Some Generalizations of the Total Irregularity of Graphs

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Abstract: A novel concept is outlined by which the total irregularity $\text{irr}_t(G)$, introduced recently by Abdo and Dimitrov, can be extended. It is demonstrated on examples that starting with this concept several generalized versions of the total irregularity can be established.

Keywords: Irregularity measure; total irregularity; nonregular graph.

1 Introduction

We consider only simple connected graphs without loops and parallel edges. For a graph G with n vertices and m edges, $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. Let $d(u)$ be the degree of vertex u of G and denote by uv an edge of G connecting vertices u and v . Let $\Delta = \Delta(G)$ and $\delta = \delta(G)$ be the maximum and the minimum degrees, respectively, the graph G . In what follows, we use the standard terminology of graph theory, for notation not defined here we refer the reader to [1, 2].

For a connected graph G , denote by $\{n_i = n_i(G) : n_i > 0, 1 \leq i \leq \Delta(G)\}$ the set of the numbers n_i of vertices of G with degree i . For simplicity, the numbers $n_1, n_2, \dots, n_\Delta$ are called the vertex-parameters of G .

Two connected graphs G_1 and G_2 are said to be vertex-degree equivalent if they have identical vertex-degree sequence. A graph is called regular, if all its vertices have the same degree. A graph which is not regular is called a nonregular graph. A connected graph G is said to be *bidegreed* if its vertex-degree sequence has exactly two elements.

A connected bidegreed bipartite graph $G(\Delta, \delta)$ is called *semiregular* if no two vertices in the same part of bipartition have different degrees. A connected graph G is said to be *harmonic* (pseudo-regular) [3, 4], if there exists a positive integer $p(G)$ such that each vertex u in G has the same average neighbor-degree number identical with $p(G)$. The spectral radius $\rho(G)$ of a harmonic graph G is equal to $p(G)$. It is obvious that any connected r -regular graph G_r is a harmonic graph with $p(G_r) = \rho(G_r) = r$.

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2 Preliminary considerations

An irregularity measure (IM) of a connected graph G is a non-negative graph invariant satisfying the property: $IM(G) = 0$ if and only if G is regular. There exist several degree-based and eigenvalue-based graph irregularity measures [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The following irregularity measure, defined for a connected (n, m) -graph G , is called Collatz-Sinogowitz irregularity index [6]:

$$CS(G) = \lambda(G) - \frac{2m}{n}$$

where λ is the spectral radius of the graph G . Due to simple computations, the majority of irregularity measures are degree-based. In the mathematical chemistry literature, a widely used graph irregularity measure is the variance of degree $Var(G)$ introduced by Bell [7]

$$Var(G) = \frac{1}{n} \sum_{u \in V} (d(u))^2 - \frac{1}{n^2} \left(\sum_{u \in V} d(u) \right)^2 = \frac{M_1(G)}{n} - \left(\frac{2m}{n} \right)^2$$

where $M_1(G)$ is the first Zagreb index [24, 25, 26, 27, 28, 29] of a graph G . Using the first Zagreb index, several irregularity measures of new type can be constructed. For example, one of such simple irregularity measures is [23]:

$$IRM_1(G) = \sqrt{\frac{M_1(G)}{n}} - \frac{2m}{n} = Var(G) \left(\frac{M_1(G)}{n} + \frac{2m}{n} \right)^{-1}$$

Additionally, the irregularity measure $IRV(G)$, a modified version of $Var(G)$, was also considered in [23]:

$$IRV(G) = n^2 Var(G)$$

The irregularity measure $IRV(G)$ will play a central role in the following investigations. It is easy to see that if the graphs G_A and G_B have identical degree sequences then equalities $IRM_1(G_A) = IRM_1(G_B)$ and $IRV(G_A) = IRV(G_B)$ hold.

The following proposition represents the relation between the Collatz-Sinogowitz irregularity measure and the Bell's degree variance of an n -vertex connected graph G .

Proposition 2.1. [23] *Let G be a connected (n, m) -graph with a spectral radius λ . Then*

$$IRV(G) = n^2 Var(G) \leq (2mn + \lambda n^2) CS(G)$$

with equality holds if G is regular or semiregular.

3 The total irregularity

Some years ago, Abdo et. al. [9, 10] established a novel irregularity measure of graphs, defined by

$$irr_{1,t}(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d(u) - d(v)| \quad (1)$$

It was named as the “total irregularity”. Equation (1) can be rewritten in the following alternative form:

$$irr_{1,t}(G) = \sum_r \sum_{s < r} n_r n_s (r - s)$$

If $V(G) = \{u_1, u_2, \dots, u_n\}$ and $d(u_1) \leq d(u_2) \leq \dots \leq d(u_n)$, then $irr_{1,t}(G)$ can be reformulated [10] as

$$irr_{1,t}(G) = \sum_{j>i} (d(u_j) - d(u_i)).$$

From the above formulas it follows that the total irregularity of a graph depends only on its degree sequence. This means that graphs possessing identical degree sequences have the same total irregularity. In general, the converse statement does not hold. There exist n -vertex graphs with equal total irregularities, but different degree sequences [11].

Proposition 3.1. [9, 10] *For an n -vertex connected graph G_n with a maximal total irregularity, it holds that*

$$irr_{1,t}(G_n) = \left\{ \begin{array}{ll} \frac{1}{12} (2n^3 - 3n^2 - 2n + 3) & \text{if } n \text{ is odd,} \\ \frac{1}{12} (2n^3 - 3n^2 - 2n) & \text{if } n \text{ is even.} \end{array} \right\}$$

4 Generalized versions of the total irregularity

The extended version of the total irregularity is defined in the following general form

$$IR(G; p, q, w) = \frac{1}{2} \sum_{u,v \in V(G)} w(u, v) |(d(u))^p - (d(v))^p|^q \quad (2)$$

where p, q are positive real numbers and $w(u, v)$ is an appropriately selected non-negative weight function, for which $w(u, v) = w(v, u)$ holds. It is easy to see that if $p = 1, q = 1$ and $w(u, v) = 1$, the total irregularity $irr_{1,t}(G)$ yields as a particular case. In what follows, we study some extended versions of the total irregularity.

If $p = 1, q = 2$ and $w(u, v) = 1$, we obtain the irregularity measure $irr_{2,t}(G)$ defined by

$$irr_{2,t}(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d(u) - d(v))^2$$

As can be seen, $irr_{2,t}(G)$ can be considered as a natural extension of the total irregularity $irr_{1,t}(G)$.

Lemma 4.1. *From the previous considerations it follows that*

$$irr_{2,t}(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d(u) - d(v))^2 = \frac{1}{2} \sum_{u,v \in V(G)} (d^2(u) + d^2(v)) - \sum_{u,v \in V(G)} d(u)d(v).$$

Lemma 4.2. [30] *Let G be a connected graph with m edges. Then*

$$\sum_{u \in V(G)} \sum_{v \in V(G)} d(u)d(v) = 4m^2.$$

Lemma 4.3. *If G is a connected (n, m) -graph, then*

$$\sum_{u,v \in V(G)} (d^p(u) + d^p(v)) = 2n \sum_{u \in V(G)} d^p(u).$$

Proof. Let $d = (d_1, d_2, \dots, d_j, \dots, d_n)$ be the vertex-degree sequence of G . Denote by $\mathbf{S}_p$ the $n \times n$ matrix whose elements s_{ij} are defined by $s_{i,j} = d_i^p + d_j^p$ for $1 \leq i, j \leq n$. Let $S(i)$ be the sum of elements $s_{i,j}$ in the i th row of $\mathbf{S}_p$. Then

$$S(i) = \sum_j s_{i,j}$$

for $i = 1, 2, \dots, n$. Consequently, we have $S(i) = \sum_j (d_j^p + nd_i^p)$ and

$$\sum_{u,v \in V(G)} (d^p(u) + d^p(v)) = \sum_{i=1}^n S(i) = \sum_{i=1}^n \left(\sum_{j=1}^n d_j^p + nd_i^p \right) = 2n \sum_{i=1}^n d_i^p. \quad (3)$$

□

It should be noted that in the formula given in Lemma 4.3, $\sum_{i=1}^n d_i^p$ is a special case of the general zeroth-order Randić index [18].

From Equation (3), next result follows.

Corollary 4.4. *For an (n, m) -graph G , it holds that*

$$\sum_{u,v \in V(G)} (d(u) + d(v)) = 4nm,$$

and

$$\sum_{u,v \in V(G)} (d^2(u) + d^2(v)) = 2nM_1(G).$$

Proposition 4.5. *Let G be a connected graph with n vertices and m edges. Then*

$$\text{irr}_{2,t}(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d(u) - d(v))^2 = nM_1(G) - 4m^2 = n^2 \text{Var}(G)$$

Proof. From Lemmas 4.1, 4.2 and 4.3, one obtains

$$\text{irr}_{2,t}(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d^2(u) + d^2(v)) - \sum_{u,v \in V(G)} d(u)d(v) = nM_1(G) - 4m^2.$$

From this the result follows. □

Lemma 4.6. *Let a, b, p be positive real numbers. Then*

$$\frac{2(ab)^p}{a^{2p} + b^{2p}} \leq 1$$

with equality if and only if $a = b$.

Proof. The result follows from the inequality between geometric and arithmetic means. □

Let $q = 2$ and $w(u, v) = 1 / (d(u^{2p}) + d^{2p}(v))$, and consider the irregularity measure $IR_p(G)$ defined by

$$IR_p(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(d^p(u) - d^p(v))^2}{d^{2p}(u) + d^{2p}(v)}.$$

Proposition 4.7. *Let G be an n -vertex connected graph. Then*

$$IR_p(G) = \frac{1}{2} \left\{ n^2 - \sum_{u,v \in V(G)} \frac{2(d(u)d(v))^p}{d^{2p}(u) + d^{2p}(v)} \right\} \geq 0$$

with equality if and only if G is a regular graph.

Proof. Graph invariant $IR_p(G)$ can be rewritten in the following form

$$2IR_p(G) = \sum_{u,v \in V(G)} \frac{(d^p(u) - d^p(v))^2}{d^{2p}(u) + d^{2p}(v)} = \sum_{u,v \in V(G)} \frac{d^{2p}(u) + d^{2p}(v)}{d^{2p}(u) + d^{2p}(v)} - \sum_{u,v \in V(G)} \frac{2(d(u)d(v))^p}{d^{2p}(u) + d^{2p}(v)}.$$

Consequently, from Lemma 4.6 it follows that

$$IR_p(G) = \frac{1}{2} \left\{ n^2 - \sum_{u,v \in V(G)} \frac{2(d(u)d(v))^p}{d^{2p}(u) + d^{2p}(v)} \right\} \geq 0.$$

□

Corollary 4.8. *If G is an n -vertex connected graph, then*

$$IR_{\frac{1}{2}}(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(\sqrt{d(u)} - \sqrt{d(v)})^2}{d(u) + d(v)} = \frac{1}{2} \left\{ n^2 - \sum_{u,v \in V(G)} \frac{2\sqrt{d(u)d(v)}}{d(u) + d(v)} \right\} \geq 0,$$

and

$$IR_1(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(d(u) - d(v))^2}{d^2(u) + d^2(v)} = \frac{1}{2} \left\{ n^2 - \sum_{u,v \in V(G)} \frac{2d(u)d(v)}{d^2(u) + d^2(v)} \right\} \geq 0.$$

In the both cases, equality holds if and only if G is a regular graph.

5 Additional remarks

The general concept, discussed in the previous section, can be applied to generate a set of novel graph invariants.

1. Denote by $\mu(u)$ the average of degrees of vertices adjacent to an arbitrary vertex u of G . In this case, we obtain the following graph invariant

$$TI_{\mu,p}(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(\mu^p(u) - \mu^p(v))^2}{\mu^{2p}(u) + \mu^{2p}(v)}.$$

Consequently, we have

$$TI_{\mu,p}(G) = \frac{1}{2} \left\{ n^2 - \sum_{u,v \in V(G)} \frac{2(\mu(u)\mu(v))^p}{\mu^{2p}(u) + \mu^{2p}(v)} \right\} \geq 0.$$

In the above formula equality holds if G is a regular or harmonic (pseudo-regular) graph.

2. A connected graph G is an eccentricity-based-regular graph if and only all the vertices of G have the same eccentricity. Clearly, the radius and diameter of an eccentricity-based-regular graph G is equal to $\varepsilon(u)$, that is, the eccentricity of a vertex $u \in V(G)$. We consider the following graph invariant

$$TI_{\varepsilon,p}(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(\varepsilon^p(u) - \varepsilon^p(v))^2}{\varepsilon^{2p}(u) + \varepsilon^{2p}(v)}.$$

It follows that

$$TI_{\varepsilon,p}(G) = \frac{1}{2} \left\{ n^2 - \sum_{u,v \in V(G)} \frac{2(\varepsilon(u)\varepsilon(v))^p}{\varepsilon^{2p}(u) + \varepsilon^{2p}(v)} \right\} \geq 0,$$

where equality sign holds if and only if G is an eccentricity-based-regular graph.

3. Starting with the general formula represented in Equation (2), let $p = 1/2$, $q = 2$ and $w(u, v) = d(u, v)$ where $d(u, v)$ is the distance between vertices u and v of G . In this particular case, we obtain the irregularity measure $TI_d(G)$ defined by

$$TI_d(G) = \frac{1}{2} \sum_{u, v \in V(G)} \left(\sqrt{d(u)} - \sqrt{d(v)} \right)^2 d(u, v) \geq 0.$$

It follows that

$$TI_d(G) = \frac{1}{2} \sum_{u, v \in V(G)} (d(u) + d(v)) d(u, v) - \sum_{u, v \in V(G)} \sqrt{d(u)d(v)} d(u, v) \geq 0,$$

where equality holds if and only if G is a regular graph. In the above formula

$$DD(G) = \frac{1}{2} \sum_{u, v \in V(G)} (d(u) + d(v)) d(u, v) = \sum_{u \in V(G)} d(u) D_G(u)$$

is the *degree distance* of a graph G [31, 32]. As it is known

$$D_G(u) = \sum_{v \in V(G)} d(u, v)$$

is the transmission of a vertex u of G . It is worth noting that if the diameter of a connected graph G with n vertices and m edges is equal to 2, then [32]

$$DD(G) = 2m(2n - 2) - M_1(G).$$

4. The summations in all the novel graph invariants considered in this paper till now, are taken over all the possible vertices pairs of the graph under consideration. Taking these summations over all the edges of the considered graph, yield a new set of invariants. For example, by applying this idea to Eq. (2), we have

$$IR^*(G; p, q, w) = \sum_{uv \in E(G)} w(u, v) |(d(u))^p - (d(v))^p|^q \quad (4)$$

where p, q are positive real numbers and $w(u, v)$ is an appropriately selected non-negative weight function, for which $w(u, v) = w(v, u)$ holds. It is easy to see that if $p = 1$, $q = 2$ and $w(u, v) = 1$, the sigma index [22] yields as a particular case. Also, we would like to remark that the substitutions $p = 1/2$, $q = 2$ and $w(u, v) = \frac{1}{d(u) + d(v)}$ in Eq. (4) gives

$$IR^* \left(G; \frac{1}{2}, 2, \frac{1}{d(u) + d(v)} \right) = \sum_{uv \in E(G)} \frac{\left(\sqrt{d(u)} - \sqrt{d(v)} \right)^2}{d(u) + d(v)} = m - GA(G),$$

where m is the size of the graph G and $GA(G)$ is the geometric-arithmetic index [33] of G .

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