Iterative learning control (ILC) is one of the recent topics in control theories and it is suitable for controlling a wider class of mechatronic systems - it is especially suitable for the motion control of robotic systems. This paper addresses the problem of application of fractional order ILC for fractional order singular system. Particularly, we study fractional order singular systems in the pseudo-state space. An closed-loop fractional order P\text{D}_\alpha type ILC of the fractional-order singular system is investigated. Also, open-closed loop of the fractional order P-\text{PD}_\alpha type ILC is considered. Sufficient conditions for the convergence in the time domain of the proposed ILC schemes are given by the corresponding theorems and proved. Finally, numerical simulations show the feasibility and effectiveness of the proposed approach.

**Keywords:** control theory, iterative control, learning control, fractional order, singular system, method convergence, robotic system.

### Introduction

Iterative learning control (ILC) is one of the most active fields in control theory and it is a powerful intelligent control concept that iteratively improves behavior of the processes that are repetitive in nature [1-3]. Since the early 80’s, ILC [4,5] has been one of the very effective control strategies in dealing with repeated tracking control with the aim of improving tracking performance for the systems that work in a repetitive mode. As opposed to traditional controllers, ILC is a simple and effective control and can progressively reduce tracking errors and improve system performance from iteration to iteration. Namely, ILC is a trajectory tracking improvement technique for control systems, which can perform the same task repetitively in a finite time interval to improve the transient response of a system using the previous motion. For the purpose of emulating human learning, ILC uses knowledge obtained from the previous trial to adjust the control input for the current trial so that a better performance can be achieved. ILC is a memory based control technique since the input-output data should be stored after each iteration for updating the control input for the next iteration. Therefore, ILC requires less a priori knowledge about the controlled system in the controller design phase and also less computational effort than many other kinds of control. Besides, in terms of how to use tracking error signal of the previous iteration to form the control signal of the current iteration, ILC updating schemes can be classified as P-type, D-type, PD-type, and PID type. A typical ILC in the time domain is a simple open-loop control (off-line learning control) that only uses tracking error information in the previous iterations to form the control signal used in the current iteration and it cannot suppress the unanticipated, non-repeating disturbances. So, ILC is a technique of controlling systems operating in a repetitive mode with the additional requirement that a specified output trajectory \(y_d(t)\) in an interval \([0, T]\) is followed to a high precision and through improving the performance from trial to trial in the sense that the tracking error is sequentially reduced. The basic strategy is to use an iteration of the form

\[
i_{i+1} = f(u_i(t), e_i(t)), \quad e_i(t) = y_d(t) - y_i(t),
\]

where \(f(\cdot, \cdot)\) defines the learning algorithm and remains to be specified, \(y_d(t)\) is the output at the \(i\)-th operation resulting from an input \(u_i(t)\), and \(y_i(t)\) represents the desired output. The new control input \(u_{i+1}(t)\) should make the system closer to the desired result in the next execution cycle.

In the real application, to overcome such drawbacks, an ILC scheme is usually performed together with a proper feedback controller for compensation [6], where we often design a learning operator for the closed-loop (on-line ILC) systems that have achieved a good performance. Since the theories and learning algorithms on ILC were firstly proposed, ILC has attracted considerable interests [3] due to its simplicity and effectiveness of the learning algorithm, and its ability to deal with the problems associated with nonlinear, time-delay, uncertainties, and, recently, singular systems. Besides, during the past years, singular systems have attracted attention of a lot of researchers from the mathematics and
control communities due to the fact that singular systems can describe behavior of some physical systems better than regular systems such as: electrical network models [7], mechanical models [8, 9], etc. Naturally, many theoretical results for regular systems have been extended to singular cases. For example, the robot control systems can generally be described by some nonlinear ordinary differential equations. However, when the robots contact with the objects and the environment, they will usually be depicted by some end-point constraints. In that way, the constraints are generally described by nonlinear differential-algebraic equations which are modeled as singular systems, [9]. It is well known that the issues of concern for singular systems are much more complicated than those for regular systems, because for singular systems we need to consider not only stability, but also regularity and the absence of impulses at the same time [10]. Actually, elimination of algebraic constraints needs a suitable feedback control [11]. From the control point of view, it is also necessary to study the ILC for singular systems. Until now, there are few results reported on introducing ILC methods to studying of control for singular systems [12, 13].

Recently, increasing attentions are paid to fractional differential equations and their applications in various science and engineering fields [14, 15]. Moreover, an increasing attention has been paid to the fractional calculus (FC) and its application in control and modeling of fractional-order singular systems [16, 17]. It is not difficult to conclude that other dynamic systems (robotic systems of fractional-order, etc.) [18] can be displayed in the singular form, especially in realization of various robotic tasks.

Recently, the application of ILC to the fractional-order systems has become a new topic [19-22]. Among different fractional order controllers, fractional order iterative learning controller (FOILC), the fractional order version of iterative learning control (ILC), is of interest in this paper. Also, in [23, 24] are presented new results for P-PD type of robust ILC for a given class of fractional order uncertain time delay system. Moreover, for the first time, in the paper [25] an iterative learning feedback control is considered for the fractional-order singular systems as well as in the paper [26] a robust iterative learning feedback control of the second-order for fractional-order singular systems is considered. Motivated by the mentioned investigations of ILC algorithms for ILC fractional order control in the tracking problems of these systems, (open)-closed-loop iterative learning control for given fractional-order singular systems described in the form of state space and output equations. The sufficient convergent conditions of the proposed ILC will be derived in time-domain and formulated by a theorem. A rigorous mathematical proof for the convergence of the iterative learning process is presented. Finally, the simulation results are presented to illustrate the performance of the proposed P-PD ILC scheme.

The remainder of this paper is arranged as follows: in the Section Preliminaries and basics of fractional calculus, some preliminaries as well as the fractional Caputo operators are presented. In Section Open-loop fractional-order iterative learning control, the first main result is derived where the convergence is guaranteed by mathematical proof rigorously, which includes the extensions of some of the basic result ILC of singular fractional-order systems with order $\alpha \in (0, 1)$ to uncertain fractional-order singular system. In the next section Open-closed-loop fractional-order iterative learning control the second main result is presented in the same manner where the open-closed-loop fractional-order ILC is introduced for the same singular fractional order system.

**Preliminaries and basics of fractional calculus**

**The $\lambda$ - norm, maximum norm, induced norm**

For a later use in proving the convergence of the proposed learning control, the following norms are introduced [3] for the $n$-dimensional Euclidean space $\mathbb{R}^n$: the sup-norm $\|x\|_\infty = \sup_{i=1,2,\ldots,n} |x_i|$, $\|x\| = \left\{x_1, x_2, \ldots, x_n \right\}$, $\|x\|$-absolute value; the $\lambda$-norm $\|x\|_\lambda = \max_{i=1,2,\ldots,n} |x_i|$; the matrix norm as $\|A\|_\infty = \max_{1 \leq i \leq m} \left( \sum_{j=1}^{n} |a_{ij}| \right)$, $A = [a_{ij}]_{m \times n}$ and the $\lambda$-norm for a real function:

$$h(t), \ (t \in [0, T]), \ h : [0, T] \rightarrow \mathbb{R}^n$$

$$\|h(t)\|_\infty = \sup_{t \in [0, T]} \|e^{\lambda t} \| h(t) \| , \ \lambda > 0$$

A useful property associated with the $\lambda$-norm is the following inequality.

**Property 1:** $\lambda$ norm has the next property

$$\sup_{t \in [0, T]} \|e^{\lambda t} \| f(t) \| = \int_0^T \| f(t) \| e^{\lambda T - \lambda t} \| dt$$

$$\sup_{t \in [0, T]} \int_0^T \| f(t) \| e^{\lambda T - \lambda t} \| dt \leq \frac{1}{\lambda - \alpha} \int_0^\infty \| f(t) \| e^{(\alpha - \lambda)T} \| dt$$

The induced norm of the matrix $A$ is defined as:

$$\|A\| = \sup_{\|x\| \neq 0} \|Ax\| / \|x\|$$

where $\|x\|$ denotes an arbitrary vector norm. In case $\|x\|$, it follows that

$$\|Ax\| \leq \|A\| \|x\|$$

where $\|A\|_{\infty}$ denotes the maximum value of the matrix $A$. For the previous norms, note that

$$\|h(t)\|_\infty \leq \|h(t)\|_{\lambda} \leq e^{\lambda T} \|h(t)\|_\lambda$$

The $\lambda$ - norm is thus equivalent to the $\infty$ - norm. For simplicity, in applying the norm $\|x\|_{\lambda}$ the index $\infty$ will be omitted. Before giving the main results, we first give the following Lemma 1, [27].

**Lemma 1.** Suppose a real positive series $\{a_n\}_{n=1}^{\infty}$ satisfies

$$a_n \leq p_{1}a_{n+1} + p_{2}a_{n+2} + \ldots + p_{N}a_{n-N} + \epsilon$$

(6)

$$n = N + 1, N + 2, \ldots$$
where \( \rho_i \geq 0 \) for \( i = 1, 2, \ldots, N \) and \( \rho = \sum_{i=1}^{N} \rho_i < 1 \). Then the following holds:

\[
\lim_{n \to \infty} a_n \leq \varepsilon / (1 - \rho)
\]

**Fractional calculus - Caputo operator**

Fractional calculus (FC) is a generalization of classical calculus concerned with the operations of integration and the differentiation of non-integer (fractional) order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The three most frequently used definitions for the general fractional differential integral are: the Grunwald-Letnikov (GL) definition, the Riemann-Liouville (RL) and the Caputo definitions, [14, 15]. In this paper, Caputo fractional-order operator is used, where definition of the left Caputo fractional-derivative is given [14, 15] as follows:

\[
\frac{C}{t}D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,
\]

where \( f^{(n)}(\tau) = d^n f(\tau) / d\tau^n \), \( n-1 \leq \alpha < n \in \mathbb{Z}^+ \), and \( \Gamma(.) \) is the well-known Euler’s gamma function. In the case \( n-1 \leq \alpha < 1 \) we have \( 0 \leq \alpha < 1 \) as well as

\[
\frac{C}{t}D^\alpha f(t) = \frac{1}{(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} df(\tau) / d\tau d\tau.
\]

In the following sections, \( D^\alpha \) will denote \( \frac{C}{t}D^\alpha \) for brevity of notation.

**Fractional-order autonomous linear singular system**

Consider the following autonomous, singular, fractional-order system (SFOS) described by the state and output equations, respectively

\[
ED^\alpha x(t) = Ax(t), \quad n-1 < \alpha < n,
\]

\[
y(t) = Cx(t),
\]

where admissible initial conditions for (10) are given by

\[
x^{(k)}(0) = x_{0,k} \quad k = 0, 1, 2, \ldots, n-1.
\]

Here, \( \frac{C}{t}D^\alpha = D^\alpha \) denotes the \( \alpha \) th-order Caputo fractional derivative with respect to \( t \), while \( E, A, \) and \( C \) are matrices with appropriate dimensions \([28, 29]\). In solving a singular problem, assuming regularity of the system, it is necessary to ensure the existence and uniqueness of the solution.

**Definition 1.**

a) The SFOS system (10) is said to be regular if

\[
\det(s^E A - E) \neq 0,
\]

b) The SFOS system (10) is said to be impulse free if (10) applies and

\[
\deg(\det(s^E A - E)) = \text{rank}E.
\]

**Lemma 1.** The triplet \((E, A, \alpha)\) is called regular if and only if \( \det(s^E A - E) \neq 0 \) for some \( s \in \mathbb{C} \) \([28, 29]\). Also, if triplet \((E, A, \alpha)\) is regular, we call SFOS system (10) regular, and consequently SFOS system is solvable.

**Lemma 2.** If the function \( f(t,x) \) is continuous, then the initial value problem

\[
\begin{align*}
\frac{C}{t}D^\alpha x(t) &= f(t,x(t)), \quad 0 < \alpha < 1 \\
y(t_0) &= x(0)
\end{align*}
\]

is the equivalent to the following nonlinear Volterra integral equation:

\[
x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s,x(s)) ds
\]

and its solutions are continuous, \([30]\).

**Closed-loop fractional-order iterative learning control**

The fractional-order non-autonomous singular linear system

A non-integer (fractional) linear, singular system described in the form of pseudo state space and output equations is considered. The considered class of fractional-order \( \alpha \in (0,1) \) non-autonomous singular linear system can be written as the state space equation and output equation

\[
ED^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha < 1
\]

\[
y(t) = Cx(t).
\]

Here, \( t \) is the time within the operation interval \( J = [t_0, t_0 + T] \), \( J \subseteq \mathbb{R} \), while \( A, B, \) and \( C \) are matrices having appropriate dimensions. It is assumed that \( det E = 0 \) and that SFOS system is regular.

Also, the initial conditions of fractional differential equations which were compared to the given fractional derivatives were considered by different authors \([29, 31]\), assuming that there was no difficulty regarding the questions of existence, uniqueness, and continuity of solutions with respect to initial data. The following assumptions on the system (16), (17) are imposed.

A1. The desired trajectories \( y_d(t), x_d(t) \) are continuously differentiable in \([0, T]\).

A2. For the given desired output trajectory \( y_d(t) \), there exists a control input \( u_d(t) \) such that

\[
ED^\alpha x_d(t) = Ax_d(t) + Bu_d(t), \quad 0 < \alpha < 1
\]

\[
y_d(t) = Cx_d(t).
\]

A3. SFOS system is controllable and observable.

A4. Resetting the initial conditions holds for all iterations, i.e. \( x_k(0) = x_0(0), \quad k = 0, 1, 2, \ldots \).

**Convergence Analysis**

Here, it is suggested the closed-loop fractional order \( PD^\alpha \) learning algorithm which comprises control law a \( PD^\alpha \) feedback law. Moreover, it was shown in \([32]\) that the
tracking speed was the fastest when the system and iterative learning scheme have the same order. In the feedback loop ILC, the \( PD^\alpha \) controller provides stability of the system and norm to both

\[
\dot{x}_{i+1}^{(\alpha)}(t) = A_0 \dot{x}_{i+1}^{(\alpha)}(t) + B_0 \delta u_{i+1}(t)
\]

After, rearranging (22) it becomes

\[
(E + B \Gamma C) \dot{x}_{i+1}^{(\alpha)} = (A - B \Gamma C) x_{i+1} + B \delta u_{i+1}
\]

Using suitable gain matrix \( \Pi \) as well as taking into account previously introduced assumptions, matrix \( (E + B \Pi C) \) is invertible, i.e. exists \( (E + B \Pi C)^{-1} \). Multiplying on the left side expression (29) by \( (E + B \Pi C)^{-1} \) we obtain (30) in the form

\[
\delta x_{i+1}^{(\alpha)} = (E + B \Pi C)^{-1} (A - B \Gamma C) x_{i+1} + B \delta u_{i+1}
\]

If one adopts

\[
\bar{A} = (E + B \Pi C)^{-1} (A - B \Gamma C), \quad \bar{B} = (E + B \Pi C)^{-1} B
\]

then (30) becomes

\[
\delta x_{i+1}^{(\alpha)} = \bar{A} \delta x_{i+1} + \bar{B} \delta u_{i+1}
\]

By replacing (32) into (26), we have

\[
\delta u_{i+1} = \|I - \Pi C \bar{B}\| \delta u_{i} - \|C + \Pi C \bar{A}\| \delta x_{i+1}
\]

Estimating the norms of (33) with \( \|\cdot\| \) and using the condition of Theorem 1 one gets

\[
\|\delta u_{i+1}\| \leq \|I - \Pi C \bar{B}\| \delta u_{i} + \|C + \Pi C \bar{A}\| \|\delta x_{i+1}\| = \rho \|\delta u_{i}\| + \beta \|\delta x_{i+1}\|
\]

Also, one can write the solutions of (32) in the form of the equivalent Volterra integral equations using assumption A4, as:

\[
\delta x_{i+1}(t) = \frac{1}{\Gamma^{(\alpha)}(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} (A \delta x_{i+1}(s) + \bar{B} \delta u_{i}(s)) ds
\]

Applying the norm \( \|\cdot\| \) on the equation (35), if it is uniqueness solution, [29, 31] one obtains:

\[
\|\delta x_{i+1}(t)\| \leq \frac{1}{\Gamma^{(\alpha)}(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \|A\| \|\delta x_{i+1}(s)\| ds +
\]

\[
+ \frac{1}{\Gamma^{(\alpha)}(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \|\bar{A}\| \|\delta u_{i}(s)\| ds
\]

where \( \alpha = \|A\|, \quad b = \|\bar{B}\| \). Moreover, applying \( \lambda \) norm to both sides of the previous (36), it follows
Further, the case of the fractional order $\alpha \in (0,1)$ singular system non-autonomous singular linear system can be written as the state space equation and output equation is also discussed here:

$$ED^\alpha x(t) = (A + \Delta A)x(t) + Bu(t),$$

$$0 < \alpha < 1$$

$$y(t) = Cx(t),$$

(49)

(50)

Here, $t$ is time in the operation interval $J = [t_0, t_0 + T]$, $J \subset R$, as well as $A, B$ and $C$ are matrices with the appropriate dimensions; $\Delta A$ is unknown real norm-bounded matrix which represent parameter uncertainty in the system model.

**Theorem 2.** For the fractional order singular system (49), (50) with the PD$^\alpha$-type ILC scheme (21), and the assumptions A1-A4 where the convergence condition is given by (22), then when $i \to \infty$ the bounds of the tracking errors $|x_d(t) - x_i(t)|$, $|y_d(t) - y_i(t)|$, $|y_d(t) - y(t)|$, converge asymptotically to a residual ball centered at the origin.

**Proof:** The proof follows from the proof of Theorem 1. Namely, from (49), (50) one can easily find that

$$E\delta x_{i+1}^\alpha = (A + \Delta A)\delta x_{i+1} + B\delta u_i - \Delta Ax_d$$

(52)

Multiplying on the left side expression (52) by $(E + BIC)^{-1}$ we obtain (53) in the form

$$\delta x_{i+1}^\alpha = (\bar{A} + \Delta \bar{A})\delta x_{i+1} + \bar{B}\delta u_i - \Delta \bar{A}x_d$$

(53)

where

$$\bar{A} = (E + BIC)^{-1}(A - BIC), \quad \bar{B} = (E + BIC)^{-1}B,$$

(54)

$$\Delta \bar{A} = (E + BIC)^{-1}\Delta A.$$  

By replacing (53) into (26), we have

$$\delta u_i = \left[I - \Pi C\bar{B}\right]\delta u_i - \left[\Gamma C + \Pi C\left(\bar{A} + \Delta \bar{A}\right)\right]\delta x_{i+1} + \Pi C\Delta \bar{A}x_d$$

(55)

Estimating the norms of (55) with $\|\cdot\|$ and using the condition of Theorem 2 one gets

$$\|\delta u_i\| \leq \rho\|\delta u_i\| + \rho\|\delta x_{i+1}\|$$

(56)

where it is fulfilled, $\|x_d(t)\| \leq c, \forall t \in [0, T]$. Also, one can write the solutions of (53) in the form of the equivalent Volterra integral equations using the assumption A4, as:

$$\lim_{i \to \infty} x_i(t) = x_d(t), \lim_{i \to \infty} y_i(t) = y_d(t).$$

(48)
\[ \delta x_{i+1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( [\bar{A} + \Delta \bar{A}] \delta x_{i+1}(s) + \bar{B} \delta u_i(s) \right) ds - \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \Delta \bar{A} x_i(s) ds \]  

(57)

In a similar manner, applying the norm \( \| \cdot \| \) on the equation (57), if a uniqueness solution exists, \([29, 31]\) where \( a = [\bar{A}] \), \( b = [\bar{B}] \), \( \alpha = [\bar{A}] \), and applying \( \lambda \) norm, we have

\[ \| \delta x_{i+1}(t) \| \leq \left( (a + a_\lambda) \| \delta x_{i+1}(t) \| + b \| \delta u_i(t) \| + a_\lambda c \right) \cdot \frac{1}{1 - e^{-2\lambda t}} \delta^\alpha \| \eta \| \lambda \Gamma(\alpha + 1) \]  

or, one may conclude

\[ \| \delta x_{i+1}(t) \| \leq \frac{b O(\lambda^{-1}) \| \delta u_i(t) \| + a_\lambda O(\lambda^{-1})}{(1 - (a + a_\lambda) O(\lambda^{-1}))} \leq O_{\lambda} \left( \lambda^{-1} \right) \| \delta u_i(t) \| + \epsilon' \left( \lambda^{-1} \right) \]  

(58)

Finally, taking the \( \lambda \)-norm of the expression (56) leads to:

\[ \| \delta u_{i+1} \| \leq \rho \| \delta u_i \| + \rho_0 \| \delta x_{i+1} \| + \rho_0 c \]  

(60)

or, taking into account (59) we obtain

\[ \| \delta u_{i+1} \| \leq \left( \rho + \rho_0 O(\lambda^{-1}) \right) \| \delta u_i \| + \rho_0 c \]  

(61)

So that, there exists a sufficient large \( \lambda \) satisfying

\[ \rho' = \left( \rho + \rho_0 O(\lambda^{-1}) \right) \leq 1 \]  

(62)

Therefore, taking into account Lemma 1, [3] it yields:

\[ \lim_{i \to \infty} \| \delta u_i \|_{\lambda} \leq \frac{1}{1 - \rho} \epsilon, \]  

(63)

This completes the proof of Theorem 2.

**Remark 1.** In the case of no parameter uncertainty, i.e. \( \Delta \lambda = 0 \), one can obtain that when \( i \to \infty \) bounds of the tracking errors \( \| y_d(t) - x_i(t) \| \), \( \| y_d(t) - y_i(t) \| \), and \( \| u_d(t) - u_i(t) \| \) converge asymptotically to zero, as stated in Theorem 1, (i.e \( \Delta \lambda = 0 \), i.e \( \epsilon = 0 \)).

**Open-closed-loop fractional-order iterative learning control**

Also, for the singular system defined by (10), open-closed-loop \( P-D^{\alpha} \)-type iterative learning algorithm is proposed as follows:

\[ u_{i+1}(t) = u_i(t) + \Gamma_1 e_i(t) + \Gamma_2 \left( c D_0^\alpha e_i(t) + \Pi_2 e_i(t) \right), \]  

(64)

where \( u_i(t) \) and \( y_i(t) \) are, respectively, the system input and output in the \( i \)-th iteration, \( e_i(t) = y_d(t) - y_i(t) \) is the trajectory tracking error at \( i \)-th iteration, \( u_{i+1}(t) \) is the system input of the \( (i+1) \)-th trial, \( y_{d,i}(t) = C x_d(t) \) denotes desired output trajectory, and \( \Gamma_1, \Gamma_2, \Pi_2 \) are open-closed-loop learning matrices. In the closed loop, the \( P-D^{\alpha} \) controller \( \Gamma_2 \left( c D_0^\alpha e_i(t) + \Pi_2 e_i(t) \right) \) provides stability of the system and keeps its state errors within uniform bounds. A sufficient condition for convergence of the proposed open-closed-loop ILC is given by Theorem 3. The proof is as follows:

**Theorem 3:** Suppose that the update law defined by (64) is applied to the non-autonomous singular linear system (16), (17) and assumptions \( A_i, i = 1, 2, 3, 4 \) are satisfied. If matrix \( \Gamma_2 \) exists such that

\[ \left\| I - \Gamma_2 C B \right\| \leq \rho < 1, \]  

(65)

where \( B = (E + B T_{2} C)^{-1} B \) and matrix \( \Gamma_2 \) is such that \((E + B \Gamma_2 C)\) is invertible, then, when \( i \to \infty \), the bounds of the tracking errors \( \| x_i(t) - x_i(t) \| \), \( \| y_d(t) - y_i(t) \| \), \( \| u_d(t) - u_i(t) \| \) converge asymptotically to a residual ball centered at the origin.

**Proof.**

The proof is similar to the proofs of the previous two theorems. Taking the proposed control law gives:

\[ \delta u_{i+1} = u_d - u_{i+1} = \delta u_i - \Gamma_1 e_i - \Gamma_2 \left( e_i^{(0)} + \Pi_2 e_i \right), \]  

(66)

or, based on equation (24), it follows:

\[ \delta u_{i+1} = \delta u_i - \Gamma_1 C \delta x_{i+1} - \Gamma_2 C \delta x_{i+1}^{(0)} - \Gamma_2 \Pi_2 C \delta x_{i+1}, \]  

(67)

as well as taking into account (27) one can find that

\[ (E + B T_{2} C) \delta x_{i+1}^{(0)} = \]  

(68)

By choosing suitable gain matrix \( \Gamma_1 \), as well as by taking into account the previously introduced assumptions, the matrix \((E + B T_{2} C)\) is invertible, i.e. there exists \((E + B T_{2} C)^{-1}\). By multiplying the expression (68) by \((E + B T_{2} C)^{-1}\), we obtain

\[ \delta x_{i+1}^{(0)} = \delta x_{i+1} - \delta x_{i+1} = \delta x_{i+1} - \delta x_{i+1} \]  

(69)

where

\[ \delta x_{i+1} = \Delta \delta x_{i+1} + \bar{A} \delta x_{i+1} + \bar{B} \delta u_i, \]  

(70)

And after replacing (69) into (67), we have

\[ \delta u_{i+1} = \left[ I - \Gamma_2 C \bar{B} \right] \delta u_i - \left[ \Gamma_2 C \bar{A} + \Gamma_2 \Pi_2 C \right] \delta x_{i+1} - \left[ \Gamma_1 C + \Gamma_2 C \bar{A} \right] \delta x_{i+1}, \]  

(71)

Taking the norm of both sides of the equation (71) and using the condition of Theorem 3, this reduces to:

\[ \| \delta u_{i+1} \| \leq \rho \| \delta u_i \| + \| \left[ \Gamma_2 C \bar{A} + \Gamma_2 \Pi_2 C \right] \| \| \delta x_{i+1} \| + \| \left[ \Gamma_1 C + \Gamma_2 C \bar{A} \right] \| \| \delta x_{i+1} \| + \beta \| \delta x_{i+1} \| \]  

(72)
Again, we can obtain the solutions of (25) in form of the equivalent Volterra integral equations using the assumption A4, as:

$$\delta x_{i+1}(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \left( \delta \delta x_{i+1}(s) + \delta \delta x_i(s) \right) ds. \quad (73)$$

Taking norms and using their properties, we have

$$\|\delta x_{i+1}(t)\| \leq \frac{q}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \|\delta x_{i+1}(s)\| ds +$$

$$+ \frac{a}{\Gamma(\alpha+1)} \int_0^t (t-s)^{-\alpha} \|\delta x_i(s)\| ds + \frac{b}{\Gamma(\alpha+1)} \int_0^t (t-s)^{-\alpha} \|\delta u_i(s)\| ds +$$

where $$q = \|\delta\|$$, $$b = \|\beta\|$$. Furthermore, the next relation is fulfilled:

$$t \in [0, T], \quad \|\delta x_i(t)\| = \|\delta x_i(t) + x_i(t) - x_i(t)\| \leq \|\delta x_i(t)\| + \|x_i(t) - x_i(t)\|. \quad (75)$$

Here, we may introduce $$\eta_{i+1} = \sup_{t \in [0, T]} \|x_i(t) - x_i(t)\|$$, and

$$\|\delta x_i(t)\| \leq \|\delta x_i(t)\| + \eta_{i+1}$$

and applying $$\lambda$$ norm to both sides leads to

$$\|\delta x_{i+1}(t)\| \leq \sup_{0 \leq t \leq T} \left\{ \int_0^t e^{(a_i + a_i)^{\alpha-1}} \left[ (a_i + a_i) \|\delta x_{i+1}(s)\| + b \|\delta u_i(s)\| \right] ds \right\}$$

$$+ \sup_{0 \leq t \leq T} \left\{ \int_0^t e^{(a_i + a_i)^{\alpha}} \|\delta x_i(s)\| ds \right\}$$

$$\leq \sup_{0 \leq t \leq T} \left\{ \int_0^t e^{(a_i + a_i)^{\alpha-1}} \left[ (a_i + a_i) \|\delta x_{i+1}(s)\| + b \|\delta u_i(s)\| \right] ds \right\}$$

$$+ \frac{a_i \eta_{i+1}}{\Gamma(\alpha+1)}.$$
It is easy to show that the pair \((E; A)\) is regular.

**Figure 2.** The tracking performance of the system output \((y_1(t); \text{solid line, } y_{d1}(t); \text{bold line})\)

**Figure 3.** The tracking performance of the system output \((y_2(t); \text{solid line, } y_{d2}(t); \text{bold line})\)

Simulation results in Figures 2-5 show the effectiveness of the developed ILC control scheme for the system (16), (17). The ILC rule (21) is used, (Figures 3, 4) show the tracking performance of the ILC system outputs on the interval \(t \in [0,1] \). Also, we can find (see Figures 4, 5) that proposed requirement of tracking performance is achieved at the seventh iteration.

**Figure 4.** The sup-norm of tracking error \(e_1(t)\) in each iteration

**Figure 5.** The sup-norm of tracking error \(e_2(t)\) in each iteration

Now, we consider the same singular system where we apply open-closed-loop \(P-PD^{\alpha}\)-type iterative learning algorithm (64). In the simulation, we select the following gain matrices:

\[
\Gamma_1 = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0.95 & 0.4 \\ 0 & 0.95 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \tag{88}
\]

To determine values of the gain matrices, it is necessary to satisfy the convergence condition of Theorem 2 and make a comprehensive consideration of the convergence speed. It is easy to show that the pair \((E; A)\) is regular and \(|I - \Gamma_2 CB| = 0.7287 < 1\).

**Figure 6.** The tracking performance of the system output \((y_1(t); \text{solid line, } y_{d1}(t); \text{bold line})\)

**Figure 7.** The tracking performance of the system output \((y_2(t); \text{solid line, } y_{d2}(t); \text{bold line})\)
Conclusions

In this paper, a fractional order (P)-PDα type of ILC is proposed for a given class of fractional order singular systems and, using simulations, the effectiveness of the proposed ILC controller was investigated. Particularly, we considered two cases of ILC: closed-loop PDα type of ILC as well as open-loop control (P)-PDα type of ILC. Sufficient conditions for the convergence in the time domain of a proposed ILC were given by the corresponding theorems and proved.

Finally, improved ILC performances by including (open)-closured ILC controller are illustrated by numerical simulations.

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Literature


Figure 8. The sup-norms of tracking errors e(t) and e(t) in each iteration.

Here the ILC rule (64) is used, where Figures 6, 7 show the tracking performance of the ILC system outputs over the interval t∈[0,1]. Also, we can find (see Fig.8) that the proposed requirement for the tracking performance is achieved at the fifth iteration.

Compared with the results shown in Figures 4 and 5, the ILC tracking errors presented in Fig.8 are bounded to a lower level. Beside using suggested open-closed ILC control as well as learning gains matrices, one may improve the speed convergence and transient behavior of the proposed ILC fractional order systems.
Iterative learning control for fractional-order systems

Iterative learning control (ILC) is a domain important in the theory of control and is suitable for the control of a large class of mechatronic systems. This paper focuses on the use of fractional order ILC control for singular systems of fractional order. In particular, the article discusses the case of fractional order ILC control for systems with time delay. The results are obtained through the use of fractional-order calculus and the concept of fractional-order convergence. The conclusions are supported by numerical simulations that demonstrate the feasibility and efficiency of the approach proposed.

Keywords: iterative learning control, fractional-order control, singular system, fractional-order convergence.