cosh\left(\sqrt{\alpha_1 \cdot \beta_1 \cdot t}\right) - Y_0 \cdot \sqrt{\frac{\beta_1}{\alpha_1}} \cdot \sinh\sqrt{\alpha_1 \cdot \beta_1 \cdot t} + Q_0 \cdot \left(\frac{\alpha_2}{\alpha_1}\right) \cdot \left[\cosh\sqrt{\alpha_1 \cdot \beta_1 \cdot t} - 1\right] - Z_0 \cdot \left(\frac{\beta_2}{\sqrt{\alpha_1 \cdot \beta_1}}\right)] \cdot \sinh\sqrt{\alpha_1 \cdot \beta_1 \cdot t} \tag{8}$$

$$Y_j = Y_0 \cdot \cosh\left(\sqrt{\alpha_1 \cdot \beta_1 \cdot t}\right) - X_0 \cdot \sqrt{\frac{\beta_1}{\alpha_1}} \cdot \sinh\sqrt{\alpha_1 \cdot \beta_1 \cdot t} + Z_0 \cdot \left(\frac{\beta_2}{\beta_1}\right) \cdot \left[\cosh\sqrt{\alpha_1 \cdot \beta_1 \cdot t} - 1\right] - Q_0\left(\frac{\alpha_2}{\sqrt{\alpha_1 \cdot \beta_1}}\right)] \cdot \sinh\sqrt{\alpha_1 \cdot \beta_1 \cdot t} \tag{9}$$

In the case of solving the „scenario" problem in this paper, the qualities of the Lanchester type model are used, as a very useful and practical method for simple model simulation. This simulation considers opponents with two (heterogenic) types of forces: Air Force and Army. According to the scenario this force can be used in the following types of operations: offensive and defensive counterair operations, air support and ground/joint offensive or defensive operations. The results can help in the analysis of the factors for victory in the strategic operation (campaign) of war and the creation of an optimal operational combat concept, as the basis of the operational Military doctrine. Practical solving of combat problems was carried out using three types of analytical equations. In the case of fighting independent parts of the opponent's forces (homogeneous forces), the following were used: Lanchester's linear law (3) and (4) and Lanchester's quadratic law (6) and (7). The combat programming of heterogeneous forces (Joint operations and air support operations) was quite complicated. Programming was done according to Hsiao and Guu (2004), based on Lanchester's quadratic law, but for heterogeneous force structure (8) and (9).

3. METHODOLOGY –MATHEMATICAL MODELING

War plan is based on operational Military doctrine, which makes mindset and shapes the outlook for future operational combat concept (Concept, 2006). Strategic operations (campaign) are very complex and consist of lower level operations such as air, land or naval operations or combined offensive and defensive operations. Air operations are very significant and many authors agree that no one can win in a modern war without air superiority over battlefield or if opponent can successfully use its air power (Saunders, 2020). However, this is often forgotten, even by the great powers, as we have witnessed throughout the history of warfare in the 20th and 21st centuries. Air power creates air superiority over battlefield and supports land/sea force maneuver. Joint operations are force multiplier, which can create favorably operational environment which bring win to war (Doctrine, 2003).

The correct use of combined forces is a very delicate process and it is important to know the right way to use them, which important factors must be taken into account, such as combat capabilities and the ability to decisively strike the enemy in a given operational environment. The operation planning process (Handbook, 2018) is similarly applied when making plans for all types of combat operations. On the other hand, plans as products of this process and their execution can be very different. Operational environment and important operational factors such as: operational objective, combat capabilities, weather and geography, have a strong influence on mission success, measured by defined criteria (Vlatkovic, 1988) and can be very different. Process of modeling as a methodology is of great importance as a tool for designing combat operation. It is very useful for building up the armed forces and preparing an adequate response to various security challenges. It offers expertise in the battle planning process taking into account the complexities of the wartime operational environment

Theoretically, all types of forces can be engaged in offensive or defensive combat operations (Vego, 2002). The mode of action depends on the operational situation, the operational capabilities of the forces and the combat capabilities of the combat systems in use. On the other hand, the method of engagement and the type of combat operations also affect the outcome of the battle, the duration of the battle and the mutual attrition of forces. In this paper, three types of air operations and two types of army and joint operations and their variations will be discussed (Doctrine, 2010).

Offensive counterair operations (Doctrine, 2008) are the primary means of achieving air superiority over a battlefield or area of operation. It is characterized by the initiative to influence enemy ground forces to defend their own territory and be passive. It is an essential tool for transferring the combat deep into enemy territory, blocking airfields, neutralizing air defenses, destroying logistics and command systems. All these actions have the effect of disorganizing the enemy's air force as a system and create the effects of air supremacy or air superiority. Defensive counterair operation (Doctrine, 2008) is another form of combat use of air power that is characterized by the passive use of air power in combat over one's own territory. The essential meaning for this type of operation is defensive and protection of ground forces and capital infrastructure and resources. The most important difference compared to an offensive operation is that the initiative passes to the enemy. This has been the basic use of any air force since its first appearance over the battlefield in WWI. Strategic air operation (Doctrine, 2007) and air support operation are two ways of properly using air forces as air power. These operations enable the combat and destruction of enemy force and strategic resources, deep above enemy territory.

Army force may be engaged in attack or defense operations (Doctrine, 2022). There are some variations in the use of army force in offensive or defensive operations. The first case is characterized by engagement of only army against the enemy army. Another case is a Joint operation in which both army and air force inflict losses on the enemy army. Army force can perform various types of maneuvers, but for this paper it is only important to mention the decisive and elusive defense. In the first case, it means fighting without retreat and in the second case, delaying the fight until the air force gains air supremacy.

There are many possible courses of action (CA), given the different modes of tactical engagement, based on the combat capabilities of the forces to use different modes of battle formation and tactics. It is always useful to create a scenario (Doctrine, 2011), according to a real or hypothetical situation. The reason for this methodological approach is familiarity with the problem and ways of possible solutions.

In this case, the hypothetical war scenario (Doctrine, 2007) considers two opponents: The Red side is the aggressor and conducts the attack but the Blue side may choose to defend by using a defensive or offensive course of action in separate or simultaneous air, ground or joint operations. According to the battle scenario and the stated assumptions, six characteristic courses of combat action of the Red and Blue forces developed by the author should have been considered.

The course of action (1) is basic example. Blue conducts strategic defensive operation (campaign). The Blue Army is engaged in a defensive ground operation and the Red army is conducting an offensive ground operation. The Blue Air Force is engaged in defensive counterair operation over its own territory. The Red Air Force was engaged in offensive counterair operations against the Blue Air Force. This is an example to describe the basic model of the Lanchester's equations and the attrition of forces according to the square law (Caldwell et al., 2000). The influence of parameters such as the rate of attrition and the number of forces engaged in combat is noticeable.

The course of action (2) considers a strategic operation in two phases (Doctrine, 2021). In the phase 1 both ground and air forces conduct ground and air operations separately, as in CA (1). After one of the opposing air forces is defeated in Phase 2, the winner shifts the mission to an air support operation.

The course of Action (3) is the joint strategic defense operation of the Blue force. The Blue Army and Air Force conduct a joint defense operation (Operations, 2017). The Blue Air Force supports its ground forces in air operations, but does not fight against the

Red Air Force. The Red Army conducts an offensive ground operation and the Red Air Force conducts an offensive counterair operation. If Red ground forces are defeated Blue wins. Otherwise, if the Red Air Force defeats the Blue Air Force, it could shift its mission to air support. In the end, the side whose ground forces survived the battle wins.

The course of action (4) is characterized by two phases of the Blue Defense Operation. The Red Air Force and the Red Army are conducting an offensive joint operation. The Red Air Force supports its ground forces in the attack in phase 1 of the operation. The Blue Air Force protects its Army in a defensive counterair operation. If the Blue Army survives and the Red Air Force is destroyed, the Blue Air Force switches operations to air support.

The course of action (5) is characterized by multirole tactical operations by the air forces of both sides. This means that air forces of both sides operate simultaneously as air superiority fighters and as air support for ground forces. Their combat capabilities are lower in this case, which implies a lower rate of ground force attrition for both sides. Red forces are engaged in a joint offensive operation and Blue forces are conducting a joint defensive operation. The winner is the side that ultimately possesses the army power.

The course of action (6) considers defensive-offensive engagement of the Blue forces. This case is specific because the Blue Army evades direct combat with Red Army as long as the Blue Air Force gains air superiority in offensive counterair operation. With the Red Air Force defeated, the Blue Air Force diverts tactical operations to air support for its ground forces. The Blue Army with the support of their air force is conducting a joint offensive operation.

Courses of action are compared and analyzed according to the sets of defined criteria for success:

- the effective execution of the mission (the enemy army is annihilated),
- the exhaustion and numbers of the rest of the army and air forces and
- the duration of the operation.

It is important to note that the focus in the comparison of results is not only a table with numerical results, although it is important that the solution is optimal. More important was the confirmation of the simplicity of applying the Lanchester’s equation in the decision-making process and the analysis of their applicability based on the comparison of results with historical facts and empirical knowledge of strategic facts and operational (combat) principles.

Assumptions for creating an adequate model:

- full engagement of forces in the operation;
- there is a possibility of postponing combat engagement for both land and air forces;
- no reinforcement or replenishment of forces during combat operations;
- army force can engage only opposing army force;
- both air forces have multirole capabilities;
- the combat capability of both air forces is equal in air support or engagement in an air superiority mission;
- the combat capability of both air forces is higher when engaged in only one mission or lower in multi-purpose engagement;
- If one opponent's army force is destroyed, the fight is over and the other side is the winner.

According to the scenario, parameters of forces with numerous states and attrition rate coefficient for each type of forces are shown in table (2).

Table 2. Parameters of forces (strength and attrition rate coefficient)[1]

Forces by type	**Army Force**		**Air Force**	
Belligerent parties	Blue (X)	Red (Y)	Blue (Z)	Red (Q)
The strength of forces	100	200	200	100
Coefficients of attrition	α_1	β_1	α_2/α_3	β_2/β_3
	0.3	0.25		
			0.5	0.3
			0.6	0.4

Mathematical description of the problem, according to the scenario and defined courses of action:

All forces in combat suffer losses according to the square law of combat (6) and (7).

Course of action 1

$$\dot{X}_1(t) = -\beta_1 * Y_1, \qquad X_1(t_0) = X_0 \qquad (10)$$

$$\dot{Y}_1(t) = -\alpha_1 * X_1, \qquad Y_1(t_0) = Y_0 \qquad (11)$$

$$\dot{Z}_1(t) = -\beta_3 * Q_1 \qquad Z_1(t_0) = Q_0 \qquad (12)$$

$$\dot{Q}_1(t) = -\alpha_3 * Z_1, \qquad Q_1(t_0) = Z_0 \qquad (13)$$

Course of action 2

Phase 1

$$\dot{X}_1(t) = -\beta_1 * Y_1, \qquad X_1(t_0) = X_0 \qquad (14)$$

$$\dot{Y}_1(t) = -\alpha_1 * X_1, \qquad Y_1(t_0) = Y_0 \qquad (15)$$

$$\dot{Z}_1(t) = -\beta_3 * Q_1, \qquad Z_1(t_0) = Z_0 \qquad (16)$$

$$\dot{Q}_1(t) = -\alpha_3 * Z_1, \qquad Q_1(t_0) = Q_0 \qquad (17)$$

[1] The variable attrition rate of the Air Forces are due to the different conditions of combat engagement, according to the scenario and course of action

In the first phase of the operation, all mutual actions of the forces are described by the Lanchester's quadratic law of combat according to (6) and (7).

In the second phase of the operation, the Red Air Force is defeated, so the Blue Air Force has no losses and the Red Army suffers losses according to Lanchester's quadratic law of heterogeneous forces combat (9).

The Red Army suffers losses according to the quadratic law of heterogeneous forces, equation (10). The Red Air Force has no losses while the Blue suffers losses according to the linear law (3) because it is engaged in air support only which makes it easy prey. The Army forces of Blue suffer losses according to the quadratic law of homogenous forces (6).

Phase 2

$$\dot{X}_2(t) = -\beta_1 * Y_1, \qquad X_2(t_1) = X_1 \tag{18}$$

$$\dot{Y}_2(t) = -\alpha_1 * X_1 - \alpha_3 * Z_2, \qquad Y_2(t_1) = Y_1 \tag{19}$$

$$\dot{Z}_2(t) = 0, \qquad Z_2(t_1) = Z_1 \tag{20}$$

$$\dot{Q}_2(t) = 0, \qquad Q_2(t_1) = Q_1 = 0 \tag{21}$$

Course of action 3

$$\dot{X}_1(t) = -\beta_1 * Y_1, \qquad X_1(t_0) = X_0 \tag{22}$$

$$\dot{Y}_1(t) = -\alpha_1 * X_1 - \alpha_3 * Z_1, \qquad Y_1(t_0) = Y_0 \tag{23}$$

$$Z_1(t) = -\beta_3, \qquad Z_1(t_0) = Z_0 \tag{24}$$

$$\dot{Q}_1(t) = 0, \qquad Q_1(t_0) = Q_0 \tag{25}$$

Course of action 4

Phase I

$$\dot{X}_1(t) = -\beta_1 * Y_1 - \beta_3 * Q_1, \qquad X_1(t_0) = X_0 \tag{26}$$

$$\dot{Y}_1(t) = -\alpha_1 * X_1, \qquad Y_1(t_0) = Y_0 \tag{27}$$

$$\dot{Z}_1(t) = 0, \qquad Z_1(t_0) = Z_0 \tag{28}$$

$$\dot{Q}_1(t) = -\alpha_3, \qquad Q_1(t_0) = Q_0 \tag{29}$$

In the first phase of the operation, the Blue Army suffers losses according to the quadratic law of heterogeneous forces (8). The Red Army suffers losses according to the quadratic law of battle of homogeneous forces of equation (7). The Blue Air Force has no losses, because it is conducting a counterair operation while the Red Air force suffers losses according to the linear law of battle of homogeneous forces (4).

In the second phase, the Red Air Force is destroyed, which is why the Red Army suffers losses according to the law of heterogeneous forces (9), and the Blue Air Force has no losses. The Blue Army suffers losses according to the quadratic law of combat of homogeneous forces (6).

The air forces of both adversaries simultaneously provide air support and conduct counterair air defense, while suffering losses according to the square law of the battle of homogeneous forces (6) and (7). Ground forces of both adversaries suffer losses according to the quadratic law of heterogeneous force combat (8) and (9).

In the first phase of the operation, there

Phase II

$$\dot{X}_2(t) = -\beta_1 * Y_1, \qquad X_2(t_1) = X_1 \qquad (30)$$

$$\dot{Y}_2(t) = -\alpha_1 * X_2 - \alpha_3 * Z_2, \qquad Y_2(t_1) = Y_1 \qquad (31)$$

$$\dot{Z}_2(t) = 0, \qquad Z_2(t_1) = Z_1 \qquad (32)$$

$$\dot{Q}_2(t) = 0, \qquad Q_2(t_1) = Q_1 = 0 \qquad (33)$$

Course of action 5

$$\dot{X}_1(t) = -\beta_1 * Y_1 - \beta_2 * Q_1, \qquad X_1(t_0) = X_0 \qquad (34)$$

$$\dot{Y}_1(t) = -\alpha_1 * X_1 - \alpha_2 * Z_1, \qquad Y_1(t_0) = Y_0 \qquad (35)$$

$$\dot{Z}_1(t) = -\beta_2 * Q_1, \qquad Z_1(t_0) = Z_0 \qquad (36)$$

$$\dot{Q}_2(t) = -\alpha_2 * Z_1, \qquad Q_1(t_0) = Q_0 \qquad (37)$$

Course of action 6

Phase I

$$\dot{X}_1(t) = 0, \qquad X_1(t_0) = X_0 \qquad (38)$$

$$\dot{Y}_1(t) = 0, \qquad Y_1(t_0) = Y_0 \qquad (39)$$

$$\dot{Z}_1(t) = -\beta_3 * Q_1, \qquad Z_1(t_0) = Z_0 \qquad (40)$$

$$\dot{Q}_1(t) = -\alpha_3 * Z_1, \qquad Q_1(t_0) = Q_0 \qquad (41)$$

were no losses of ground forces due to the delay of the battle. The air forces of both opponents suffer losses according to the square law of the battle of homogeneous forces of equations (6) and (7).

In the second phase, the Red Air Force is destroyed, due to which the Red Army suffers losses according to the square law of the battle of heterogeneous forces (9). The Blue Air Force has no losses. The Blue Army suffers losses according to the square law of the battle of homogeneous forces (3).

Phase II

$$\dot{X}_2(t) = -\beta_1 * \dot{Y}_2, \qquad X_2(t_1) = X_1 \qquad (1)$$

$$\dot{Y}_2(t) = -\alpha_1 * X_2 - \alpha_3 * Z_2, \qquad Y_2(t_1) = Y_1 \qquad (2)$$

$$\dot{Z}_2(t) = 0, \qquad Z_2(t_1) = Z_1 \qquad (3)$$

$$\dot{Q}_2(t) = 0, \qquad Q_2(t_1) = Q_1 = 0 \qquad (4)$$

4. EXPERIMENTAL RESULTS

Research results for practical solving of problem are given with six solutions, according to the scenario and the chosen course of action. The results for each course of action are shown in diagrams (1) to (6).

The comparison of courses of actions according to given criteria are shown in diagrams (7) and (8). Results comparing according to criteria seems very simple but it need to be analyzed. Operation time lasting

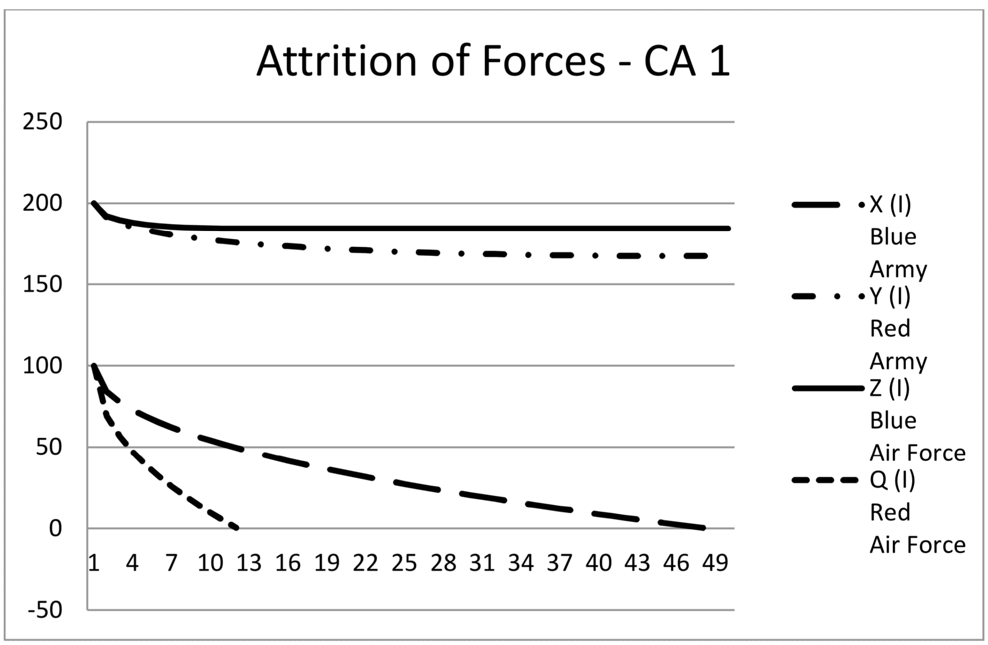


Figure 1. Attrition of Forces CA 1

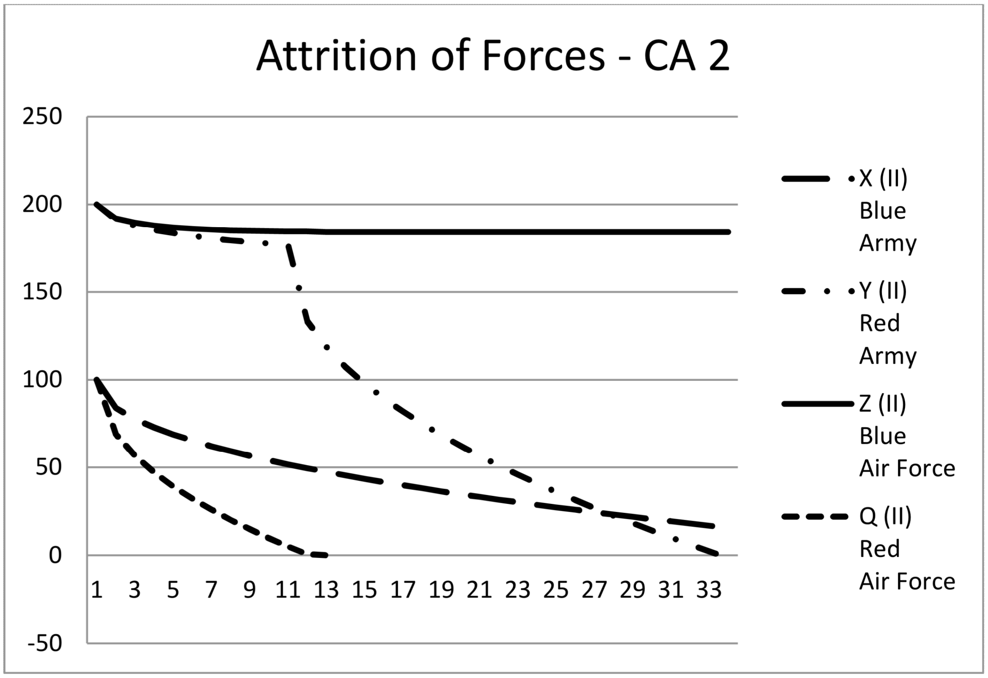


Figure 2. Attrition of Forces CA 2

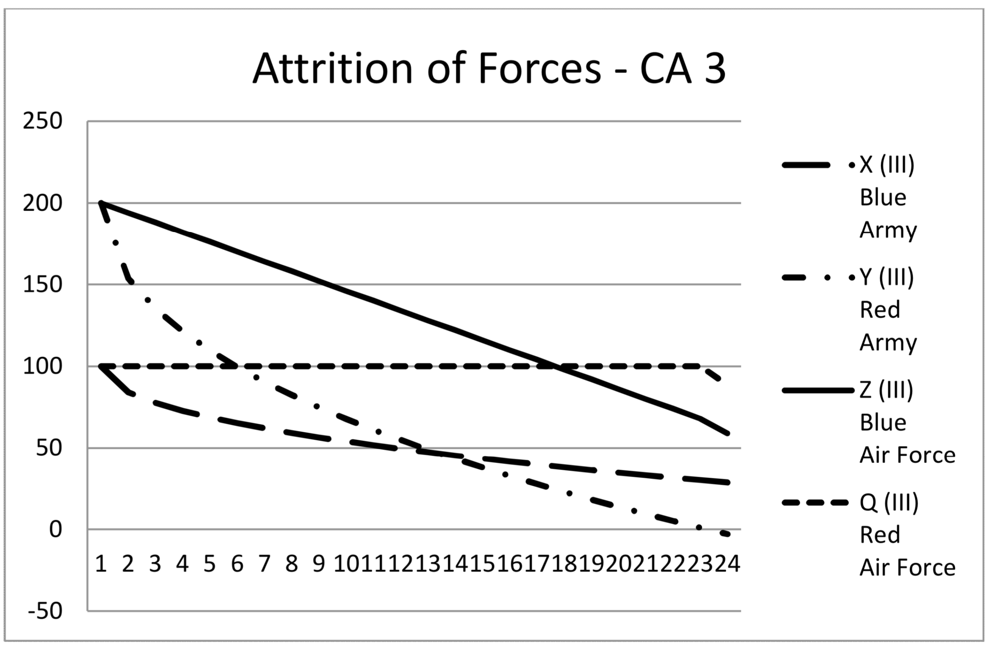


Figure 3. Attrition of Forces CA 3

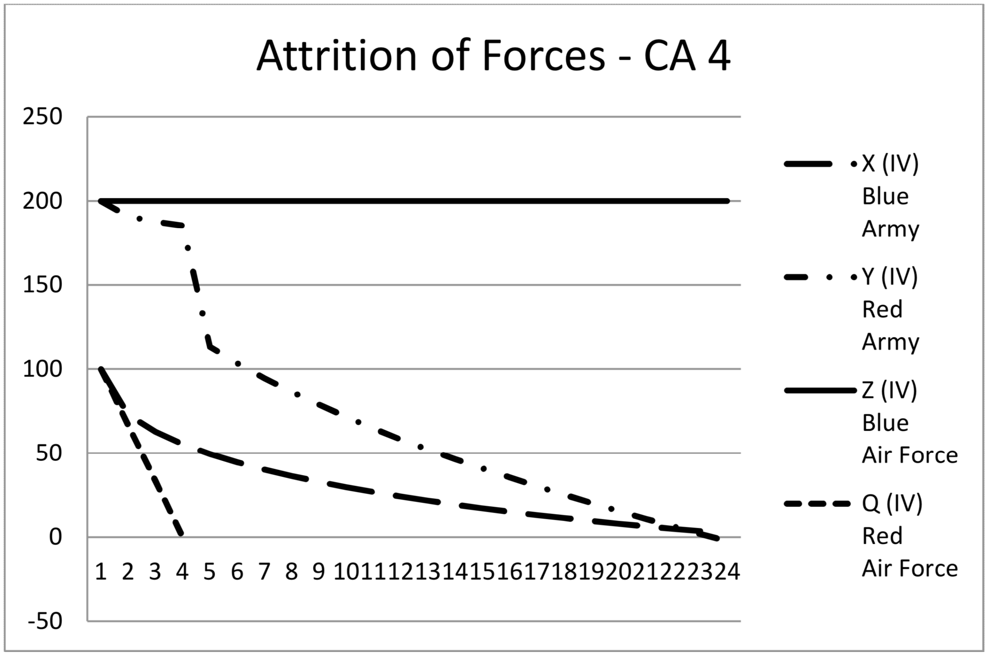


Figure 4. Attrition of Forces CA 4

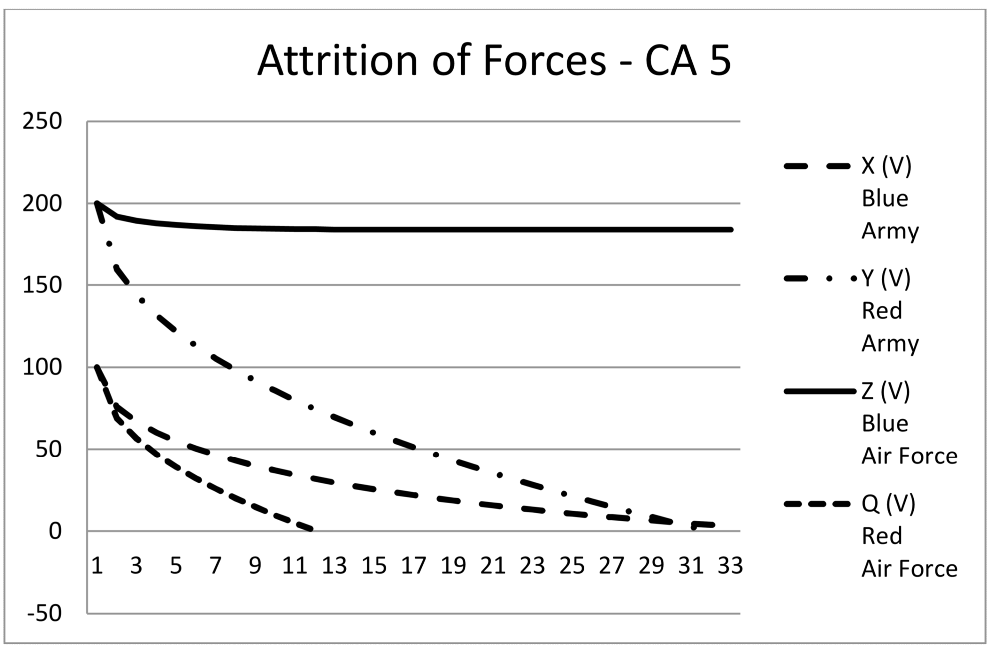


Figure 5. Attrition of Forces CA5

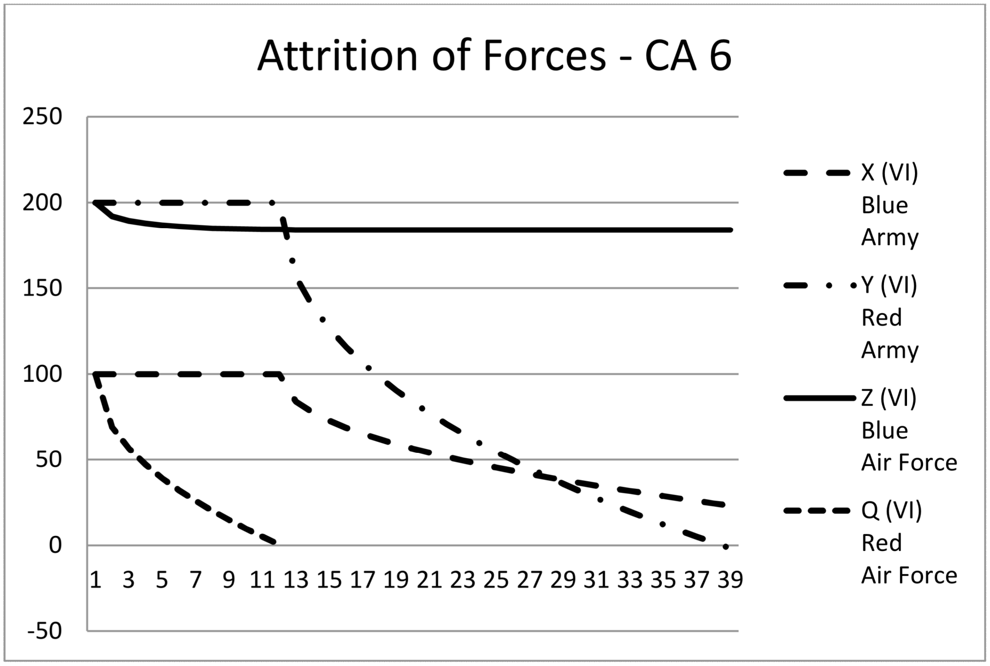
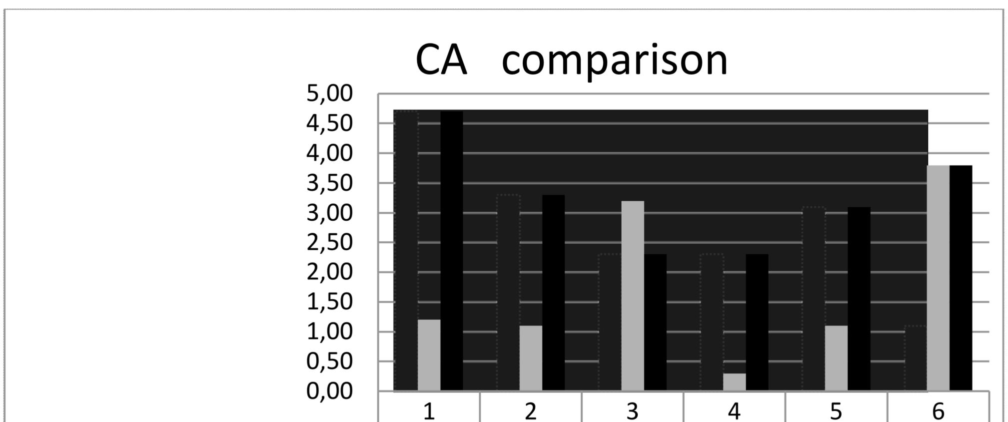
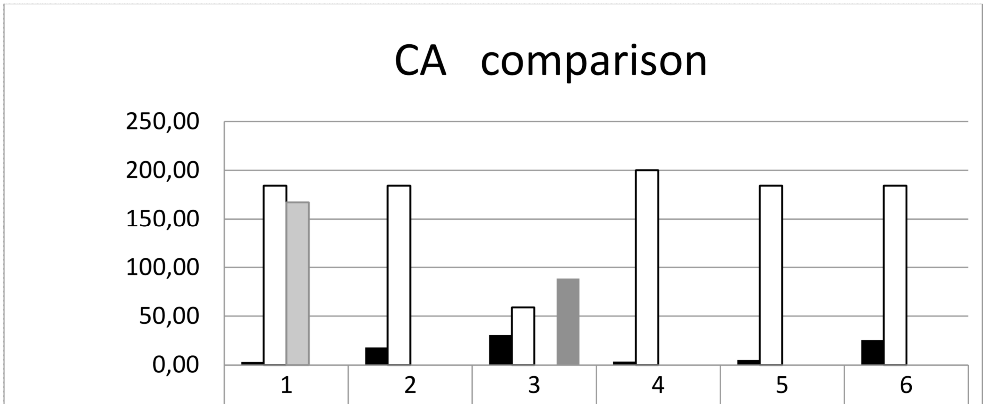


Figure 6. Attrition of Forces CA 6



	1	2	3	4	5	6
Land operation	4,70	3,30	2,30	2,30	3,10	1,10
Air operation	1,20	1,10	3,20	0,30	1,10	3,80
Time duration of campaign	4,70	3,30	2,30	2,30	3,10	3,80

Figure 7. Operation Time Lasting



	1	2	3	4	5	6
■ Blue Army	3,27	18,01	30,72	3,30	4,98	25,34
□ Blue Air Force	184,41	184,39	59,00	200,00	184,39	184,39
Red Army	167,35	0,00	0,00	0,00	0,00	0,00
Red Air Force	0,00	0,00	88,67	0,00	0,58	0,58

Figure 8. Survived Forces

is not exactly given (days, hours etc.) and should be consider only conditional for comparison only.

5. ANALYSIS AND CONCLUSIONS

As already mentioned, the goal of this paper is not exclusively the best solution according to criteria, but the analysis and familiarization of the problem as well as the factors that significantly affect the optimal use of forces. Considering the results according to the defined criteria, "mission success", the worst solution is CA (1). The reason for this is that the Blue Army was clearly outnumbered by the Red Army and destroyed, regardless of its superior fighting capabilities. This is what essentially defines Lanchester's square law of combat, the importance of concentration of forces. Solution CA (3) at first glance is worse than the others. The reason for this is the highest level of losses of the Blue Air Force and the long duration of land and air operations. However, this solution will be comparable to the other and will be very intriguing.

Solutions CA (4) and CA (5) are equally bad, as the Blue Army forces barely survived. CA (4) is better because in this case Blue Air force has no losses and the duration is shorter than CA (5). The paradox is that CA (4) is the best solution for the Air Force because there are no combat losses, but it lacks more important criteria. A comparison of CA (2) and CA (6) implies the advantage of CA (6) due to less attrition of the Blue Army forces and shorter ground operation time regardless of a slightly longer strategic campaign. Both solutions have the same Blue Air Force losses and both are better compared to CA (4) and CA (5). Finally, comparing CA (6) and CA (3) is a rather complicated problem in terms of the decision-making process. If we look strictly at the criteria, the lowest losses of the Blue Army dictate that CA (3) is the best solution. Finally, CA (3) takes less time for both ground operations and strategic campaigning. But if we make a "deeper" analytical comparison of all criteria and results, we can give preference to CA (6) due to Blue Air Force losses. This evidently emphasizes the complexity of the situation in the strategic decision-making process for leaders in that position.

If we hypothetically consider a second phase for CA (3) due to an almost intact Red Air Force, the end of the operation would probably be worse for CA (3) than CA (6). The Red Air Force could destroy the Blue Air Force and then shift the operation to battlefield isolation and air support, as Israel did in the 1973 Arab-Israeli War. This was also the case with the Desert Storm campaign, where Coalition's force planners abandoned the standard doctrine and first defeated the Iraqi air force and then the ground forces. Taking these arguments into account, the final score table for courses of action would be from optimal to less optimal: CA (6), CA (3), CA (2), CA (4), CA (5), and CA (1).

More important than the final result of the courses of action is the consideration of strategy and conclusions that correspond to historical facts, systematized in the theory and operational art of warfare. According to the parameters of the scenario, the force structure and combat capability, the strategy and the operational skill of warfare, one of the principles is the offensive use of aviation to achieve air superiority over the battlefield of the operation. If this is not possible due to various factors such as lack of combat capability, the second principle is the defensive use of air power to defend ground power and air parity in the area of operation. A less desirable strategic option is air support for Army forces in a joint operation without air superiority or in the event that the enemy does not have effective air defenses. The Army force in that case is the main force and has to carry out its mission and conquer a certain area in a certain period of time, where the opponent cannot show its effective use of air power. Any attempt to fight on the ground with enemy air superiority is a strategic or operational disaster. For such a scenario, the last solution is guerilla warfare, which means a completely different strategy. In mathematical theory, this means the application of Dichman's law of mixed combat, which enables the dynamic simulation of combat between qualitatively different or heterogeneous opponents. It is fighting the enemy by applying Lanchester's linear law of combat to the square law. In that case, according to the law of mixed combat, the weaker side would be able to sustain less loss of forces than the stronger side.

The historical fact that confirms these statements is the operation Desert Storm in Iraq in 1991 (Keaney & Cohen, 1993). The coalition forces led by the armed forces of the United States (ground and air forces) as the main force, were supposed to fight in the doctrinal way of "air-ground battles". This concept could be described as an offensive ground operation supported by the Air Force both on the front line and deep in the enemy's rear. This type of combat model is comparable to CA (3), (4) and (5). The planners of the operation concluded that this way of engaging would cause a great loss of their own forces. They planned a battle similar to the CA case (6). First, they won air superiority over the battlefield. After that, they isolated the Iraqi ground forces and finally carried out a joint air-ground campaign (Surwaey II, 1993). It was a brilliant victory especially because of the almost equal ground forces. The attrition of the coalition forces was minor.

An example of the impact of Lanchester's laws of combat can be seen in the example of the current war in Ukraine. It seems like a strange situation because the Russians supposedly have better Army and a stronger and more modern Air Force. However, Ukraine possesses numerical superiority in ground forces and conducts defensive ground operations according to Dichmann's law of mixed combat. The Russian Air Force did not achieve supremacy in the airspace of the battlefield, but only tactical superiority in certain areas. The reason for this may be a deviation from the principle of concentration in the primary mission - air superiority in favor to other missions or simply failure for other reasons such as inadequate combat capability. In any case, this is reason why there is a lack of air support for the maneuver of ground troops and air strikes for the operational isolation of the battlefield. Air strikes are replaced by missile strikes, which are efficient on targets but only partially effective for strategic goals like energy system. On land, the slow advance of the Russian army, even retreat can be seen, showing little or barely sufficient air support. In the conditions of insufficient numbers and strength of ground forces, it caused an increase in losses and failure to achieve the objectives of the operation. The Air Force of Ukraine is still flying. This is the case of CA (5) from the model. According to Lanchester's model, the Russians have a problem with the choice of strategy and after certain duration of the operation, they will probably be overwhelmed by the number of ground forces of Ukraine.

The applied method and the obtained results confirm the doctrinal principles on the use of forces in combat operations. The simulated combat model proves the possibility of applying the Lanchester's equations as an operational management tool for quick operational assessment but in a simplified combat situation. For detailed and thorough modeling, planners need simulation based on approximate methods such as Euler-Cauchy numerical methods (Kress, 2020), and others. However, the essence of applying this method is that managers know the mathematical basis of the method, which is usually not the case with simulation programs bought on the market, and that it is simple and quick to apply.

The development of these models will be the further work of the author.

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ЛАНЧЕСТЕРОВЕ ДИФЕРЕНЦИЈАЛНЕ ЈЕДНАЧИНЕ КАО ОПЕРАТИВНИ АЛАТИ ЗА ДОНОШЕЊЕ ОДЛУКА

Младен Костић, Аца Јовановић

Извод

Овај рад истражује примену Ланчестерових једначина као научног метода и алата за испитивање функционисања оружаних снага као сложених организационих система у борби. Важно је проценити поузданост сазнања добијених овом методом, о чињеницама оперативног окружења и ефикасности употребе снага, како би се подржао процес планирања и доношења оптималних одлука, у условима неизвесности и ризика, који су својствени ратовању. Према овој хипотези, развијен је математички модел на основу познатих Ланчестерових једначина, које су дефинисале квадратни и линеарни закон борбе између два противника са хетерогеном структуром снага (ваздухопловство и војска). Креирани модел омогућава исправну поједностављену анализу у процесу доношења одлука. Права ратна и борбена дејства су веома сложена и захтевају употребу сложених симулатора, чија је методолошка позадина често непозната доносиоцима одлука, због чега су поуздане апроксимативне методе симулације и моделирања неопходне и пожељне.

Кључне речи: диференцијалне једначине, моделирање ратовања, процес трошења снага

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