Fixed Point Results in G_q -Metric Spaces with W -distance

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Abstract: The aim of this paper is to introduce the concept of G_q -metric space. We establish the existence of fixed points of Jachymski type mappings in the framework of G_q -metric spaces equipped with w -distance. Our results generalize and extend various results in the existing literature. Several examples to support our results are also presented.

Keywords: Fixed point, Jachymski function, w -distance, G_q -metric space.

1 Introduction and Preliminaries

The notion of distance plays an important role in geometric properties of Banach spaces as well as in different decision processes made in optimization theory. Gahler [7, 8] introduced 2-metric space in an attempt to generalize the concept of usual metric space. Ha et al. [11] showed the fact that 2-metric is not a continuous mapping with respect to its variables. There is irrelevance with the results between in ordinary metric spaces and in 2-metric spaces. These observations motivated Dhage [4] to introduce the concept of D -metric space (see also, [3, 5, 6]). Mustaf and Sims [14] mentioned some flaws in Dhage's theory of D -metric spaces. In 2006, Mustafa and Sims [15] introduced the concept of G -metric space to overcome the drawbacks in the existing theories concerning the generalizations of ordinary metric spaces. Based on G -metric spaces, several authors have obtained fixed and common fixed point results for mappings satisfying certain contractive conditions (see for example, [1, 16] and references contained therein). Saadati et al. [16] introduced the notion of Ω -distance on G -metric spaces and obtained fixed point results by utilizing this concept. Recently, Alegre *et al.*[2] obtained an interesting result for the mapping satisfying some generalized contractive conditions on complete quasi-metric spaces involving w -distances. In this paper, we introduce the concept of G_q -metric space which generalizes the one of G -metric space. We obtain some fixed point results of Jachymski type mappings in the setup

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of such spaces involving w -distances. Our results extend and strengthen previous known comparable results in the existing literature.

In the sequel we denote $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}^*$ as the set of all positive real numbers, the set of natural numbers and the set of all positive integers, respectively.

Definition 1 Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \rightarrow \mathbb{R}^+$ satisfies:

1. $G(x, y, z) = 0$ if $x = y = z \in X$;
2. $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
3. $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables);
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a G -metric on X and (X, G) is called a G -metric space.

Definition 2 Let X be a nonempty set. Suppose that a mapping $G_q : X \times X \times X \rightarrow \mathbb{R}^+$ satisfies:

1. $G_q(x, y, z) = 0$ implies that $x = y = z$ for all $x, y, z \in X$;
2. $G_q(x, x, y) \leq G_q(x, y, z)$ for all $x, y, z \in X$;
3. $G_q(x, y, z) \leq G_q(x, a, a) + G_q(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G_q is called a G_q -metric or G -quasi-metric on X and (X, G_q) is called a G_q -metric space or G -quasi metric space.

Remark 1 Every G -metric space is a G_q -metric space but the converse is not true.

Example 1 If $X = [0, \infty)$, then $G_q : X \times X \times X \rightarrow \mathbb{R}^+$ defined by

$$G_q(x, y, z) = \max\{|x - y|, |y - z|\}$$

is a G_q -metric on X . However, G_q is not a G -metric on X . Indeed, $G_q(1, -1, 0) \neq G_q(1, 0, -1)$.

Definition 3 Let (X, G_q) be a G_q -metric space and $\{x_n\}$ a sequence in X . Then

- (a) $\{x_n\}$ is called G_q -convergent if there exists $x \in X$ such that $G_q(x_n, x_m, x) \rightarrow 0$ as $n, m \rightarrow \infty$;

- (b) $\{x_n\}$ is called a G_q -Cauchy sequence if for any $\varepsilon > 0$, there is a positive integer n_0 such that $G_q(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l > n_0$;
- (c) (X, G_q) is called G_q -complete if every G_q -Cauchy sequence in X is G_q -convergent in X .

Saadati et al. [16] introduced the notion of Ω -distance on G -metric spaces. We similarly define w -distance on G_q -metric spaces as follows:

Definition 4 Let X be a G_q -metric space. A function $q : X \times X \times X \rightarrow \mathbb{R}^+$ is said to be a w -distance on X if for any $x, y, z, a \in X$, the following conditions hold:

- (W1) $q(x, y, z) \leq q(x, a, a) + q(a, y, z)$;
- (W2) $q(x, y, \cdot), q(x, \cdot, y) : X \rightarrow \mathbb{R}^+$ are lower semi-continuous;
- (W3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $q(x, a, a) \leq \delta$ and $q(a, y, z) \leq \delta$ imply that $G_q(x, y, z) \leq \varepsilon$.

For definitions and examples of w -distances on G -metric space, we refer to [2] and references mentioned therein.

Consistent with Alegre *et al.* [2], the following definition will be needed in the sequel.

Definition 5 A mapping $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be a Meir-Keeler function if $\phi(0) = 0$, and for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \in \mathbb{R}^+$,

$$\varepsilon \leq t < \varepsilon + \delta \Rightarrow \phi(t) < \varepsilon, \quad (1.1)$$

Clearly, if ϕ is a Meir-Keeler function, then $\phi(t) < t$ for all $t > 0$.

Definition 6 A mapping $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be a Jachymski function if $\phi(0) = 0$, and for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon < t < \varepsilon + \delta \text{ implies } \phi(t) \leq \varepsilon, \text{ for all } t \in \mathbb{R}^+.$$

Note that Meir-Keeler function definitely be a Jachymski function. However, the converse does not hold in general (see [2]).

Example 2 Let $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined by

$$\phi(t) = \begin{cases} \frac{t}{3}, & \text{if } t \in [2, 3], \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\phi(0) = 0$. For $\varepsilon > 0$, we consider the following cases:

- (1) If $0 < \varepsilon < 2$, then $\delta = 2 - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = 2$ implies that $\phi(t) = 0 < \varepsilon$.
- (2) If $\varepsilon = 2$, then $\delta = 1$ with $\varepsilon \leq t < \varepsilon + \delta = 3$ gives that $\phi(t) = \frac{t}{3} < 1 < \varepsilon$;
- (3) If $2 < \varepsilon < 3$, then choose $\delta = 3 - \varepsilon$. From $\varepsilon \leq t < \varepsilon + \delta = 3$, it follows that $\phi(t) = \frac{t}{3} < 1 < \varepsilon$;
- (4) When $\varepsilon \geq 3$. Take δ such that $\varepsilon \leq t < \varepsilon + \delta$. As $\phi(3) = 1$ and $\phi(t) = 0$ for $t > 3$, so $\phi(t) < \varepsilon$.

Thus ϕ is a Meir-Keeler mapping and hence is a Jachymski function, too.

Alegre *et al.* [2] proved the following result in the setting of quasi-metric spaces.

Theorem 1 *Let T be a self-mapping on a complete quasi-metric space X . Suppose that there exist a w -distance q on X and a Jachymski function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$. If for any $x, y \in X$, the following condition holds:*

$$q(Tx, Ty) \leq \phi(q(x, y)).$$

Then T has a unique fixed point $z \in X$ and $q(z, z) = 0$.

2 Main result

We use the following lemma to prove the main result of this paper.

Lemma 1 *Let X be a G_q -metric space and q a w -distance on X . Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in X , $\{\alpha_n\}$ and $\{\beta_n\}$ are null sequences in $[0, \infty)$ and $x, y, z, a \in X$. Then we have the following:*

1. *If $q(y, x_n, x_n) \leq \alpha_n$ and $q(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$. Then $G_q(y, y, z) < \varepsilon$.*
2. *If $q(y_n, x_n, x_n) \leq \alpha_n$ and $q(x_n, y_m, z) \leq \beta_n$ for $m > n$. Then $G_q(y_n, y_m, z) \rightarrow 0$.*
3. *If $q(x_n, x_m, x_l) \leq \alpha_n$ for $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$. Then $\{x_n\}$ is a G_q -Cauchy sequence.*
4. *If $q(x_n, a, a) \leq \alpha_n$ for $n \in \mathbb{N}$. Then $\{x_n\}$ is a G_q -Cauchy sequence.*

Proof Let $\varepsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for $n > n_0$. Then for $m > n \geq n_0$, we have $q(y_n, x_n, x_n) \leq \alpha_n < \delta$ and $q(x_n, y_m, z) \leq \beta_n < \delta$. Since q is a w -distance on X , by (W3) it follows that $G_q(y_n, y_m, z) \rightarrow 0$ and hence $y_n \rightarrow z$. Thus (1) is completed, subsequently, (2) follows immediately from (1).

To prove (3), let $l > m > n \geq n_0$. Then

$$q(x_n, x_{n+1}, x_{n+1}) \leq \alpha_n < \delta$$

and

$$q(x_{n+1}, x_m, x_l) \leq \alpha_{n+1} < \delta.$$

Using (W3), we get $G_q(x_n, x_m, x_l) \leq \varepsilon$ and $\{x_n\}$ is a G_q -Cauchy sequence. Whereas, (4) is a special case of (3).

■

Now we shall prove the main results of this paper as follows.

Theorem 2 Let X be a complete G_q -metric space, and T a self-mapping on X . Suppose that there exist a w -distance q on X and a Jachymski function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\phi(t) < t$, for all $t > 0$. If for all $x, y \in X$, the following condition holds:

$$q(Tx, Ty, Tz) \leq \phi(q(x, y, z)). \quad (2.1)$$

Then T has a unique fixed point $z \in X$. Moreover, $q(z, z, z) = 0$.

Proof Let x_0 be a given point in X . Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. By (2.1), it follows that

$$q(x_{n+1}, x_{n+2}, x_{n+2}) = q(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \phi(q(x_n, x_{n+1}, x_{n+1})). \quad (2.2)$$

Put $r_n = q(x_n, x_{n+1}, x_{n+1})$ for all $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that $r_{n_0} = 0$, then by repeated application of (2.2), we have $r_n = 0$ for all $n \geq n_0$. In particular,

$$r_{n_0} = q(x_{n_0}, x_{n_0+1}, x_{n_0+1}) = 0 \quad (2.3)$$

gives that

$$q(x_{n_0+1}, x_m, x_l) = 0 \text{ for } l = m \geq n_0 + 2. \quad (2.4)$$

As q is a w -distance on X , so from (2.3) and (2.4) it follows that

$$G_q(x_{n_0}, x_m, x_l) = 0 \text{ for } l = m \geq n_0 + 2.$$

Thus $x_n = x_{n_0}$ for $n \geq n_0 + 2$. In this case, x_{n_0} is a fixed point of T .

Assume that $r_n > 0$. Making using of (2.2), we arrive at

$$r_{n+1} \leq \phi(r_n) < r_n \text{ for } n \in \mathbb{N}. \quad (2.5)$$

Thus $\{r_n\}$ converges to some $r \in \mathbb{R}^+$. Obviously, $r_n > r$ for all $n \in \mathbb{N}$. We prove that $r = 0$. Indeed, if $r > 0$, then there exists $\delta = \delta(r)$ such that $r < t < r + \delta$. By the given assumption, we have $\phi(t) \leq r$. Take $n_0 \in \mathbb{N}$ such that $r < r_n < r + \delta$, for any $n \geq n_0$, hence $\phi(r_n) \leq r$.

Also, from (2.2), it establishes that $r_{n+1} \leq r$ for all $n > n_0$. This a contradiction with (2.5). Accordingly, $r = 0$.

Now we show that $\{x_n\}$ is a G_q -Cauchy sequence. Let $\varepsilon > 0$. Take $\delta = \delta(\varepsilon) \in (0, \varepsilon)$ such that conditions (W3) and (1.1) hold. Also, for $\frac{\delta}{2} > 0$, choose $\delta_1 = \delta_1(\frac{\delta}{2}) \in (0, \frac{\delta}{2})$ such that (W3) and (1.1) hold. As $r_n \rightarrow 0$, there exists $k_0 \in \mathbb{N}$ such that $r_n < \delta_1$ for all $n \geq k_0$. By induction, we shall prove that for each $k \geq k_0$ and $n \in \mathbb{N}$, the following inequality (2.6) holds:

$$q(x_k, x_{n+k}, x_{n+k}) < \frac{\delta}{2} + \delta_1. \quad (2.6)$$

For any fixed $k \geq k_0$, we have

$$q(x_k, x_{1+k}, x_{1+k}) < \delta_1. \quad (2.7)$$

Hence (2.6) holds for $n = 1$. Assume that (2.6) holds for some $n \in \mathbb{N}$. We consider the following two cases:

(i) If

$$q(x_k, x_{n+k}, x_{n+k}) > \frac{\delta}{2}, \quad (2.8)$$

then by (1.1), we have

$$\phi(q(x_k, x_{n+k}, x_{n+k})) \leq \frac{\delta}{2}. \quad (2.9)$$

From (2.2) we have

$$q(x_{k+1}, x_{n+k+1}, x_{n+k+1}) \leq \frac{\delta}{2}. \quad (2.10)$$

By condition (W1), (2.7) and (2.10), we have

$$q(x_k, x_{n+k+1}, x_{n+k+1}) \leq q(x_k, x_{k+1}, x_{k+1}) + q(x_{k+1}, x_{n+k+1}, x_{n+k+1}) < \frac{\delta}{2} + \delta_1.$$

(ii) If

$$q(x_k, x_{n+k}, x_{n+k}) \leq \frac{\delta}{2}. \quad (2.11)$$

Then there are two possibilities: $q(x_k, x_{n+k}, x_{n+k}) = 0$ or $q(x_k, x_{n+k}, x_{n+k}) > 0$.

If $q(x_k, x_{n+k}, x_{n+k}) = 0$. Using (2.1), we have

$$q(x_{k+1}, x_{n+k+1}, x_{n+k+1}) = 0. \quad (2.12)$$

From (W1), (2.7) and (2.12), we get

$$\begin{aligned} q(x_k, x_{n+k+1}, x_{n+k+1}) &\leq q(x_k, x_{k+1}, x_{k+1}) + q(x_{k+1}, x_{n+k+1}, x_{n+k+1}) \\ &\leq \delta_1 + 0 < \frac{\delta}{2} + \delta_1. \end{aligned}$$

If $q(x_k, x_{n+k}, x_{n+k}) > 0$. Then we acquire that

$$\phi(q(x_k, x_{n+k}, x_{n+k})) < q(x_k, x_{n+k}, x_{n+k}) \leq \frac{\delta}{2}. \quad (2.13)$$

Using (W1), (2.2), (2.8) and (2.13), we have

$$\begin{aligned} q(x_k, x_{n+k+1}, x_{n+k+1}) &\leq q(x_k, x_{k+1}, x_{k+1}) + q(x_{k+1}, x_{n+k+1}, x_{n+k+1}) \\ &\leq q(x_k, x_{k+1}, x_{k+1}) + \phi(q(x_k, x_{n+k}, x_{n+k})) \\ &< \frac{\delta}{2} + \delta_1. \end{aligned}$$

Thus (2.6) is proved.

If we take $i \in \mathbb{N}$ with $i > k$. Then $i = n + k$ for some $n \in \mathbb{N}$. As $\delta_1 \in (0, \frac{\delta}{2})$, from (2.6) we have

$$q(x_k, x_i, x_i) = q(x_k, x_{n+k}, x_{n+k}) < \frac{\delta}{2} + \delta_1 < \delta.$$

As a result, by Lemma 2.1, $\{x_n\}$ is a G_q -Cauchy sequence in X . As X is G_q -complete, there exists an element $z \in X$ such that $\lim_{n \rightarrow \infty} G_q(x_n, z, z) = 0$. Next, we shall show that $\lim_{n \rightarrow \infty} q(x_n, z, z) = 0$. Actually, for each $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that

$$q(x_k, x_{n+k}, x_{n+k}) < \varepsilon, \forall k \geq k_0, n \in \mathbb{N}.$$

Using the lower semi-continuity of w -distance q , we have

$$q(x_k, z, z) \leq \liminf_{n \rightarrow \infty} q(x_k, x_{n+k}, x_{n+k}) < \varepsilon.$$

Thus $q(x_k, z, z) < \varepsilon$ for all $k \geq k_0$ and hence the claim is satisfied. From (2.1) it follows that $\lim_{n \rightarrow \infty} q(x_{n+1}, Tz, Tz) = 0$. Using (W3), we have $\lim_{n \rightarrow \infty} G_q(x_n, Tz, Tz) = 0$. Consequently, $z = Tz$.

Note that $q(z, z, z) = 0$. If not, then

$$q(z, z, z) = q(Tz, Tz, Tz) \leq \phi(q(z, z, z)) < q(z, z, z)$$

gives a contradiction. Hence $q(z, z, z) = 0$. To prove the uniqueness of fixed point of T , let u be another fixed point of T such that $u \neq z$. If $q(u, z, z) > 0$, then we have

$$q(u, z, z) = q(Tu, Tz, Tz) \leq \phi(q(u, z, z)) < q(u, z, z),$$

a contradiction. Hence $q(u, z, z) = 0$. Similarly $q(u, u, u) = 0$. Using (W3), we have $G_q(u, z, z) = 0$. Thus $u = z$.

■

Now we present some corollaries of the above Theorem.

Corollary 1 *Let X be a complete G_q -metric space and T a self-mapping on X . Suppose that there exist a w -distance q on X and a Jachymski function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$. If for all $x, y, z \in X$, the following condition holds:*

$$\int_0^{q(Tx, Ty, Tz)} \lambda(s) ds \leq \int_0^{\phi(q(x, y, z))} \lambda(s) ds, \quad (2.14)$$

where $\lambda : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue-integrable, non-negative mapping on every compact subset of $[0, \infty)$ such that $\int_0^\varepsilon \lambda(s) ds > 0$ for each $\varepsilon > 0$. Then T has a unique fixed point $z \in X$.

Proof Define a mapping $\Psi : [0, \infty) \rightarrow [0, \infty)$ by $\Psi(t) = \int_0^t \lambda(s) ds$. Using (2.14) we obtain that

$$\Psi(q(Tx, Ty, Tz)) \leq \Psi(\phi(q(x, y, z)))$$

which implies that $q(Tx, Ty, Tz) \leq \phi(q(x, y, z))$. Thus by Theorem 2, T has a unique fixed point. ■

Corollary 2 *Let X be a complete G_q -metric space, and T a self-mapping on X . Suppose that there exist a w -distance q on X and a Meir-Keeler function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$. If for all $x, y, z \in X$, the following condition holds:*

$$q(Tx, Ty, Tz) \leq \phi(q(x, y, z)).$$

Then T has a unique fixed point $z \in X$.

Proof Since every Meir-Keeler function is Jachymski function, so Corollary 2.2 follows from Theorem 2.1. ■

Corollary 3 *Let X be a complete G_q -metric space, and T a self-mapping on X . Suppose that there exist a w -distance q on X and a non-decreasing function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\phi(0) = 0$, and $\phi^n(r) \rightarrow 0$ ($n \rightarrow \infty$), $\forall r > 0$. If for all $x, y, z \in X$, the following condition holds:*

$$q(Tx, Ty, Tz) \leq \phi(q(x, y, z)).$$

Then T has a unique fixed point $z \in X$.

Proof To prove this, it is enough to show that ϕ is Meir-Keeler function. Assume on the contrary that ϕ is not a Meir-Keeler function. There exists $\varepsilon > 0$ and a sequence $\{r_n\}$ of positive real numbers such that

$$\varepsilon \leq r_n < \varepsilon + \delta \text{ and } \phi(r_n) \geq \varepsilon, \forall n \in \mathbb{N}.$$

Since ϕ is non-decreasing, then $\phi(r_n) \geq \varepsilon$. Hence $\phi^m(r_n) \geq \varepsilon$ ($\forall m \in \mathbb{N}$), a contradiction with the fact that $\phi^m(r) \rightarrow 0$ ($m \rightarrow \infty$) for all $r > 0$. Therefore, ϕ is a Meir-Keeler and hence the result follows from Corollary 2. ■

The following example shows that the condition $\phi(t) < t$ ($\forall t > 0$) from Theorem 2.1 can not be omitted.

Example 3 Let $X = \{0, 2\}$ and $G_q : X \times X \times X \rightarrow \mathbb{R}^+$ be a mapping defined by

$$G_q(x, y, z) = |x - y| + |x - z|.$$

Then (X, G_q) is a complete G_q -metric space. Define the mappings $q : X \times X \times X \rightarrow \mathbb{R}^+$, $T : X \rightarrow X$ and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$q(x, y, z) = |x - y| + |x - z|,$$

$$Tx = \begin{cases} 2, & \text{if } x = 0, \\ 0, & \text{if } x = 2. \end{cases} \quad \phi(t) = \begin{cases} 2, & \text{if } t = 2, \\ 0, & \text{if } t = 4, \\ 0, & \text{otherwise.} \end{cases}$$

For a given $\varepsilon > 0$, we consider the following cases:

- (1) If $0 < \varepsilon < 2$, then $\delta = 2 - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = 2$ implies $\phi(t) = 0 < \varepsilon$;
- (2) If $\varepsilon = 2$, take $\delta = \frac{1}{3}$, then from $\varepsilon < t < \varepsilon + \delta = \frac{7}{3}$ we have $\phi(t) = 0 < \varepsilon$;
- (3) If $2 < \varepsilon < 4$, then $\delta = 4 - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = 4$ gives that $\phi(t) = 0 < \varepsilon$;
- (4) If $\varepsilon = 4$, choose $\delta = \frac{1}{3}$, then from $\varepsilon < t < \varepsilon + \delta = \frac{13}{3}$ it follows that $\phi(t) = 0 < \varepsilon$;
- (5) If $\varepsilon \geq 4$, then for any fixed δ with $\varepsilon \leq t < \varepsilon + \delta$ implies that $\phi(t) \leq \varepsilon$.

Thus ϕ is a Jachymski function. Consider the following table:

This table shows that condition (2.1) in Theorem 2 is satisfied but T has no fixed point.

Example 4 Let $X = \{0, \frac{1}{2}\}$ and G_q be a mapping defined by

$$G_q(x, y, z) = \max\{|x - y|, |y - z|\}.$$

Then X is a complete G_q -metric space. Define the mappings $q : X \times X \times X \rightarrow \mathbb{R}^+$, $T : X \rightarrow X$ and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $Tx = \frac{1}{2}$, $q(x, y, z) = \max\{|x - y|, |y - z|\}$, and

$$\phi(t) = \begin{cases} \frac{1}{4}, & \text{if } t = \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

x	y	z	Tx	Ty	Tz	$q(x, y, z)$	$q(Tx, Ty, Tz)$	$\phi(q(x, y, z))$
0	0	0	2	2	2	0	0	0
0	0	2	2	2	0	2	2	2
0	2	0	2	0	2	2	2	2
0	2	2	2	0	0	4	4	4
2	0	0	0	2	2	4	4	4
2	0	2	0	2	0	2	2	2
2	2	0	0	0	2	2	2	2
2	2	2	0	0	0	0	0	0

Clearly $\phi(0) = 0$. For $\varepsilon > 0$, consider the following cases:

(1) If $0 < \varepsilon < \frac{1}{2}$, then $\delta = \frac{1}{2} - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = \frac{1}{2}$ implies that $\phi(t) = 0 < \varepsilon$;

(2) If $\varepsilon = \frac{1}{2}$, then $\delta = \frac{1}{3}$. From $\varepsilon < t < \varepsilon + \delta = \frac{5}{6}$, it follows that $\phi(t) = 0 < \varepsilon$;

(3) If $\varepsilon \geq \frac{1}{2}$, then for any fixed δ with $\varepsilon \leq t < \varepsilon + \delta$ gives $\phi(t) < \varepsilon$.

So ϕ is a Jachymski function. From the following table, we are easy to see all the conditions of Theorem 2.1 are satisfied, hence T has a unique fixed point $x = \frac{1}{2}$.

x	y	z	Tx	Ty	Tz	$q(x, y, z)$	$q(Tx, Ty, Tz)$	$\phi(q(x, y, z))$
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$
0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0

Example 5 Let $X = \{0, 1, 2\}$ and let the mappings $G_q : X \times X \times X \rightarrow \mathbb{R}^+$, $q : X \times X \times X \rightarrow \mathbb{R}^+$ be defined by

$$q(x, y, z) = G_q(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{|x|, |y|\} + \max\{|x|, |z|\}, & \text{otherwise.} \end{cases}$$

Then (X, G_q) is a complete G_q -metric space. Let the mappings $T : X \rightarrow X$ and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

be defined by

$$T(x) = \begin{cases} 0, & \text{if } x = 1, \\ \frac{x}{2}, & \text{if } x = \{0, 2\}. \end{cases} \quad \phi(t) = \begin{cases} 1, & \text{if } t = 2, \\ 2, & \text{if } t = 3, \\ 3, & \text{if } t = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\phi(0) = 0$. For $\varepsilon > 0$ we consider the following cases:

- (1) If $0 < \varepsilon < 1$, then $\delta = 1 - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = 1$ implies that $\phi(t) = 0 < \varepsilon$;
- (2) If $\varepsilon = 1$. Choose $\delta = \frac{1}{3}$, then $\varepsilon < t < \varepsilon + \delta = \frac{4}{3}$ gives $\phi(t) = 0 < \varepsilon$;
- (3) If $1 < \varepsilon < 2$, then $\delta = 2 - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = 2$ implies that $\phi(t) = 0 < \varepsilon$;
- (4) If $\varepsilon = 2$, take $\delta = \frac{1}{3}$. From $\varepsilon < t < \varepsilon + \delta = \frac{7}{3}$ it follow that $\phi(t) = 0 < \varepsilon$;
- (5) If $2 < \varepsilon < 3$, then $\delta = 3 - \varepsilon$ with $\varepsilon \leq t < \varepsilon + \delta = 3$ gives that $\phi(t) = 0 < \varepsilon$;
- (6) If $\varepsilon = 3$, $\delta = \frac{1}{3}$ with $\varepsilon < t < \varepsilon + \delta = \frac{10}{3}$, then $\phi(t) = 0 < \varepsilon$;
- (7) If $\varepsilon \geq 3$, then for any fixed δ with $\varepsilon \leq t < \varepsilon + \delta$ implies that $\phi(t) \leq \varepsilon$.

x	y	z	Tx	Ty	Tz	$q(x, y, z)$	$q(Tx, Ty, Tz)$	$\phi(q(x, y, z))$
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0	0
0	0	2	0	0	1	2	1	1
0	1	0	0	0	0	1	0	0
0	1	1	0	0	0	2	0	1
0	1	2	0	0	1	3	1	2
0	2	0	0	1	0	2	1	1
0	2	1	0	1	0	3	1	2
0	2	2	0	1	1	4	2	3
1	0	0	0	0	0	2	0	1
1	0	1	0	0	0	2	0	1
1	0	2	0	0	1	3	1	2
1	1	0	0	0	0	2	0	1
1	1	1	0	0	0	2	0	1
1	1	2	0	0	1	3	1	2
1	2	0	0	1	0	3	1	2
1	2	1	0	1	0	3	1	2
1	2	2	0	1	1	4	2	3
2	0	0	1	0	0	4	2	3
2	0	1	1	0	0	4	2	3
2	0	2	1	0	1	4	2	3
2	1	0	1	0	0	4	2	3
2	1	1	1	0	0	4	2	3

2	1	2	1	0	1	4	2	3
2	2	0	1	1	0	4	2	3
2	2	1	1	1	0	4	2	3
2	2	2	1	1	1	4	2	3

As a consequence, ϕ is a Jachymski function. From the above table, we are easy to see that all the conditions of Theorem 2 are satisfied and therefore $x = 0$ is the unique fixed point of T .

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