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Some properties of meromorphic solutions of higher order linear difference equations

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Abstract: In this paper, we investigate the growth of solutions of the linear difference equations

$$A_k(z)f(z+c_k)+A_{k-1}(z)f(z+c_{k-1})+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=0,$$

$$A_k(z)f(z+c_k)+A_{k-1}(z)f(z+c_{k-1})+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=F(z),$$

where $A_k(z),\cdots,A_0(z)$, $F(z)(\not\equiv 0)$ are entire functions and $c_k,\cdots,c_1$ are distinct non-zero complex numbers. We extend some precedent results due to Liu and Mao [15].

Keywords: Complex linear difference equation, meromorphic solution, iterated $p-$order, iterated $p-$type.

1 Introduction and main results

In this paper, we use the standard notations of Nevanlinna's value distribution theory (see [7], [11], [17]). Recently, study of properties of meromorphic solutions of complex difference equations have become a subject of great interest from the viewpoint of Nevanlinna theory, due to the apparent role of the existence of such solutions of finite order for the integrability of discrete difference equations (see, e.g., [1,4,5,13,14,15,16,18,19]). The key result here is the difference analogue of the lemma on the logarithmic derivative obtained by Halburd-Korhonen [9,10] and Chiang-Feng [6], independently.
In the rest of the paper, the linear measure of a set $E\subset(0,+\infty)$ is defined as

$$m(E)=\int_0^{+\infty}\chi_E(t)dt,$$

and the logarithmic measure of a set $F\subset(1,+\infty)$ is defined by

$$lm(F)=\int_1^{+\infty}\frac{\chi_F(t)}{t}dt,$$

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where $\chi_H(t)$ is the characteristic function of a set H. Moreover, the upper and the lower densities of a set $E \subset (0,+\infty)$ are defined respectively by

$$\overline{dens}E = \limsup_{r\to+\infty} \frac{m(E\cap[0,r])}{r},$$

$$\underline{dens}E = \liminf_{r\to+\infty} \frac{m(E\cap[0,r])}{r},$$

and the upper and the lower logarithmic densities of a set $F \subset (1,+\infty)$ are defined respectively by

$$\overline{\log dens}F = \limsup_{r\to+\infty} \frac{lm(F\cap[1,r])}{\log r},$$

$$\underline{\log dens}F = \liminf_{r\to+\infty} \frac{lm(F\cap[1,r])}{\log r}.$$

Proposition 1.1 [2] *For all $H \subset [1,+\infty)$ the following statements hold :*
(i) *If $lm(H) = \infty$, then $m(H) = \infty$;*
(ii) *If $\overline{dens}H > 0$, then $m(H) = \infty$;*
(iii) *If $\overline{\log dens}H > 0$, then $lm(H) = \infty$.*

In the following, we recall some fundamental definitions which are used later.

For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$, see [12].

Definition 1.1 [12] Let $p \geq 1$ be an integer. Then, the iterated p-order $\rho_p(f)$ of a meromorphic function f is defined by

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r},$$

where $T(r,f)$ is the characteristic function of Nevanlinna (see, [7,11,16]). For $p = 1$, this notation is called order and hyper-order when $p = 2$. The iterated p-order $\rho_p(f)$ of an entire function f is defined by

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Definition 1.2 [3] Let f be a meromorphic function of iterated p-order $(0 < \rho_p(f) < \infty)$, the iterated $p-$type $\tau_p(f)$ of f is defined by

$$\tau_p(f) = \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{r^{\rho_p(f)}} \ (p \geq 1 \text{ an integer}).$$

In recent paper [6], Chiang and Feng investigated meromorphic solutions of the linear difference equation

$$A_k(z)f(z+k)+A_{k-1}(z)f(z+k-1)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=0, \qquad (1.1)$$

where $A_k(z),\cdots;A_0(z)$ are entire functions and proved the following result.

Theorem A [6] *Let $A_0(z),\cdots,A_k(z)$ be polynomials. If there exists an integer $l\,(0 \leq l \leq k)$ such that*

$$\deg(A_l) > \max_{0\leq l\leq k, j\neq l}\{\deg(A_j)\}$$

holds, then every meromorphic solution $f\,(\not\equiv 0)$ of equation (1.1) satisfies $\rho(f) \geq 1$, where deg(A_l) denotes the degree of the polynomial A_l.

Theorem B [6] *Let $A_0(z),\cdots,A_k(z)$ be entire functions. If there exists an integer $l\,(0 \leq l \leq k)$ such that*

$$\rho(A_l) > \max_{0\leq l\leq k, j\neq l}\{\rho(A_j)\},$$

holds, then every meromorphic solution $f\,(\not\equiv 0)$ of equation (1.1) satisfies $\rho(f) \geq \rho(A_l)+1$.

Note that in Theorems A and B, equation (1.1) has only one dominating coefficient A_l. For the case when there is no dominating coefficient and all coefficients are polynomials in equation (1.1), Chen [4] obtained an improvement of Theorem A.

Theorem C [4] *Let $A_0(z),\cdots,A_k(z)$ be polynomials such that*

$$\deg(A_0+\cdots+A_k) = \max_{0\leq j\leq k}\{\deg(A_j)\} \geq 1.$$

Then every finite order meromorphic solution $f(\not\equiv 0)$ of equation (1.1) satisfies $\rho(f) \geq 1$.

For the case when there is more than one of coefficients which have the maximal order, Laine and Yang [13] obtained the following result.

Theorem D [13] *Let $A_0(z),\cdots,A_k(z)$ be entire functions of finite order such that among those having the maximal order*

$$\rho = \max_{0\le j\le k}\{\rho(A_j)\},$$

exactly one has its type strictly greater than the others. Then for every meromorphic solution $f\,(\not\equiv 0)$ of equation

$$A_k(z)f(z+c_k)+A_{k-1}(z)f(z+c_{k-1})+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=0, \tag{1.2}$$

where $c_k,\cdots,c_1$ are non-zero distinct complex numbers, we have

$$\rho(f)\ge\rho+1.$$

In the present paper, we continue to study the growth of solutions of some linear difference equations, we improve and extend Theorem A, Theorem B, Theorem C and Theorem D by using the concept of the iterated $p-$order for equation (1.2). We obtain the following results.

Theorem 1.1 *Let H be a complex set satisfying $\overline{\log dens}\{r=|z|:z\in H\}>0$, and let $A_0(z),\cdots,A_k(z)$ be entire functions of iterated $p-$order satisfying $\max_{0\le j\le k}\{\rho_p(A_j)\}\le\rho$. If there exists an integer $l\,(0\le l\le k)$ such that for some constants $0\le\beta<\alpha$ and δ $(0<\delta<\rho)$ sufficiently small, we have*

$$|A_l(z)|\ge \exp_p\{\alpha r^{\rho-\delta}\}, \tag{1.3}$$

$$|A_j(z)|\le \exp_p\{\beta r^{\rho-\delta}\},\quad j=0,\cdots,k,\ j\ne l, \tag{1.4}$$

as $z\to\infty$ for $z\in H$, then every meromorphic solution $f\,(\not\equiv 0)$ of equation (1.2) satisfies

$$\begin{cases} \rho(f)\ge\rho(A_l)+1, & \text{for } p=1,\\ \rho_p(f)\ge\rho_p(A_l), & \text{for } p\ge 2.\end{cases}$$

Remark 1.1 The Theorem 1.1 was obtained by Liu and Mao [15] when $p=1$ and for equation (1.1) with H is a complex set satisfying $\overline{dens}\{r=|z|:z\in H\}>0$.

Theorem 1.2 *Let H be a complex set satisfying $\overline{\log dens}\{r=|z|:z\in H\}>0$, and let $A_0(z),\cdots,A_k(z)$ be entire functions satisfying $\max_{0\le j\le k}\{\rho_p(A_j)\}\le\rho$. If there exists an integer $l\,(0\le l\le k)$ such that for some constants $0\le\beta<\alpha$ and δ $(0<\delta<\rho)$ sufficiently small, we have*

$$T(r,A_l)\ge \exp_{p-1}\{\alpha r^{\rho-\delta}\}, \tag{1.5}$$

$$T(r,A_j)\le \exp_{p-1}\{\beta r^{\rho-\delta}\},\quad (j=0,\cdots,k,\ j\ne l), \tag{1.6}$$

as $z \to \infty$ for $z \in H$. Then the following statements hold:
(i) *If $p = 1$ and $0 \le k\beta < \alpha$, then every meromorphic solution $f \not\equiv 0$ of equation* (1.2) *satisfies $\rho(f) \ge \rho(A_l) + 1$.*
(ii) *If $p \ge 2$ and $0 \le \beta < \alpha$, then every meromorphic solution $f \not\equiv 0$ of equation* (1.2) *satisfies $\rho_p(f) \ge \rho_p(A_l)$.*

In the following theorem, we will add a condition on the type. When there exists more than one coefficient having the order ∞ in equation (1.2), we obtain the following result. Note that in this case Theorem D is invalid for $p = 1$.

Theorem 1.3 *Let $A_0(z), \cdots, A_k(z)$ be entire functions, and let $p \ge 1$ be an integer. If there exists an integer $l\,(0 \le l \le k)$ such that*

$$\max\{\rho_{p+1}(A_j) : j = 0, \cdots, k,\ j \ne l\} \le \rho_{p+1}(A_l),\ (0 < \rho_{p+1}(A_l) < \infty),$$

$$\max\left\{\tau_{p+1}(A_j) : \rho_{p+1}(A_j) = \rho_{p+1}(A_l)\right\} < \tau_{p+1}(A_l),\ (0 < \tau_{p+1}(A_l) < \infty),$$

then every meromorphic solution $f\,(\not\equiv 0)$ of equation (1.2) *satisfies*

$$\rho_p(f) = \infty \quad \textit{and} \quad \rho_{p+1}(f) \ge \rho_{p+1}(A_l).$$

Next we consider the properties of meromorphic solutions of the non-homogeneous linear difference equation corresponding to (1.2)

$$A_k(z)f(z+c_k) + \cdots + A_1(z)f(z+c_1) + A_0(z)f(z) = F(z), \tag{1.7}$$

where where $A_k(z), \cdots, A_0(z), F(z)(\not\equiv 0)$ are entire functions and $c_k, \cdots, c_1$ are distinct non-zero complex numbers.

Theorem 1.4 *Let $A_j(z)\,(j = 0, \cdots, k)$ satisfy the hypothesis of Theorem 1.3, and let $F(z)$ be an entire function. Then*
(i) *If $\rho_{p+1}(F) < \rho_{p+1}(A_l)$ or $\rho_{p+1}(F) = \rho_{p+1}(A_l)$, $\tau_{p+1}(F) < \tau_{p+1}(A_l)$, then every meromorphic solution $f\,(\not\equiv 0)$ of equation* (1.7) *satisfies*

$$\rho_p(f) = \infty \quad \textit{and} \quad \rho_{p+1}(f) \ge \rho_{p+1}(A_l).$$

(ii) *If $\rho_{p+1}(F) > \rho_{p+1}(A_l)$, then every meromorphic solution $f\,(\not\equiv 0)$ of equation* (1.7) *satisfies*

$$\rho_p(f) = \infty \quad \textit{and} \quad \rho_{p+1}(f) \ge \rho_{p+1}(F).$$

Remark 1.2 The Theorems 1.3 and 1.4 were obtained by Liu and Mao [15] when $p = 1$ and for equation (1.1). For some related results when $A_j(z)\,(j = 0, \cdots, k)$, $F(z)$ are meromorphic functions, see [19].

2 Preliminary Lemmas

Our proofs depend mainly upon the following lemmas.

Lemma 2.1 [6] *Let f be a meromorphic function, η a non-zero complex number, and let $\gamma > 1$, and $\varepsilon > 0$ be given real constants. Then there exists a subset $E_1 \subset (1, +\infty)$ of finite logarithmic measure, and a constant A depending only on γ and η, such that for all $|z| = r \notin E_1 \cup [0, 1]$, we have*

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\right),$$

where $n(t) = n(t, \infty, f) + n(t, \infty, 1/f)$.

Lemma 2.2 [8] *Let f be a transcendental meromorphic function, let j be non-negative integer, let x be a value in the extended complex plane, and let $\mu > 1$ be a real constant. Then there exists a constant $R > 0$ such that for all $r > R$, we have*

$$n\left(r, x, f^{(j)}\right) \leq \frac{2j+6}{\log \mu} T(\mu r, f). \tag{2.1}$$

Lemma 2.3 *Let f be a meromorphic function, η a non-zero complex number, and $\varepsilon > 0$ be given real constants. Then there exists a subset $E_2 \subset (1, +\infty)$ of finite logarithmic measure, such that if f has finite iterated $p-$order $\rho_p(f) = \rho$, then for all $|z| = r \notin [0, 1] \cup E_2$, we have*

(i) *If $p = 1$, then*

$$\exp\{-r^{\rho-1+\varepsilon}\} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\{r^{\rho-1+\varepsilon}\}. \tag{2.2}$$

(ii) *If $p \geq 2$, then*

$$\exp_p\{-r^{\rho+\varepsilon}\} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp_p\{r^{\rho+\varepsilon}\}. \tag{2.3}$$

Proof. We prove only (ii). For the proof of (i) see [6]. Let $p \geq 2$. By Lemma 2.1, there exist a subset $E_2 \subset (1, +\infty)$ of finite logarithmic measure, and a constant A depending only on γ and η, such that for all $|z| = r \notin E_2 \cup [0, 1]$, we have

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\right), \tag{2.4}$$

where $n(t) = n(t, \infty, f) + n(t, \infty, 1/f)$. By using (2.1) and (2.4), we obtain

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r, f)}{r}\right.$$

$$+\frac{12}{\log\mu}\frac{T\left(\mu\gamma r,f\right)}{r}\log^{\gamma}r\log^{+}\left(\frac{12}{\log\mu}T\left(\mu\gamma r,f\right)\right)\Bigg)$$

$$\leq B\left(T\left(\lambda r,f\right)\frac{\log^{\lambda}r}{r}\log T\left(\lambda r,f\right)\right), \tag{2.5}$$

where $B>0$ is some constant and $\lambda=\mu\gamma>1$. Since f has finite iterated $p-$order $\rho_p(f)=\rho$, so given ε, $0<\varepsilon<2$, we have for sufficiently large r

$$T(r,f)\leq \exp_{p-1}\{r^{\rho+\frac{\varepsilon}{2}}\}. \tag{2.6}$$

Then by using (2.5) and (2.6), we obtain

$$\left|\log\left|\frac{f\left(z+\eta\right)}{f\left(z\right)}\right|\right|\leq B\left(T\left(\lambda r,f\right)\frac{\log^{\lambda}r}{r}\log T\left(\lambda r,f\right)\right)$$

$$\leq B\exp_{p-1}\{(\lambda r)^{\rho+\frac{\varepsilon}{2}}\}\frac{\log^{\lambda}r}{r}\log\exp_{p-1}\{(\lambda r)^{\rho+\frac{\varepsilon}{2}}\}$$

$$=B\exp_{p-1}\{(\lambda r)^{\rho+\frac{\varepsilon}{2}}\}\frac{\log^{\lambda}r}{r}\exp_{p-2}\{(\lambda r)^{\rho+\frac{\varepsilon}{2}}\}\leq \exp_{p-1}\{r^{\rho+\varepsilon}\}. \tag{2.7}$$

From (2.7) we easily obtain (2.3).

Lemma 2.4 *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1\neq\eta_2$, and let f be a meromorphic function of finite iterated $p-$order $\rho_p(f)=\rho$. Let $\varepsilon>0$ be given, then there exists a subset $E_3\subset(0,+\infty)$ with finite logarithmic measure such that for all $|z|=r\notin E_3\cup[0,+\infty]$, we have*
(i) *If $p=1$, then*

$$\exp\{-r^{\rho-1+\varepsilon}\}\leq\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right|\leq\exp\{r^{\rho-1+\varepsilon}\}.$$

(ii) *If $p\geq 2$, then*

$$\exp_p\{-r^{\rho+\varepsilon}\}\leq\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right|\leq\exp_p\{r^{\rho+\varepsilon}\}.$$

Proof. We prove only (ii). For the proof of (i) see [6]. We can write

$$\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right|=\left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right|,\quad(\eta_1\neq\eta_2).$$

Then by using Lemma 2.3, we obtain for any given $\varepsilon>0$ and all $|z+\eta_2|=R\notin[0,1]\cup E_2$, such that $lm(E_2)<\infty$

$$\exp_p\{-r^{\rho+\varepsilon}\}\leq\exp\left\{-\left(|z|+|\eta_2|\right)^{\rho+\frac{\varepsilon}{2}}\right\}\leq\exp_p\{-R^{\rho+\frac{\varepsilon}{2}}\}$$

$$\leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| = \left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right|$$

$$\leq \exp_p\{R^{\rho+\frac{\varepsilon}{2}}\} \leq \exp_p\{(|z|+|\eta_2|)^{\rho+\frac{\varepsilon}{2}}\} \leq \exp_p\{r^{\rho+\varepsilon}\},$$

where $|z| = r \notin [0,1] \cup E_3$ and E_3 is a set of finite logarithmic measure.

Lemma 2.5 [9] *Let f be a non-constant meromorphic function, $c \in \mathbb{C}$, $\delta < 1$ and $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|,f)^{1+\varepsilon}}{r^{\delta}}\right),$$

for all r outside of a possible exceptional set E_4 with finite logarithmic measure $\int_{E_4} \frac{dr}{r} < \infty$.

Remark 2.1 [7] *Let f be a meromorphic function, c be a non-zero complex constant. Then we have that for $r \longrightarrow +\infty$*

$$(1+o(1))T(r-|c|, f(z)) \leq T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f(z)).$$

Consecontly for $p \in \mathbb{N}_+ = \{1,2,\cdots\}$, $\rho_p(f(z+h)) = \rho_p(f)$.

Lemma 2.5 and Remark 2.1 lead to the following lemma.

Lemma 2.6 [9] *Let f be a non-constant meromorphic function, c, $h \in \mathbb{C}$, $c \neq h$, $\delta < 1$, $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z+c)}{f(z+h)}\right) = o\left(\frac{(T(r+|c-h|+|h|,f))^{1+\varepsilon}}{r^{\delta}}\right),$$

holds for all r outside of a possible exceptional set E_5 with finite logarithmic measure $\int_{E_5} \frac{dr}{r} < \infty$.

Lemma 2.7 [3] *Let f be a meromorphic function with iterated $p-$order $0 < \rho_p(f) < \infty$ and iterated $p-$type $0 < \tau_p(f) < \infty$. Then for any given $\beta < \tau_p(f)$, there exists a subset $E_6 \subset [1,+\infty)$ of infinite logarithmic measure such that*

$$\log_{p-1} T(r,f) > \beta r^{\rho_p(f)},$$

holds for all $r \in E_6$.

Lemma 2.8 [6] *Let μ, R, R' be real numbers such that $0 < \mu < 1, R > 0$, and let η be a non-zero complex number. Then there is a positive constant C_μ depending only on μ such* *that for a given meromorphic function f we have, when $|z|=r$, $\max\{1,r+|\eta|\}<R<R'$, the estimate*

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta)}\right)\leq\frac{2|\eta|R}{(R-r-|\eta|)^2}\left(m(R,f)+m\left(R,\frac{1}{f}\right)\right)$$

$$+\frac{2R'}{(R'-R)}\left(\frac{|\eta|}{R-r-|\eta|}+\frac{C_\mu|\eta|^\mu}{(1-\mu)r^\mu}\right)\left(N(R',f)+N\left(R',\frac{1}{f}\right)\right).$$

Lemma 2.9 *Let η_1, η_2 be two complex numbers such that $\eta_1\neq\eta_2$ and let f be a finite iterated $p-$order meromorphic function. Let $\rho_p(f)=\rho$ be the iterated $p-$order of f. Then for each $\varepsilon>0$, we have*
(i) *If $p=1$, then*

$$m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right)=O\left(r^{\rho-1+\varepsilon}\right).$$

(ii) *If $p\geq 2$, then*

$$m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right)=O\left(\exp_{p-1}\{r^{\rho+\varepsilon}\}\right).$$

Proof. We prove only (ii). For the proof of (i) see [6]. Let $p\geq 2$. We have

$$m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right)\leq m\left(r,\frac{f(z+\eta_1)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta_2)}\right)$$

$$\leq m\left(r,\frac{f(z+\eta_1)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta_1)}\right)$$

$$+m\left(r,\frac{f(z)}{f(z+\eta_2)}\right)+m\left(r,\frac{f(z+\eta_2)}{f(z)}\right). \tag{2.8}$$

Since f has finite iterated $p-$order $\rho_p(f)=\rho<+\infty$, so given ε, $0<\varepsilon<2$, we have

$$T(r,f)\leq\exp_{p-1}\{r^{\rho+\frac{\varepsilon}{2}}\} \tag{2.9}$$

for all r. By using Lemma 2.8, we obtain from equation (2.8)

$$m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right)\leq\frac{2|\eta_1|R}{(R-r-|\eta_1|)^2}\left(m(R,f)+m\left(R,\frac{1}{f}\right)\right)$$

$$+\frac{2R'}{(R'-R)}\left(\frac{|\eta_1|}{R-r-|\eta_1|}+\frac{C_\mu|\eta_1|^\mu}{(1-\mu)r^\mu}\right)\left(N(R',f)+N\left(R',\frac{1}{f}\right)\right)$$

$$+\frac{2|\eta_2|R}{(R-r-|\eta_2|)^2}\left(m(R,f)+m\left(R,\frac{1}{f}\right)\right)$$

$$+\frac{2R'}{(R'-R)}\left(\frac{|\eta_2|}{R-r-|\eta_2|}+\frac{C_\mu\,|\eta_2|^\mu}{(1-\mu)\,r^\mu}\right)\left(N\left(R',f\right)+N\left(R',\frac{1}{f}\right)\right)$$

$$=\left(\frac{2|\eta_1|R}{(R-r-|\eta_1|)^2}+\frac{2|\eta_2|R}{(R-r-|\eta_2|)^2}\right)\left(m\left(R,f\right)+m\left(R,\frac{1}{f}\right)\right)$$

$$+\frac{2R'}{(R'-R)}\left(\frac{|\eta_1|}{R-r-|\eta_1|}+\frac{C_\mu\,|\eta_1|^\mu}{(1-\mu)\,r^\mu}\right.$$

$$\left.+\frac{|\eta_2|}{R-r-|\eta_2|}+\frac{C_\mu\,|\eta_2|^\mu}{(1-\mu)\,r^\mu}\right)\left(N\left(R',f\right)+N\left(R',\frac{1}{f}\right)\right). \tag{2.10}$$

By choosing $\mu=1-\dfrac{\varepsilon}{2}$, $R=2r$, $R'=3r$ and $r>\max\{|\eta_1|,|\eta_2|,1/2\}$ in (2.10), we get

$$m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right)\le\left(\frac{4|\eta_1|r}{(r-|\eta_1|)^2}+\frac{4|\eta_2|r}{(r-|\eta_2|)^2}\right)\left(m\left(2r,f\right)+m\left(2r,\frac{1}{f}\right)\right)$$

$$+6\left(\frac{|\eta_1|}{r-|\eta_1|}+\frac{2C_\mu\,|\eta_1|^{1-\frac{\varepsilon}{2}}}{\varepsilon r^{1-\frac{\varepsilon}{2}}}+\frac{|\eta_2|}{r-|\eta_2|}+\frac{2C_\mu\,|\eta_2|^{1-\frac{\varepsilon}{2}}}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\right)\left(N\left(3r,f\right)+N\left(3r,\frac{1}{f}\right)\right)$$

$$\le 4\left[\frac{4|\eta_1|r}{(r-|\eta_1|)^2}+\frac{4|\eta_2|r}{(r-|\eta_2|)^2}\right.$$

$$\left.+\,6\left(\frac{|\eta_1|}{r-|\eta_1|}+\frac{|\eta_2|}{r-|\eta_2|}+\frac{2C_\mu\left(|\eta_1|^{1-\frac{\varepsilon}{2}}+|\eta_2|^{1-\frac{\varepsilon}{2}}\right)}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\right)\right]T\left(3r,f\right).$$

From this, by using the estimate (2.9), we have

$$m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right)\le 4K\left[\frac{4|\eta_1|r}{(r-|\eta_1|)^2}+\frac{4|\eta_2|r}{(r-|\eta_2|)^2}\right.$$

$$\left.+\,6\left(\frac{|\eta_1|}{r-|\eta_1|}+\frac{|\eta_2|}{r-|\eta_2|}+\frac{2C_\mu\left(|\eta_1|^{1-\frac{\varepsilon}{2}}+|\eta_2|^{1-\frac{\varepsilon}{2}}\right)}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\right)\right]\exp_{p-1}\{(3r)^{\rho+\frac{\varepsilon}{2}}\}$$

$$\le M\exp_{p-1}\{r^{\rho+\varepsilon}\},$$

where $K>0$, $M>0$ are some constants. This completes the proof.

Lemma 2.10 Under the assumptions of Theorem 1.1 or Theorem 1.2, we have $\rho_p(A_l)=\rho$.

Proof. By Theorem 1.1, we have $\rho_p(A_l)\le\rho$. Suppose that $\rho_p(A_l)=\mu<\rho$. Then, for any given $\varepsilon>0$ and sufficiently large r, we have

$$|A_l(z)|\le\exp_p\{r^{\mu+\varepsilon}\}. \tag{2.11}$$

On the other hand, by the hypotheses of Theorems 1.1, there exist positive constants $0 \leq \beta < \alpha$ and δ $(0 < \delta < \rho)$ sufficiently small, such that

$$|A_l(z)| \geq \exp_p\{\alpha r^{\rho-\delta}\} \tag{2.12}$$

as $z \to \infty$ for $z \in H$. From (2.11) and (2.12), we obtain for $z \in H$, $|z| = r \to +\infty$

$$\exp_p\{\alpha r^{\rho-\delta}\} \leq |A_l(z)| \leq \exp_p\{r^{\mu+\varepsilon}\}$$

and by ε is arbitrary with $0 < \varepsilon < \rho - \mu - 2\delta$, this is a contradiction as $r \to +\infty$. Hence, $\rho_p(A_l) = \rho$.

By Theorem 1.2, we have $\rho_p(A_l) \leq \rho$. Suppose that $\rho_p(A_l) = \mu < \rho$. Then, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$T(r, A_l) \leq \exp_{p-1}\{r^{\mu+\varepsilon}\}. \tag{2.13}$$

On the other hand, by the hypotheses of Theorems 1.2, there exist positive constants $0 \leq \beta < \alpha$ and δ $(0 < \delta < \rho)$ sufficiently small, such that

$$T(r, A_l) \geq \exp_{p-1}\{\alpha r^{\rho-\delta}\} \tag{2.14}$$

as $z \to \infty$ for $z \in H$. From (2.13) and (2.14), we obtain for $z \in H$, $|z| = r \to +\infty$

$$\exp_{p-1}\{\alpha r^{\rho-\delta}\} \leq T(r, A_l) \leq \exp_{p-1}\{r^{\mu+\varepsilon}\}$$

and by ε is arbitrary with $0 < \varepsilon < \rho - \mu - 2\delta$, this is a contradiction as $r \to +\infty$. Hence, $\rho_p(A_l) = \rho$.

3 Proofs of main results

Proof of Theorem 1.1. First case**:** When $p = 1$, let $f\,(\not\equiv 0)$ be a meromorphic solution of equation (1.2). Suppose that $\rho(f) < \rho + 1$. Then by Lemma 2.4 (i), for any given $\varepsilon\,(0 < \varepsilon < \rho + 1 - \rho(f) - 2\delta)$, there exists a set $E_3 \subset (0, +\infty)$ with finite logarithmic measure such that for all $|z| = r \notin E_3 \cup [0, 1]$, we have

$$\left|\frac{f(z+c_j)}{f(z+c_l)}\right| \leq \exp\{r^{\rho(f)-1+\varepsilon}\} < \exp\{r^{\rho-2\delta}\}, \quad (j = 1, \cdots, k,\, j \neq l) \tag{3.1}$$

and

$$\left|\frac{f(z)}{f(z+c_l)}\right| \leq \exp\{r^{\rho(f)-1+\varepsilon}\} < \exp\{r^{\rho-2\delta}\}. \tag{3.2}$$

We divide through equation (1.2) by $f(z+c_l)$ to get

$$-A_l(z) = A_k(z)\frac{f(z+c_k)}{f(z+c_l)} + \cdots + A_{l-1}(z)\frac{f(z+c_{l-1})}{f(z+c_l)}$$

$$+\cdots + A_1(z)\frac{f(z+c_1)}{f(z+c_l)} + A_0(z)\frac{f(z)}{f(z+c_l)}. \tag{3.3}$$

Rewrite equation (3.3) in the form

$$-1 = \sum_{j=1, j\neq l}^{k} \frac{A_j(z) f(z+c_j)}{A_l(z) f(z+c_l)} + \frac{A_0(z) f(z)}{A_l(z) f(z+c_l)},$$

it follows that

$$1 \leq \sum_{j=1, j\neq l}^{k} \left|\frac{A_j(z)}{A_l(z)} \cdot \frac{f(z+c_j)}{f(z+c_l)}\right| + \left|\frac{A_0(z)}{A_l(z)} \cdot \frac{f(z)}{f(z+c_l)}\right|. \tag{3.4}$$

From the conditions of Theorem 1.1, there is a set H of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ such that for $z \in H$, we have (1.3) and (1.4) as $|z| \to +\infty$. Set $H_1 = \{r = |z| : z \in H\}$, since $\overline{\log dens}\{|z| : z \in H\} > 0$, then H_1 is a set of r with $\int_{H_1} \frac{dr}{r} = \infty$. Substituting (1.3), (1.4) (when $p = 1$), (3.1) and (3.2) into (3.4), we get for $z \in H_1 \setminus (E_3 \cup [0,1])$

$$1 \leq k\frac{\exp\{\beta r^{\rho-\delta}\}}{\exp\{\alpha r^{\rho-\delta}\}} \exp\{r^{\rho-2\delta}\} = \exp\{(\beta-\alpha) r^{\rho-\delta} + r^{\rho-2\delta}\} \to 0,\ r \to +\infty$$

which is a contradiction. Hence, we get $\rho(f) \geq \rho + 1$. By Lemma 2.10, we know that $\rho(A_l) = \rho$. So, $\rho(f) \geq \rho(A_l) + 1$.

Second case**:** For $p \geq 2$, let $f\,(\not\equiv 0)$ be a meromorphic solution of equation (1.2). Suppose that $\rho_p(f) < \rho$. Then by Lemma 2.4 (ii), for any given $\varepsilon\,(0 < \varepsilon < \rho - \rho_p(f) - 2\delta)$, there exists a set $E_3 \subset (0, +\infty)$ with finite logarithmic measure such that for all $|z| = r \notin E_3 \cup [0,1]$, we have

$$\left|\frac{f(z+c_j)}{f(z+c_l)}\right| \leq \exp_p\{r^{\rho_p(f)+\varepsilon}\} < \exp_p\{r^{\rho-2\delta}\}, \quad (j = 1, \cdots, k,\ j \neq l) \tag{3.5}$$

and

$$\left|\frac{f(z)}{f(z+c_l)}\right| \leq \exp_p\{r^{\rho_p(f)+\varepsilon}\} < \exp_p\{r^{\rho-2\delta}\}. \tag{3.6}$$

Substituting (1.3), (1.4), (3.5) and (3.6) into (3.4), we get for $z \in H_1 \setminus (E_3 \cup [0,1])$

$$1 \leq k\frac{\exp_p\{\beta r^{\rho-\delta}\}}{\exp_p\{\alpha r^{\rho-\delta}\}} \exp_p\{r^{\rho-2\delta}\} \to 0,\ r \to +\infty$$

which is a contradiction. Hence, we get $\rho_p(f) \geq \rho$. By Lemma 2.10, we know that $\rho_p(A_l) = \rho$. So, $\rho_p(f) \geq \rho_p(A_l)$.

Proof of Theorem 1.2
First case**:** When $p = 1$, let $f\,(\not\equiv 0)$ be a meromorphic solution of equation (1.2). Suppose that

$$\rho(f) < \rho + 1.$$

Since $A_0(z), \cdots, A_k(z)$ are entire functions, then by (3.3), we have

$$\begin{aligned} m(r,A_l(z)) = T(r,A_l(z)) \leq \sum_{j=0,j\neq l}^{k} m(r,A_j(z)) \\ + \sum_{j=1,j\neq l}^{k} m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + O(1) \\ = \sum_{j=0,j\neq l}^{k} T(r,A_j(z)) + \sum_{j=1,j\neq l}^{k} m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + O(1). \end{aligned} \tag{3.7}$$

By Lemma 2.9 (i) and (3.7), we obtain for any given $\varepsilon\,(0 < \varepsilon < \rho + 1 - \rho(f) - 2\delta)$

$$\begin{aligned} T(r,A_l(z)) \leq \sum_{j=0,j\neq l}^{k} T(r,A_j(z)) + \sum_{j=1,j\neq l}^{k} m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) \\ + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + O(1) \leq \sum_{j=0,j\neq l}^{k} T(r,A_j(z)) + O(r^{\rho(f)-1+\varepsilon}). \end{aligned} \tag{3.8}$$

Substituting (1.5) and (1.6) (when $p = 1$) into (3.8), we get for $|z| = r \to +\infty$, $z \in H$

$$(\alpha - k\beta)\, r^{\rho-\delta} \leq O(r^{\rho(f)-1+\varepsilon}).$$

By $\alpha - k\beta > 0$, it follows that

$$1 \leq O(1) r^{\rho(f)-1+\varepsilon-\rho+\delta} \to 0,\ r \to +\infty$$

which is a contradiction. Hence we get $\rho(f) \geq \rho + 1$. By Lemma 2.10, we know that $\rho(A_l) = \rho$. So, $\rho(f) \geq \rho(A_l) + 1$.

Second case**:** For $p \geq 2$, let $f\,(\not\equiv 0)$ be a meromorphic solution of equation (1.2). Suppose that $\rho_p(f) < \rho$. By Lemma 2.9 (ii) and (3.7), we obtain for any given $\varepsilon\,(0 < \varepsilon < \rho - \rho_p(f) - 2\delta)$

$$T(r,A_l(z)) \leq \sum_{j=0,j\neq l}^{k} T(r,A_j(z))$$

$$+\sum_{j=1,j\neq l}^{k}\exp_{p-1}\{r^{\rho_p(f)+\varepsilon}\}+\exp_{p-1}\{r^{\rho_p(f)+\varepsilon}\}+O(1). \tag{3.9}$$

Substituting (1.5) and (1.6) (when $p=2$) into (3.9), we get for $|z|=r\to+\infty$, $z\in H$

$$\exp_{p-1}\{\alpha r^{\rho-\delta}\}\leq\sum_{j=0,j\neq l}^{k}\exp_{p-1}\{\beta r^{\rho-\delta}\}+\sum_{j=1,j\neq l}^{k}\exp_{p-1}\{r^{\rho_p(f)+\varepsilon}\}$$

$$+\exp_{p-1}\{r^{\rho_p(f)+\varepsilon}\}+O(1)\leq k\exp_{p-1}\{\beta r^{\rho-\delta}\}+k\exp_{p-1}\{r^{\rho_p(f)+\varepsilon}\}+O(1). \tag{3.10}$$

By (3.10), we obtain

$$(\alpha-\beta)r^{\rho-\delta}\leq r^{\rho_p(f)+\varepsilon}+O(1).$$

By $\alpha-\beta>0$, it follows that

$$1\leq\frac{1}{\alpha-\beta}r^{\rho_p(f)+\varepsilon-\rho+\delta}+\frac{1}{(\alpha-\beta)\,r^{\rho-\delta}}O(1)\to 0,\ r\to+\infty$$

which is a contradiction. By Lemma 2.10, we know that $\rho_p(A_l)=\rho$. Hence, we get $\rho_p(f)\geq\rho_p(A_l)$. Thus, Theorem 1.2 is proved.

Proof of Theorem 1.3. Let $f\,(\not\equiv 0)$ be a meromorphic solution of (1.2). By equation (3.7) and Lemma 2.6, we obtain

$$T(r,A_l(z))=m(r,A_l(z))\leq\sum_{j=0,j\neq l}^{k}m(r,A_j(z))+\sum_{j=1,j\neq l}^{k}m\left(r,\frac{f(z+c_j)}{f(z+c_l)}\right)$$

$$+m\left(r,\frac{f(z)}{f(z+c_l)}\right)+O(1)\leq\sum_{j=0,j\neq l}^{k}T(r,A_j(z))$$

$$+\sum_{j=1,j\neq l}^{k}o\left(\frac{(T(r+|c_j-c_l|+|c_l|,f))^{1+\varepsilon}}{r^{\delta}}\right)$$

$$+o\left(\frac{(T(r+2|c_l|,f))^{1+\varepsilon}}{r^{\delta}}\right)+O(1)\leq\sum_{j=0,j\neq l}^{k}T(r,A_j(z))+o\left(\frac{(T(r+2|c_l|,f))^{1+\varepsilon}}{r^{\delta}}\right) \tag{3.11}$$

for all r outside of a possible exceptional set E_5 with finite logarithmic measure $\int_{E_5}\frac{dr}{r}<\infty$. Let β_1, β_2 be two real numbers such that

$$\max\{\tau_{p+1}(A_j):\rho_{p+1}(A_j)=\rho_{p+1}(A_l)\}<\beta_1<\beta_2<\tau_{p+1}(A_l).$$

Then by Lemma 2.7, we know that there exists a set E_6 of infinite logarithmic measure, such that

$$T(r,A_l) > \exp_p\{\beta_2 r^{\rho_{p+1}(A_l)}\}$$

holds for all $r \in E_6$. Therefore we can take a sequence $\{r_n\}$ such that $r_n \in E_6$, $r_n \to \infty$, and

$$T(r_n,A_l) > \exp_p\{\beta_2 r_n^{\rho_{p+1}(A_l)}\}. \tag{3.12}$$

On the other hand, if $b = \max\{\rho_{p+1}(A_j) : j = 0,\cdots,k,\ j \neq l\} < \rho_{p+1}(A_l)$, then for any given ε $(0 < \varepsilon < \rho_{p+1}(A_l) - b)$ and sufficiently large r_n, we have

$$T(r_n,A_j) \le \exp_p\{r_n^{b+\varepsilon}\} \le \exp_p\{\beta_1 r_n^{\rho_{p+1}(A_l)}\}. \tag{3.13}$$

If $\max\{\tau_{p+1}(A_j) : \rho_{p+1}(A_j) = \rho_{p+1}(A_l)\} < \tau_{p+1}(A_l)$, then for sufficiently large r_n, we have

$$T(r_n,A_j) \le \exp_p\{\beta_1 r_n^{\rho_{p+1}(A_l)}\}. \tag{3.14}$$

Then substituting (3.12), (3.13) or (3.14) into (3.11), we get for $r_n \in E_6 \setminus E_5$

$$\exp_p\{\beta_2 r_n^{\rho_{p+1}(A_l)}\} < T(r_n,A_l) \le k\exp_p\{\beta_1 r_n^{\rho_{p+1}(A_l)}\} + o\left(\frac{(T(r_n+2|c_l|,f))^{1+\varepsilon}}{r_n^{\delta}}\right). \tag{3.15}$$

Then by (3.15), we get

$$(1-o(1))\exp_p\{\beta_2 r_n^{\rho_{p+1}(A_l)}\} < o\left(\frac{(T(r_n+2|c_l|,f))^{1+\varepsilon}}{r_n^{\delta}}\right).$$

Hence,

$$\rho_p(f) = \infty \quad \text{and} \quad \rho_{p+1}(f) \ge \rho_{p+1}(A_l).$$

Proof of Theorem 1.4. (i) First we consider the case

$$\rho_{p+1}(F) < \rho_{p+1}(A_l) \quad \text{or} \quad \rho_{p+1}(F) = \rho_{p+1}(A_l),\ \tau_{p+1}(F) < \tau_{p+1}(A_l).$$

Let f be a meromorphic solution of (1.7). We divide equation (1.7) by $f(z+c_l)$ to get

$$-A_l(z) = \sum_{j=1,j\neq l}^{k} A_j(z)\frac{f(z+c_j)}{f(z+c_l)} + A_0(z)\frac{f(z)}{f(z+c_l)} - \frac{F(z)}{f(z+c_l)}. \tag{3.16}$$

It follows from (3.16), Remark 2.1 and Lemma 2.6 that for any given $\varepsilon > 0$ and sufficiently large r, we have

$$T(r,A_l(z)) = m(r,A_l(z)) \le m\left(r,\frac{F(z)}{f(z+c_l)}\right) + \sum_{j=0,j\neq l}^{k} m(r,A_j(z))$$

$$+\sum_{j=1,j\neq l}^{k} m\left(r,\frac{f(z+c_j)}{f(z+c_l)}\right)+m\left(r,\frac{f(z)}{f(z+c_l)}\right)+O(1)\leq T(r,F(z))$$

$$+T(r,f(z+c_l))+\sum_{j=0,j\neq l}^{k} m(r,A_j(z))+o\left(\frac{(T(r+2|c_l|,f))^{1+\varepsilon}}{r^{\delta}}\right)$$

$$\leq T(r,F(z))+(1+o(1))T(r+|c_l|,f(z))+\sum_{j=0,j\neq l}^{k} T(r,A_j(z))$$

$$+o\left(\frac{(T(r+2|c_l|,f))^{1+\varepsilon}}{r^{\delta}}\right)\leq T(r,F(z))$$

$$+\sum_{j=0,j\neq l}^{k} T(r,A_j(z))+2T(r+|c_l|,f(z))+o\left(\frac{(T(r+2|c_l|,f))^{1+\varepsilon}}{r^{\delta}}\right) \tag{3.17}$$

for $r\to\infty$, $r\notin E_5$, where E_5 is a set of finite logarithmic measure. Let β_1,β_2 be two real numbers such that

$$\max\{\tau_{p+1}(A_j),\tau_{p+1}(F):\rho_{p+1}(A_j)=\rho_{p+1}(A_l)\}<\beta_1<\beta_2<\tau_{p+1}(A_l).$$

Then by Lemma 2.7, we can take a sequence $\{r_n\}$ such that $r_n\in E_6, r_n\to\infty$, and equations (3.12)–(3.14) also hold for sufficiently large r_n. On the other hand, for sufficiently large r_n we have

$$T(r_n,F)\leq \exp_p\{\beta_1 r_n^{\rho_{p+1}(A_l)}\}. \tag{3.18}$$

Substituting equations (3.12), (3.13) (or (3.14)) and (3.18) into (3.17), we get for $r_n\in E_6\diagdown E_5$

$$\exp_p\{\beta_2 r_n^{\rho_{p+1}(A_l)}\}<T(r_n,A_l)\leq (k+1)\exp_p\{\beta_1 r_n^{\rho_{p+1}(A_l)}\}+3\left(T(2r_n,f)\right)^2. \tag{3.19}$$

Hence, by equation (3.19), we get

$$\rho_p(f)=\infty \quad \text{and} \quad \rho_{p+1}(f)\geq\rho_{p+1}(A_l).$$

(ii) Next we consider the case $\rho_{p+1}(F)>\rho_{p+1}(A_l)$. Let f be a meromorphic solution of (1.7). By Remark 2.1 and Lemma 2.6 that for any given $\varepsilon>0$ and sufficiently large r, we have

$$T(r,F(z))\leq\sum_{j=0}^{k}T(r,A_j(z))+\sum_{j=1}^{k}T(r,f(z+c_j))+T(r,f(z))+O(1)$$

$$\leq\sum_{j=0}^{k}T(r,A_j(z))+(1+o(1))kT(r,f(z+|c_s|))+T(r,f(z))+O(1)$$

$$\leq \sum_{j=0}^{k} T(r,A_j(z)) + (2k+1)\,T(2r,f(z)) + O(1),\ |c_s| = \max_{1\leq j\leq k}\left\{\left|c_j\right|\right\}. \tag{3.20}$$

By the definition of iterated $p+1-$order, we know that there exists a sequence $\{r_n\}$ such that $r_n \to +\infty$, and for any given ε $(0 < 2\varepsilon < \rho_{p+1}(F) - \rho_{p+1}(A_l))$, we have

$$T(r_n,F) \geq \exp_p\{r_n^{\rho_{p+1}(F)-\varepsilon}\} \tag{3.21}$$

and

$$T(r_n,A_j) \leq \exp_p\{r_n^{b+\varepsilon}\} \leq \exp_p\{r_n^{\rho_{p+1}(A_l)+\varepsilon}\}\ (j = 0,\cdots,k), \tag{3.22}$$

where $b = \max\{\rho_{p+1}(A_j) : j = 0,\cdots,k,\ j \neq l\} < \rho_{p+1}(A_l)$. Substituting (3.21) and (3.22) into (3.20), we get

$$\exp_p\{r_n^{\rho_{p+1}(F)-\varepsilon}\} \leq (k+1)\exp_p\{r_n^{\rho_{p+1}(A_l)+\varepsilon}\} + (2k+1)\,T(2r,f(z)).$$

Hence,

$$\rho_p(f) = \infty \quad \text{and} \quad \rho_{p+1}(f) \geq \rho_{p+1}(F).$$

4 Examples

Next we give an example that illustrates Theorem 1.1.

Exemple 4.1. We consider the meromorphic function

$$f(z) = e^{-z^2}\tan z.$$

Then f satisfies the difference equation

$$A_2(z)f(z+2\pi) + A_1(z)f(z+\pi) + A_0(z)f(z) = 0, \tag{4.1}$$

where

$$A_2(z) = 2\exp\{2\pi z + 3\pi^2\},\ A_1(z) = -1,\ A_0(z) = -\exp\{-2\pi z - \pi^2\}.$$

We have

$$\rho(A_2) = \rho(A_0) = 1, \quad \rho(A_1) = 0$$

and

$$1 = \max_{0\leq j\leq 1}\{\rho(A_j)\} \leq \rho = 1.$$

We choose

$$H = \{z \in \mathbb{C} : z = re^{i\theta},\ r \in [1,+\infty[,\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3}\}$$

a complex set satisfying $\overline{\log dens}\{r=|z|:z\in H\}>0$, we get for δ $(0<\delta<\rho=1)$ sufficiently small

$$|A_2(z)|=|2\exp\{2\pi z+3\pi^2\}|=2\exp\{2\pi r\cos\theta+3\pi^2\}$$
$$\geq 2\exp\{\pi r+4\pi^2\}\geq \exp\{\pi r^{1-\delta}\},$$
$$|A_1(z)|=1\leq \exp\{r^{1-\delta}\}$$

and

$$|A_0(z)|=|-\exp\{-2\pi z-\pi^2\}|=\exp\{-2\pi r\cos\theta-\pi^2\}$$
$$\leq \exp\{-\pi r-\pi^2\}\leq \exp\{r^{1-\delta}\}$$

as $z\to\infty$ for $z\in H$. As we see, conditions of Theorem 1.1 are verified with $\alpha=\pi$ and $\beta=1$. We get

$$2=\rho(f)\geq\rho(A_2)+1=2.$$

Next we give an example that illustrates Theorem 1.3.

Exemple 4.2. Consider the difference equation

$$(z+2i\pi)\exp\{-\sin(z+2i\pi)\}f(z+2i\pi)$$
$$-2\left(z+i\frac{\pi}{2}\right)\exp\left\{2i\sinh 2z-\sin\left(z+i\frac{\pi}{2}\right)\right\}f\left(z+i\frac{\pi}{2}\right)$$
$$+z\exp\{-\sin z\}f(z)=0. \tag{4.2}$$

In this equation we have

$$A_2(z)=(z+2i\pi)\exp\{-\sin(z+2i\pi)\},$$
$$A_1(z)=-2\left(z+i\frac{\pi}{2}\right)\exp\left\{2i\sinh 2z-\sin\left(z+i\frac{\pi}{2}\right)\right\},$$
$$A_0(z)=z\exp\{-\sin z\}.$$

We obtain

$$\rho(A_2)=\rho(A_1)=\rho(A_0)=\infty,$$
$$\rho_2(A_2)=\rho_2(A_1)=\rho_2(A_0)=1,$$
$$\tau_2(A_2)=1,\quad \tau_2(A_1)=2,\quad \tau_2(A_0)=1.$$

As we see, conditions of Theorem 1.3 are verified

$$1=\max\{\rho_2(A_j):j=0,2\}\leq\rho_2(A_1)=1,$$
$$1=\max\{\tau_2(A_j):\rho_2(A_j)=\rho_2(A_1)\}<\tau_2(A_1)=2.$$

The meromorphic function

$$f(z)=\frac{\exp\{\sin 2iz+\sin z\}}{z}$$

is solution of equation (4.2) and f satisfies

$$\rho(f)=\infty \quad \text{and} \quad 1=\rho_2(f)\geq\rho_2(A_1)=1.$$

Next, we give an example that illustrates Theorem 1.4.

Exemple 4.3. Case (i). Consider the difference equation

$$\begin{aligned}&(z+2i\pi)\exp\{-\sin(z+2i\pi)\}f(z+2i\pi)\\&-(z+i\pi)\exp\{-\sin(z+i\pi)\}f(z+i\pi)\\&+z\exp\{-\sin 2iz\}f(z)=\exp\{\sin z\}.\end{aligned}\tag{4.3}$$

In this equation, we have

$$A_2(z)=(z+2i\pi)\exp\{-\sin(z+2i\pi)\},$$

$$A_1(z)=-(z+i\pi)\exp\{-\sin(z+i\pi)\},$$

$$A_0(z)=z\exp\{-\sin 2iz\},\ F(z)=\exp\{\sin z\}.$$

We obtain

$$\rho(A_2)=\rho(A_1)=\rho(A_0)=\rho(F)=\infty,$$

$$\rho_2(A_2)=\rho_2(A_1)=\rho_2(A_0)=\rho_2(F)=1,$$

$$\tau_2(A_2)=1,\quad \tau_2(A_1)=1,\quad \tau_2(A_0)=2,\quad \tau_2(F)=1.$$

It clear that the conditions of Theorem 1.4 (i) are satisfied

$$1=\max\{\rho_2(A_j):j=1,2\}\leq\rho_2(A_0)=1,$$

$$1=\max\{\tau_2(A_j):\rho_2(A_j)=\rho_2(A_0)\}<\tau_2(A_0)=2,$$

$\rho_2(F)=\rho_2(A_0)=1$ and $\tau_2(F)=1<\tau_2(A_0)=2$. The meromorphic function

$$f(z)=\frac{\exp\{\sin 2iz+\sin z\}}{z}$$

is solution of equation (4.3) and f satisfies

$$\rho(f)=\infty \quad \text{and} \quad 1=\rho_2(f)\geq\rho_2(A_0)=1.$$

Case (ii). We consider the meromorphic function

$$f(z) = \frac{\exp\{\sin z\}}{z}.$$

Then f satisfies the difference equation

$$A_2(z)f(z+2\pi)+A_1(z)f(z+\pi)+A_0(z)f(z) = F(z), \tag{4.4}$$

where

$$A_2(z) = (z+2\pi)\exp\left\{\frac{\sin\sqrt{z}}{\sqrt{z}}\right\},\ A_1(z) = (z+\pi)\exp\left\{\frac{\sin 3\sqrt{z}}{\sqrt{z}}\right\}$$

$$A_0(z) = -z\exp\left\{\frac{\sin\sqrt{z}}{\sqrt{z}}\right\},\ F(z) = \exp\left\{-\sin z+\frac{\sin 3\sqrt{z}}{\sqrt{z}}\right\}.$$

We have

$$\rho(A_2) = \rho(A_1) = \rho(A_0) = \rho(F) = \infty,$$

$$\rho_2(F) = 1, \quad \rho_2(A_2) = \rho_2(A_1) = \rho_2(A_0) = 1/2,$$

$$\tau_2(A_2) = \tau_2(A_0) = 1, \quad \tau_2(A_1) = 3.$$

It clear that the conditions of Theorem 1.4 (ii) are satisfied

$$1/2 = \max\{\rho_2(A_j) : j = 0,2\} \leq \rho_2(A_1) = 1/2,$$

$$1 = \max\{\tau_2(A_j) : \rho_2(A_j) = \rho_2(A_1)\} < 3$$

and $1 = \rho_2(F) > \rho_2(A_1) = 1/2$. We see that

$$\rho(f) = \infty \quad \text{and} \quad 1 = \rho_2(f) \geq \rho_2(F) = 1.$$

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