Original scientific paper
Received: 01.07.2016.
Approved: 21.09.2016.

THE APPLICATION OF THE DYNAMIC PROGRAMMING METHOD IN INVESTMENT OPTIMIZATION

Abstract. *This paper deals with the problem of investment in Measuring Transformers Factory in Zajecar and the application of the dynamic programming method as one of the methods used in business process optimization. Dynamic programming is a special case of nonlinear programming that is widely applicable to nonlinear systems in economics. Measuring Transformers Factory in Zajecar was founded in 1969. It manufactures electrical equipment, primarily low and medium voltage current measuring transformers, voltage transformers, bushings, etc. The company offers a wide range of products and for this paper's needs the company's management selected three products for each of which optimal investment costing was made. The purpose was to see which product would be the most profitable and thus proceed with the manufacturing and selling of that particular product or products.*

Key words: *Dynamic programming, Bellman's principle of optimality, investment optimization.*

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1. Introduction

The principle of dynamic programming was, in a sense, known even before Second World War, but Professor Richard Bellman is considered to be the official creator of this method. In 1957, in his book 'Dynamic Programming' (Princeton University Press, Princeton), Bellman set the basic principles of this method. What he did was the following: he looked at a particular problem by making a hierarchy of sub-problems and then went on to solve the easiest one. That is how his **principle of optimality** came into being, which is the essence of dynamic programming. The main idea in this method's application is the division of a management process into several stages after which an optimal management model is chosen for each stage. The selected management model is the one which leads to optimal process functioning. At the same time one has to bear in mind that optimal management is characterized by the following: the future decisions have to lead to optimal management considering the present state and regardless of the previous state or previous management.

Since successful management means successful fulfillment of all business tasks of which a business process is composed, it is very important to deal with the issue of the optimal allocation of investment funds into specific activities or categories. The problem of revenue management has been dealt with by D. Zhang and D. Adelman^{1, 2}, while portfolio management has been the subject matter in the works of J. Han and B. Van Roy³. In the next sections, the mathematical model of Bellman's principle of optimality for additive objective functions will be presented.

1.1. Bellman's Principle of Optimality

Suppose that the functions $f_1(x_1), f_2(x_2), \dots, f_n(x_n)$, are such that the objective function is as follows:

$$F(x_1, x_2, \dots, x_n) = f_1(x_1) + f_2(x_2) + \dots + f_n(x_n) \quad (1)$$

The problem pins down to finding the value of the variable $x_1, x_2, \dots, x_n$ when the objective function has the maximum value subject to the following constraints:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b_n$$

¹ Zhang Dan and Adelman Daniel (2007): Dynamic bid prices in revenue management, *Operations Research*, 55(4): 647-661

² Zhang Dan and Adelman Daniel (2009): An approximate dynamic programming approach to network revenue management with customer choice., *Transportation Science*, 43: 381-394

³ Farias F. Vivek and Van Roy Benjamin, *An approximate dynamic programming approach to network revenue management*, www.stanford.edu/~bvr/psfiles/adp-rm.pdf, Working paper.

Whereby $a_j \geq 0, x_j \geq 0, j = 1, \dots, n$ $b_n \geq 0$.

For arbitrary $k = 1, \dots, n$ we introduce a set of functions $\{F_k(b_k)\}$ in the following way:

$$F_1(b_1) = \max_{a_1 x_1 \leq b_1} \{f_1(x_1)\} = \max_{x_1 \leq b_1 / a_1} \{f_1(x_1)\}, \quad (2)$$

$$F_2(b_2) = \max_{a_1 x_1 + a_2 x_2 \leq b_2} \{f_1(x_1) + f_2(x_2)\}$$

$$= \max_{a_2 x_2 \leq b_2} \left\{ \max_{a_1 x_1 \leq b_2 - a_2 x_2} [f_1(x_1) + f_2(x_2)] \right\}$$

$$= \max_{a_2 x_2 \leq b_2} \left\{ f_2(x_2) + \max_{a_1 x_1 \leq b_2 - a_2 x_2} [f_1(x_1)] \right\}$$

$$= \max_{a_2 x_2 \leq b_2} \{f_2(x_2) + F_1(b_2 - a_2 x_2)\}.$$

In the general case we get the following:

$$F_k(b_k) = \max_{x_k \leq b_k / a_k} \{f_k(x_k) + F_{k-1}(b_k - a_k x_k)\} \quad k = 2, \dots, n \quad (3)$$

Whereby $F_0(b_0) = 0$.

The equations (2) and (3) are known as **the Bellman Equation**.

On the basis of these equations, the first task with n variable is broken down into n simpler problems each with one variable and in this way the task solving is done in stages. The function F_1 is determined indirectly while the functions $F_2, \dots, F_n$ are determined by the recurrence formula (relation) (3), whereby $F_n(b_n)$ stands for the optimal value.

Considering the last equation (3), where $k = n$, for the unique number $b_n \geq 0$ the value is $x_n^* = x_n(b_n)$ for which

$$(\max)F = F_n(b_n).$$

In other words, objective function F , defined in (1), reaches its maximum value in the last iteration $F_n(b_n)$.

Finally, a set of found optimal values is to be defined, that is, one or more n values $(x_1^*, \dots, x_n^*)$ whereby objective function F has the maximum value, in which case the optimal management strategy is finally defined as follows:

$$U^* = (x_1^*, \dots, x_n^*).$$

2. Business process optimization in MTF

Measuring Transformers Factory (MTF) in Zajecar was founded in 1969. Its scope of business is manufacturing electrical equipment and it has a wide range of products including various types of current, voltage protective and measuring transformers. It also supplies the market with different kinds of medium voltage supporting and bushing types of insulators and inductive reactors. The company's product order quantity differs depending on the demand; it can supply large quantities (more than 100 items or more than 50 items), medium quantities (from 50-100, or from 20-50 items) or small quantities (fewer than 50, or fewer than 20 items). The price depends on both on investment capital and on the quantity of the sold products; this further means that the profit the company makes over a period of time depends on the selling price of its products in the market.

The company's management does its best to successfully perform all its business activities. The main business goal, of course, is meeting the demand in the market; however, the decision on optimal allocation of investment regarding particular products which will ensure maximum profit is not less important.

After the company's management had agreed, we conducted the following research: three important products manufactured by MTF were singled out and we tried to define the most effective way of investing in these products which would ultimately prove to be the most profitable for the company's business.

These products are the following:

The low voltage current transformer STEM – 081 up to 0.72kV, to which we will further refer as T_1 , is ordered in medium quantities (50-100 items), large quantities (more than 100 items) and in small quantities (fewer than 50 items). The price is EUR 17, 01 per unit.

The medium voltage current transformer STEM – N 3821, up to 35 kV, to which we will further refer as T_2 , is ordered in medium quantities (20-50 items), large quantities (more than 50 items), or small quantities (fewer than 20 items). The price is EUR 331,42 per unit.

The transformer JNT sm-24/12 of 24 kV, to which we will further refer as T_3 , is ordered in medium quantities (from 20-50 items), in large quantities (more than 50 items) or small quantities (fewer than 20 items). The price is EUR 355, 53 per unit.

According to the data provided by the Company's Finance Department the percentage of sales for each of the products in relation to the quantity sold is presented in the following table:

Table 1. *The percentage of sales*

Quantities	T_1	T_2	T_3
Medium	48.16	37.14	30.42
Large	40.00	29.52	23.18
Small	64.62	52.37	44.91

Table 2, below presents the expected profit in case of investing EUR 20000 that is, EUR 40000 that is, EUR 60000 and EUR 80000 in each of the products T_1 , T_2 , and T_3 .

Table 2. *The expected profit*

Investment	T_1	T_2	T_3
20 000	9 632	7 428	6 084
40 000	16 000	11 800	9 272
60 000	38 772	31 422	26 946
80 000	41 600	43 536	39 040

The company has the amount of money of EUR 80 000 at its disposal. It is to invest this money in the production of the three transformers with the aim of making maximum profit.

In order to solve this investment problem the above-mentioned mathematical model of Bellman's principle will be used. In this case, it will have the following form:

$f_i(x_i)$ stands for the expected profit after investing the sum x_i in product i , whereby $i=1,2,3$. Objective function $F(x_1, x_2, x_3)$ stands for the overall profit achieved by investing the amount of money of EUR 80 000 in products T_1 , T_2 , and T_3 . On the basis of the introduced conditions in this problem, the mathematical model of this problem will have the following form:

$$\max F(x_1, x_2, x_3) = f_1(x_1) + f_2(x_2) + f_3(x_3) \quad (4)$$

subject to the following constraints:

$$x_1 + x_2 + x_3 = 80000, \quad x_1, x_2, x_3 \geq 0.$$

After this the application of Bellman's principle of optimality comes, whose algorithm has already been described. As already mentioned, the problem is solved in stages but the result of one stage depends on the result gained in the previous one. Functions $F_k(b_k)$ are determined successively for $k=1,2,3$. The following is also to be taken into account:

$$b_k \in \{0, 20000, 40000, 60000, 80000\}, \quad \text{for } k = 1, 2, 3.$$

The value of function $F_1(b_1)$ is found in the first stage in the following way:

$$F_1(b_1) = \max f_1(x_1), \quad (5)$$

Further, the value of the function is as follows:

$$\begin{aligned} F_1(0) &= 0, F_1(20000) = 9632, F_1(40000) = 16000, \\ F_1(60000) &= 38772, F_1(80000) = 41600 \end{aligned} \quad (6)$$

The next stage determines the optimal allocation of investment in products T_1 and T_2 , so objective function $F_2(b_2)$ will have the following form:

$$F_2(b_2) = \max_{x_2 \leq b_2} \{f_2(b_2) + F_1(b_2 - x_2)\} \quad (7)$$

Using the gained values for function $F_1(b_1)$ and the data on investment efficiency in Product T_2 , and using the formula given at (7), the function's values are as follows $F_2(b_2)$:

for $b_2 = 0$ the following is true:

$$F_2(0) = \max_{x_2 \leq 0} \{f_2(0) + f_1(0 - x_2)\} = 0, \quad x_2 = 0$$

Then for $b_2 = 20000$, we have the following :

$$\begin{aligned} F_2(20000) &= \max_{x_2 \leq 20000} \{f_2(x_2) + F_1(20000 - x_2)\} \\ &= \max \left\{ \begin{array}{l} f_2(0) + F_1(20000) \\ f_2(20000) + F_1(0) \end{array} \right\} = \max \left\{ \begin{array}{l} 0 + 9632 \\ 7428 + 0 \end{array} \right\} = 9632 \quad (x_2 = 0) \end{aligned}$$

For $b_2 = 40000$, we have the following:

$$\begin{aligned} F_2(40000) &= \max_{x_2 \leq 40000} \{f_2(x_2) + F_1(40000 - x_2)\} = \max \left\{ \begin{array}{l} f_2(0) + F_1(40000) \\ f_2(20000) + F_1(20000) \\ f_2(40000) + F_1(0) \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} 0 + 16000 \\ 7428 + 9632 \\ 11808 + 0 \end{array} \right\} = \max \left\{ \begin{array}{l} 16000 \\ 17060 \\ 11808 \end{array} \right\} = 17060 \quad \text{whereby } x_1 = x_2 = 20000. \end{aligned}$$

Now, for $b_2 = 60000$, we will have the following values:

$$\begin{aligned} F_2(60000) &= \max_{x_2 \leq 60000} \{f_2(x_2) + F_1(60000 - x_2)\} \\ &= \max \left\{ \begin{array}{l} f_2(0) + F_1(60000) \\ f_2(20000) + F_1(40000) \\ f_2(40000) + F_1(20000) \\ f_2(60000) + F_1(0) \end{array} \right\} \end{aligned}$$

$$= \max \begin{Bmatrix} 0+38772 \\ 7428+16000 \\ 11808+9632 \\ 31422+0 \end{Bmatrix} = \max \begin{Bmatrix} 38772 \\ 23428 \\ 21440 \\ 31422 \end{Bmatrix} = 38772$$

whereby $x_2 = 0$.

At the end of this stage, for $b_2 = 80000$ we will have the following:

$$F_2(80000) = \max_{x_2 \leq 80000} \{f_2(x_2) + F_1(80000 - x_2)\}$$

$$= \max \begin{Bmatrix} f_2(0) + F_1(80000) \\ f_2(20000) + F_1(60000) \\ f_2(40000) + F_1(40000) \\ f_2(60000) + F_1(20000) \\ f_2(80000) + F_1(0) \end{Bmatrix} = \max \begin{Bmatrix} f_2(0) + F_1(80000) \\ f_2(20000) + F_1(60000) \\ f_2(40000) + F_1(40000) \\ f_2(60000) + F_1(20000) \\ f_2(80000) + F_1(0) \end{Bmatrix}$$

$$= \max \begin{Bmatrix} 0+41600 \\ 7428+38772 \\ 11808+16000 \\ 31422+9632 \\ 43536+0 \end{Bmatrix} = \max \begin{Bmatrix} 41600 \\ 46200 \\ 27808 \\ 41054 \\ 43536 \end{Bmatrix} = 46200$$

whereby $x_1 = 60000$ and $x_2 = 20000$.

On the basis of these results, we can conclude that in this investment phase it is necessary to invest $x_1=60000$ in the production of transformer T_1 , while the amount of money (EUR) needed for the production of T_2 is $x_2=20000$.

Finally, in the third stage, the allocation of funds among the three products is carried out, and in that case, objective function $F_3(b_3)$ will be as follows:

$$F_3(b_3) = \max_{x_3 \leq b_3} \{f_3(x_3) + F_2(b_3 - x_3)\} \quad (8)$$

Considering (8), the calculated values of function F_3 for expected investment capital are as follows:

if $b_3 = 0$, $F_3(0) = 0$, and $x_3 = 0$.

if $b_3 = 20000$, the values are as follows:

$$\begin{aligned} F_3(20000) &= \max_{x_3 \leq 20000} \{f_3(x_3) + F_2(20000 - x_3)\} \\ &= \max \left\{ \begin{array}{l} f_3(0) + F_2(20000) \\ f_3(20000) + F_2(0) \end{array} \right\} = \max \left\{ \begin{array}{l} 0 + 9632 \\ 6084 + 0 \end{array} \right\} = 9632 \quad (x_3 = 0) \end{aligned}$$

If the values of the parameter i $b_3 = 40000$, the values of F_3 are as follows:

$$\begin{aligned} F_3(40000) &= \max_{x_3 \leq 40000} \{f_3(x_3) + F_2(40000 - x_3)\} \\ &= \max \left\{ \begin{array}{l} f_3(0) + F_2(40000) \\ f_3(20000) + F_2(20000) \\ f_3(40000) + F_2(0) \end{array} \right\} = \left\{ \begin{array}{l} 0 + 17060 \\ 6084 + 9632 \\ 9272 + 0 \end{array} \right\} = \max \left\{ \begin{array}{l} 17060 \\ 15716 \\ 9272 \end{array} \right\} = 17060 \end{aligned}$$

whereby $x_3 = 0$.

if $b_3 = 60000$, the following is true:

$$\begin{aligned} F_3(60000) &= \max_{x_3 \leq 60000} \{f_3(x_3) + F_2(60000 - x_3)\} = \max \left\{ \begin{array}{l} f_3(0) + F_2(60000) \\ f_3(20000) + F_2(40000) \\ f_3(40000) + F_2(20000) \\ f_3(60000) + F_2(0) \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} 0 + 38772 \\ 6084 + 17060 \\ 9272 + 9632 \\ 26946 + 0 \end{array} \right\} = \max \left\{ \begin{array}{l} 38772 \\ 23144 \\ 18904 \\ 26946 \end{array} \right\} = 38772 \quad \text{whereby } x_3 = 0 \end{aligned}$$

And if the invested amount of money $b_3 = 80000$, whose allocation is crucial for the solution of this problem, the values of function F_3 will be as follows:

$$\begin{aligned} F_3(80000) &= \max_{x_3 \leq 80000} \{f_3(x_3) + F_2(80000 - x_3)\} \\ &= \max \left\{ \begin{array}{l} f_3(0) + F_2(80000) \\ f_3(20000) + F_2(60000) \\ f_3(40000) + F_2(40000) \\ f_3(60000) + F_2(20000) \\ f_3(80000) + F_2(0) \end{array} \right\} = \left\{ \begin{array}{l} 0 + 46200 \\ 6084 + 38772 \\ 9272 + 17060 \\ 26946 + 9632 \\ 39040 + 0 \end{array} \right\} = \left\{ \begin{array}{l} 46200 \\ 44856 \\ 26332 \\ 36578 \\ 39040 \end{array} \right\} = 46200 \end{aligned}$$

Finally we come up with the following values: $x_3=0$, $x_2=20000$ and $x_1=60000$, which means that the amount of EUR 80 000 is to be allocated in the following manner: EUR 60 000 in the production of T_1 and EUR 20 000 in the production of T_2 , while no investment in T_3 is to be made. In this way the company would make maximum profit, that is EUR 46 200.

All the results of the dynamic analysis carried out in Measuring Transformers Factory in Zajecar are presented in the following table.

Table 3. *The results of the dynamic analysis*

Investment	x_1	$F_1(x_1)$	x_2	$F_2(x_2)$	x_3	$F_3(x_3)$
0	0	0	0	0	0	0
20 000	20 000	9 632	0	9 632	0	9 632
40 000	40 000	16 000	20 000	17 060	0	1 7060
60 000	60 000	38 772	0	35 772	0	38 772
80 000	80 000	41 600	20 000	46 200	0	46 200

The results shown in Table 3 will be the same as those acquired by the application of **Bellman** software application that will be dealt with in the next section. In that way we will practically check and prove the previously used procedures in Bellman’s principle of optimality which is, as described so far, applied in the optimization of investment allocation in Measuring Transformers Factory in Zajecar.

3. The software implementation

As there was a need for the dynamic analysis of the dynamic programming problem, we created the **original software application** of Bellman’s principle of optimality by making a **database application in Access**. This application, called **Bellman**, is used for checking the data acquired in the previous analysis.

The main principle this application relies on is a known algorithm based on Bellman’s principle of optimality which ensures optimal planning of the so-called **multi-stage process of management**. The software, whose main components are shown in Picture 1, is such that it first requires a user to enter the total investment capital, that is, the number of projects which are to be analyzed. After that, the investment value is entered as well as the expected value of profit for each level of investment and for each project respectively.

After the values have been entered, the software finds the optimal investment plan which ensures maximum profit. All the calculated parameters and the main characteristics of the DP problem in question are presented in the form of the MS Access Report as shown in Picture 2. It is clear that the data obtained

by the dynamic analysis of Bellman's principle of optimality completely match the results gained by the software analysis application.

Picture 1. Process for solving DP problems

Picture 2. Software implementation of investment optimization in MTF in Zajecar

Ulaganja	x(1)	F1	x(2)	F2	x(3)	F3
200000	200000	96320	0	96320	0	96320
400000	400000	160000	200000	170600	0	170600
600000	600000	387720	0	387720	0	387720
800000	800000	416000	200000	462000	0	462000

Maksimalna vrednost ciljne funkcije iznosi F=462000.novcanih jedinica

It goes without saying that this software application is of a general character and can be used in solving other problems when it is necessary to apply dynamic programming. It may be interesting to mention a problem many banks have when it comes to financing business plans – they are to select the most optimal business plans and finance them and not the others given the limited resources.

Conclusion

After conducting the dynamic analysis by applying Bellman's principle of optimality and after the results had been proven by **Bellman** software application, we can conclude that we solved the problem of investment optimization in Measuring Transformers Factory in Zajecar regarding the three types of transformers marked as T_1 , T_2 and T_3 in this paper.

The analysis shows that if the company's management wants to invest EUR 80 000 in the production of these three types of transformers it is most cost-effective to invest EUR 60 000 in T_1 , and EUR 20000 in T_2 , while no investing should occur in T_3 as it makes no profit. The analysis leads to the following objective function F with its maximum value:

$$(\max)F=46200$$

which means that the profit made is EUR 46 2 00.

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OPTIMIZACIJA INVESTICIJA METODOM DINAMIČKOG PROGRAMIRANJA

U ovom radu opisuje se problem investicionih ulaganja u Fabriki mernih transformatora u Zaječaru, metodom dinamičkog programiranja, kao jedne od metoda za optimizaciju upravljačkih procesa. Dinamičko programiranje je specijalni slučaj nelinearnog programiranja i kao takvo pruža velike mogućnosti u rešavanju raznih problema u ekonomiji nelinearnog tipa. Fabrika mernih transformatora u Zaječaru posluje od 1969. godine, a bavi se proizvodnjom elektro opreme, pre svega transformatora niskog i srednjeg napona, naponskih transformatora, provodnih izolatora, itd. Menadžment ove kompanije, od širokog asortimana proizvoda izdvojio je tri proizvoda, za koje treba odrediti optimalni iznos investicionih ulaganja, i to u svakog od njih posebno, kako bi se njihovom proizvodnjom i prodajom ostvario maksimalni profit kompanije.

Ključne reči: *Dinamičko programiranje, Bellmanov princip optimalnosti, optimizacija investicionih ulaganja*