A variant of McDougall-Wotherspoon method for finding simple roots of nonlinear equations

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Abstract: Recently, McDougall and Wotherspoon have developed a simple modification to the standard Newton method for finding simple roots of nonlinear equations. Their method is based on the arithmetic mean and achieves approximately 2.4 convergence order. This research presents a new iteration scheme based on the harmonic mean with the same convergence order and the comparison with the original McDougall-Wotherspoon method. Several test examples have been used and the results show good numerical behavior of the new method.

Keywords: McDougall and Wotherspoon method, root-finding, iterative method, numerical method.

1 Introduction

One of the most frequent problems in applied mathematics is finding the roots of nonlinear equation $f(x) = 0$. For a vast number of real-life applications, the exact roots cannot be found, and therefore some iterative methods are being employed to determine the approximate roots with certain level of accuracy. When a sequence $\{x_n\}$ generated by an iterative process converges to a exact root α such that the following equation holds

$$\lim_{x \to \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^p} = C,$$

for some constant $C \neq 0$, and $p \geq 1$, than it can be said that the iterative method has the convergence order p, and C is asymptotic error constant.

Probably the best-known root-finding iterative method is Newton's method with the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \tag{1}$$

If initial approximation x_0 is chosen sufficiently close to the simple root α (which means that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$), Newton's method has quadratic convergence.

Based on this method, many researchers have developed various modifications and multistep iteration schemes with improved convergence order. As a measure of efficiency, in 1960. Ostrowski [8] introduced the coefficient $I = p^{1/q}$, where p is the convergence order, and q is the number of function of derivative evaluations per iteration. Therefore, since Newton's method (1) reaches convergence order two with one function and one first derivative evaluation per iteration, its efficiency index is $2^{1/2} \approx 1.4142$. Higher convergence order or efficiency usually demand additional function/derivative evaluations. For example, Weerakoon and Fernando [11] proposed two-step method

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= x_n - \frac{f(x_n)}{\frac{1}{2}(f'(x_n) + f'(y_n))},
\end{aligned}
\tag{2}
$$

using one function evaluation and the arithmetic mean of two derivative evaluations. Using the harmonic mean in the denominator of method (2) instead of the arithmetic mean, Özban developed a similar third-order root-finding method [9].

Beside those methods, for the purpose of comparison with the new method, some Newton-type methods are presented as follows:

- Frontini-Sormani method [2]

$$
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/(2f'(x_n)))},
\tag{3}
$$

- Kou-Li-Wang method [5]

$$
x_{n+1} = x_n - \frac{f(x_n + f(x_n)/f'(x_n)) - f(x_n)}{f'(x_n)},
\tag{4}
$$

- Wang method [10]

$$
x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 3f'(x_n - 2f(x_n)/(3f'(x_n)))}.
\tag{5}
$$

All presented third-order method have efficiency index $3^{1/3} \approx 1.4422$.

In contrast to the displayed methods, there are some other root-finding techniques which reuse information from the previous iterations in order to improve convergence speed. The methods of this type are called methods with memory. The best-known method with memory is the secant method, which can be also viewed as a modification of Newton's

method (1), whereas the first derivative $f'(x_n)$ is approximated by the divided difference $(f(x_n) - f(x_{n-1}))/(x_n - x_{n-1})$. Although its efficiency index is $0.5(1+\sqrt{5}) \approx 1.618$ which is larger than method (1), the secant method has certain drawbacks described in [7] with more details.

2 McDougall-Wotherspoon method

Motivated by studies related to the algorithms for seawater analysis [3, 6], McDougall and Wotherspoon have derived a multistep iterative method with memory [7], as a new modification of the classical Newton's method, given by

$$
\begin{aligned}
\hat{x}_0 &= x_0, \\
x_1 &= x_0 - \frac{f(x_0)}{f'\left(\frac{1}{2}\left(x_0 + \hat{x}_0\right)\right)} = x_0 - \frac{f(x_0)}{f'(x_0)},
\end{aligned}
$$

followed by

$$
\begin{aligned}
\hat{x}_n &= x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}\left(x_{n-1} + \hat{x}_{n-1}\right)\right)}, \\
x_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}\left(x_n + \hat{x}_n\right)\right)},
\end{aligned}
\tag{6}
$$

for $n \geq 1$. For example, the approximation x_2 is calculated using $f(x_1)$ and the derivative evaluated at the arithmetic mean of x_1 and $\hat{x}_1$. The same value of that derivative is reused in the next step to get x_3.

McDougall and Wotherspoon have shown that method (6) achieves the convergence order $1 + \sqrt{2} \approx 2.4142$, and consequently the efficiency index 1.5538 (for the initial approximation x_0 chosen close to the root α). Furthermore, they have verified its good numerical behavior compared to existing methods through several test examples.

3 A new method and the convergence analysis

If the arithmetic mean of x_n and $\hat{x}_n$ is replaced by its harmonic mean, preserving $\hat{x}_0$ and x_1, for $n \geq 1$ the new iterative scheme can be obtained

$$
\begin{aligned}
\hat{x}_n &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_{n-1}\hat{x}_{n-1}}{x_{n-1} + \hat{x}_{n-1}}\right)}, \\
x_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_n\hat{x}_n}{x_n + \hat{x}_n}\right)}.
\end{aligned}
\tag{7}
$$

From Taylor's expansion about α, we have

$$f(x) = f'(\alpha) \cdot e \cdot (1 + c_2 e + c_3 e^2 + c_4 e^3 + ...), \tag{8}$$

and

$$f'(x) = f'(\alpha) \cdot (1 + 2c_2 e + 3c_3 e^2 + 4c_4 e^3 + ...), \tag{9}$$

where $e = x - \alpha$ and $c_i = (1/i!)f^{(i)}(\alpha)/f'(\alpha)$, for $i = 2, 3, ...$ Let the errors e_n and $\hat{e}_n$ be defined as $x_n - \alpha$ and $\hat{x}_n - \alpha$, respectively. Thus, from Taylor's expansion of the harmonic mean of x_n and $\hat{x}_n$ about (α, α), it is easy to get the following approximation

$$\frac{2x_n \hat{x}_n}{x_n + \hat{x}_n} \approx \alpha + \frac{1}{2}e_n + \frac{1}{2}\hat{e}_n. \tag{10}$$

Therefore, repeated substitutions of (8), (9) and (10) into the consecutive steps of method (7) yield to the following approximations of errors:

$$\hat{e}_0 \sim e_0, \ \ \hat{e}_1 \sim 2c_2^2 e_0^3, \ \ \hat{e}_2 \sim 2c_2^6 e_0^7, \ \ \hat{e}_3 \sim 8c_2^{16} e_0^{17}, ...$$
$$e_1 \sim c_2 e_0^2, \ \ e_2 \sim 2c_2^4 e_0^5, \ \ e_3 \sim 4c_2^{11} e_0^{12}, \ \ e_4 \sim 32c_2^{28} e_0^{29}, ...$$

Hence, the sequence of powers of e_0 for errors e_i ($i \geq 1$) taken after full iterations gives values $2, 5, 12, 29, 70, 169, 408, ...$ which is the same as the result generated by McDougall-Wotherspoon method (6). Let P_i be the $i-$th term of that sequence, than for $i \geq 2$, P_i can be calculated by

$$P_i = 2P_{i-1} + P_{i-2}. \tag{11}$$

Dividing (11) by P_{i-1} and taking the limit, we get the equation $p = 2 + 1/p$, where $p = \lim_{j \to \infty}(P_j/P_{j-1})$, $j \in \mathbf{N}$. Its positive solution $p = 1 + \sqrt{2} \approx 2.4142$ represents the convergence order of the method (7).

4 Numerical comparison

From the above discussion it is clear that methods (6) and (7) have the same theoretical properties such as the convergence order and the efficiency index. For their numerical comparison, the following test examples taken from [1, 7] have been employed

$$f_1(x) = x^2 - e^x - 3x + 2, \ \ x_0 = 3, \ \ \alpha = 0.257302854...$$
$$f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \ \ x_0 = -2, \ \ \alpha = -1.207647827...$$
$$f_3(x) = e^{x^2 + 7x - 30} - 1, \ \ x_0 = 3.25, \ \ \alpha = 3$$
$$f_4(x) = \ln(x^2 + x + 2) - x + 1, \ \ x_0 = 3, \ \ \alpha = 4.152590736...$$

Tables 1-4 display the numerical results organized in the last four columns. The first column presents the methods used for calculations. Newton's method (1), the Weerakoon-Fernando

method (2), Özban's variant of method (2), the Frontini-Sormani method (3), the Kou-Li-Wang method (4), Wang's method (5) and the McDougall-Wotherspoon method (6) are denoted by "NM", "WFM", "OM", "FSM", "KLWM", "WM" and "McDWM", respectively, while the new iterative scheme (7) is denoted by "NewM".

Table 1 Numerical results for $f_1(x)$

method	it	nFe	$|f(x_i)|$	r_c
NM	8	16	$2.28 \cdot 10^{-25}$	2.0000
WFM	6	18	$2.80 \cdot 10^{-16}$	3.0000
OM	6	18	$1.33 \cdot 10^{-22}$	3.0000
FSM	6	18	$4.85 \cdot 10^{-25}$	3.0000
KLWM	6	18	$5.65 \cdot 10^{-13}$	3.0000
WM	5	15	$1.71 \cdot 10^{-33}$	3.0000
McDWM	7	14	$5.88 \cdot 10^{-50}$	2.4147
NewM	7	14	$8.97 \cdot 10^{-55}$	2.4148

Table 2 Numerical results for $f_2(x)$

method	it	nFe	$|f(x_i)|$	r_c
NM	11	22	$1.08 \cdot 10^{-4}$	2.0000
WFM	7	21	$1.76 \cdot 10^{-4}$	3.0000
OM	7	21	$5.99 \cdot 10^{-10}$	3.0000
FSM	7	21	$4.66 \cdot 10^{-7}$	3.0000
KLWM	7	21	$2.44 \cdot 10^{-10}$	3.0000
WM	7	21	$6.22 \cdot 10^{-6}$	3.0000
McDWM	9	18	$1.19 \cdot 10^{-10}$	2.4143
NewM	9	18	$8.83 \cdot 10^{-11}$	2.4143

Columns "it" and "nFe" show the number of iteration and the number of function evaluations, respectively, that each method requires to satisfy the stopping criterion $|f(x_n)| < 10^{-100}$. Column $|f(x_i)|$ shows the absolute values of the tested function on the same level of iterative process according to the number of function evaluations. Namely, that values are given after 12 function evaluations for each method, which means that $|f(x_i)|$ is given after the sixth iteration for "NM", "McDWM" and "NewM", and after the fourth iteration for other methods. The last column denoted by "r_c" displays the values that approximate theoretically determined convergence order of methods. The formula for r_c suggested in [4] is given by

$$r_c = \frac{\log|f(x_n)/f(x_{n-1})|}{\log|f(x_{n-1})/f(x_{n-2})|}.$$

Table 3 Numerical results for $f_3(x)$

method	it	nFe	$|f(x_i)|$	r_c
NM	11	22	$1.58 \cdot 10^{-4}$	2.0000
WFM	7	21	$1.86 \cdot 10^{-4}$	3.0000
OM	7	21	$1.83 \cdot 10^{-9}$	3.0000
FSM	7	21	$2.47 \cdot 10^{-6}$	3.0000
KLWM	7	21	$2.74 \cdot 10^{-7}$	3.0000
WM	7	21	$1.53 \cdot 10^{-5}$	3.0000
McDWM	9	18	$2.95 \cdot 10^{-9}$	2.4144
NewM	9	18	$2.85 \cdot 10^{-9}$	2.4144

Table 4 Numerical results for $f_4(x)$

method	it	nFe	$|f(x_i)|$	r_c
NM	7	14	$7.03 \cdot 10^{-68}$	2.0000
WFM	4	12	$1.22 \cdot 10^{-116}$	3.0000
OM	5	15	$3.66 \cdot 10^{-88}$	3.0000
FSM	5	15	$4.74 \cdot 10^{-80}$	3.0000
KLWM	5	15	$3.39 \cdot 10^{-53}$	3.0000
WM	5	15	$3.36 \cdot 10^{-86}$	3.0000
McDWM	6	12	$2.00 \cdot 10^{-169}$	2.4139
NewM	6	12	$2.73 \cdot 10^{-168}$	2.4129

Obviously, methods McMDW and NewM have similar numerical behavior. For both methods the coefficients r_c are very close to the theoretically obtained convergence order $1 + \sqrt{2}$. In comparison to the third order methods, McMDW and NewM require more iterations, but less function evaluations to satisfy stopping criterion. Therefore, for the same number function evaluations, those two methods have significantly better results in $|f(x_i)|$ columns.

According to the all presented results, one can conclude that the new variant of the McDougall-Wotherspoon method preserves good theoretic and numeric properties of the original method.

Acknowledgment

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