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On Alberson irregularity measure of graphs

M. Matejić, B. Mitić, E. Milovanović, I. Milovanović

Abstract: Let $G=(V,E)$, $V=\{1,2,\dots,n\}$ be a simple connected graph with n vertices, m edges and a sequence of vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(i)$. The irregularity measure of graph is defined as $irr(G)=\sum_{i\sim j}\left|d_i - d_j\right|$, where $i \sim j$ denotes adjacency of vertices i and j. New upper bounds for $irr(G)$ are obtained.

Keywords: Irregularity of graph, Zagreb indices, inverse sum indeg index, Alberson index

1 Introduction

Let $G=(V,E)$, $V=\{1,2,\dots,n\}$ be a simple connected graph with $n=|V|$ vertices and $m=|E|$ edges. Denote by $i \sim j$ an edge connecting vertices i and j. Further, let $d_i = d(i)$ be the degree of a vertex i, and $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$ the sequence of vertex degrees. A graph G is said to be regular if and only if there exists an integer $k, 1 \leq k \leq n-1$, so that $d_1 = d_2 = \cdots = d_n = k$, otherwise it is irregular. A union of disjointed components of graph, that is $G = G_1 \cup G_2 \cup \cdots \cup G_r$ is regular by components if every component G_i is a regular graph. Without loss of generality we will assume that G is connected. A graph invariant $I(G)$ is measure of irregularity of graph G with the property $I(G)=0$ if and only if G is regular, and $I(G)>0$ otherwise. A number of different irregularity measures have been defined in the literature (see for example [26, 13, 14, 4, 2, 8, 1, 10]). Here we will mention only two that are of interest for the present consideration.

In [4] Bell suggested a *variance* of vertex degrees,

$$VAR(G)=\frac{1}{n}\sum_{i=1}^{n} d_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2,$$

to be taken as a measure of irregularity of G.

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M. M. Matejić is with the Faculty of Electronic Engineering, Niš, Serbia; B. Mitić is with the Fazi d.o.o., Belgrade, Serbia; E. I. Milovanović, I. Ž. Milovanović are with the Faculty of Electronic Engineering, Niš, Serbia

Denote by $e = ij$ an arbitrary edge of G which is incident to the vertices i and j. In [2] Albertson defined the *imbalance* of an edge e as $imb(e) = |d_i - d_j|$, and used it to introduce another irregularity measure

$$irr(G) = \sum_{i\sim j} |d_i - d_j| \,,$$

which is sometimes referred to as *Albertson index* [39, 40] or the *third Zagreb index* [10].

In this paper we are interested in determining upper bounds for $irr(G)$ in terms of some basic graph parameters and some other graph invariants. In what follows we outline graph invariants that will be used in the paper.

A single number that can be used to characterize some property of the graph is called a *topological index* for that graph. Obviously, the number of vertices and the number of edges are topological indices.

Two vertex-degree based topological indices, the *first* and the *second Zagreb* index, M_1 and M_2, are defined as (see [15, 16])

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \qquad \text{and} \qquad M_2 = M_2(G) = \sum_{i\sim j} d_i d_j .$$

The Zagreb indices are among the oldest and most studied molecular structure descriptors and found significant applications in chemistry.

An alternative expressions for the first Zagreb index is [24]

$$M_1(G) = \sum_{i\sim j} (d_i + d_j) \,.$$

A modification of the first Zagreb index, F, defined as the sum of third powers of vertex degrees, that is

$$F = F(G) = \sum_{i=1}^{n} d_i^3 \,,$$

was first time encountered in 1972, in the paper [15], but was eventually disregarded. Recently, it was re-considered in [12] and named the *forgotten index*.

Nowadays, there exist hundreds of papers on Zagreb indices and related matter [24, 20, 5, 3, 17].

A family of Adriatic indices was introduced in [28, 29]. An especially interesting subclass of these descriptors consists of 148 discrete Adriatic indices. A so called *inverse sum indeg* index, $ISI(G)$, was selected in [28] as a significant predictor of total surface area of octane isomers. The inverse sum indeg index is defined as

$$ISI(G) = \sum_{i\sim j} \frac{d_i d_j}{d_i + d_j} \,.$$

More on mathematical properties of this topological index can be found in [9, 27, 19].

In [29] a topological index named *symmetric division deg*, $SDD(G)$, was defined as

$$SDD(G) = \sum_{i \sim j} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right).$$

2 Preliminaries

In this section we recall some results for the upper bounds of $irr(G)$. We will compare these results to the new ones derived in this paper.

In [30] Zhou and Liu proved the inequality

$$irr(G) \leq \sqrt{m(M_1(G) - 4m^2)}. \tag{1}$$

Since

$$VAR(G) = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} d_i \right)^2 = \frac{1}{n^2}\left(nM_1(G) - 4m^2 \right),$$

the inequalty (1) can be rewritten as

$$irr(G) \leq n\sqrt{mVAR(G)}.$$

The above inequality establishes a relation between two irregularity measures, that is between $irr(G)$ and $VAR(G)$.

Goldberg [11] proved the following inequality

$$irr(G) \leq \sqrt{\frac{m\mu_1(nM_1(G) - 4m^2)}{n}} = \sqrt{nm\mu_1 VAR(G)}, \tag{2}$$

where μ_1 is the Laplacian spectral radius of G. Since $\mu_1 \leq n$, the inequality (2) is stronger than (1).

In [7] Chen *at all*, proved the following

$$irr(G) \leq \frac{n\mu_1(\Delta - \delta)}{4}. \tag{3}$$

Che and Chen [6] proved that

$$irr(G) \leq \sqrt{m(F(G) - 2M_2(G))}, \tag{4}$$

and

$$irr(G) \leq \sqrt{2mF(G) - M_1(G)^2}. \tag{5}$$

In [38] the following inequality was proven

$$0 \leq irr(G) + irr(\bar{G}) \leq \frac{1}{6}(n-1)(n+1)(2n-3),$$

where $\bar{G}$ is the complement of G. Equality holds if and only if G is a regular graph.

3 Main results

In this section we will prove some new inequalities that establish upper bounds for the $irr(G)$. But, first recall one analytical inequality for positive real number sequences proved in [25].

Lemma 1. [25] *Let $x=(x_i)$ and $a=(a_i)$, $i=1,2,\ldots,m$, be two positive real number sequences. Then for any real $r\geq 0$, holds*

$$\sum_{i=1}^{m}\frac{x_i^{r+1}}{a_i^r}\geq\frac{\left(\sum_{i=1}^{m}x_i\right)^{r+1}}{\left(\sum_{i=1}^{m}a_i\right)^{r}}\,. \tag{6}$$

Equality holds if and only if $r=0$ or $\frac{x_1}{a_1}=\frac{x_2}{a_2}=\cdots=\frac{x_m}{a_m}$.

In the next theorem we establish an upper bound for $irr(G)$ in terms of indices $M_1(G)$ and $ISI(G)$.

Theorem 1. *Let G be a simple connected graph with n vertices and m edges. Then*

$$irr(G)\leq\sqrt{M_1(G)(M_1(G)-4ISI(G))}\,. \tag{7}$$

Equality holds if and only if $\frac{|d_i-d_j|}{d_i+d_j}$ is constant for each edge of G.

Proof. For $r=1$, $x_i:=|d_i-d_j|$, $a_i:=d_i+d_j$, where the summation is performed over all edges of G, the inequality (6) becomes

$$\sum_{i\sim j}\frac{(d_i-d_j)^2}{d_i+d_j}\geq\frac{\left(\sum_{i\sim j}|d_i-d_j|\right)^2}{\sum_{i\sim j}(d_i+d_j)}=\frac{irr(G)^2}{M_1(G)}\,. \tag{8}$$

Since

$$0\leq\sum_{i\sim j}\frac{(d_i-d_j)^2}{d_i+d_j}=\sum_{i\sim j}(d_i+d_j)-4\sum_{i\sim j}\frac{d_id_j}{d_i+d_j}=M_1(G)-4ISI(G)\,,$$

from the above and (8) we arrive at (7).

For $r=1$ equality in (6) holds if and only if $\frac{x_1}{a_1}=\frac{x_2}{a_2}=\cdots=\frac{x_m}{a_m}$, which implies that equality in (8), that is (7), holds if and only if $\frac{|d_i-d_j|}{d_i+d_j}$ constant for each edge of G. □

Remark 1. *Equality in* (7) *holds, for example, if G is regular or semiregular bipartite graph.*

Since $ISI(G)\geq\frac{m^2}{n}$ (see for example [9]), we have the following corollary of Theorem 1.

Corollary 1. *Let G be a simple connected graph of order n and size m. Then*

$$irr(G) \leq \sqrt{\frac{M_1(G)(nM_1(G)-4m^2)}{n}} = \sqrt{nM_1(G)VAR(G)}\,. \tag{9}$$

Equality holds if and only if G is regular or semiregular bipartite graph.

In [21] (see also [22, 18]) the following inequality was proved

$$M_1(G) \leq \frac{4m^2}{n} + n\alpha(n)(\Delta-\delta)^2\,, \tag{10}$$

where

$$\alpha(n) = \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right) = \frac{1}{4}\left(1-\frac{(-1)^{n+1}+1}{2n^2}\right) \leq \frac{1}{4}\,.$$

Therefore we have the following corollary of Theorem 1.

Corollary 2. *Let G be a simple connected graph wit n vertices and m edges. Then*

$$irr(G) \leq \sqrt{n\alpha(n)M_1(G)}(\Delta-\delta)\,.$$

Equality holds if and only if G is regular.

Remark 2. *According to* (10) *we have*

$$VAR(G) \leq \alpha(n)(\Delta-\delta)^2\,.$$

Since $\alpha(n) \leq \frac{1}{4}$*, the above inequality is stronger than*

$$VAR(G) \leq \frac{(\Delta-\delta)^2}{4}\,,$$

which was proved in [23].

Corollary 3. *Let G be a simple connected graph with* $n \geq 2$ *vertices and m edges. Then*

$$irr(G) \leq \frac{m}{n-1}\sqrt{\frac{(n-2)((n(n-1)-2m)(2m+(n-1)(n-2))}{n}}\,. \tag{11}$$

Equality holds if and only if either $G \cong K_n$*, or* $G \cong K_{1,n-1}$

Proof. In [31] it was proven that

$$M_1(G) \leq m\left(\frac{2m}{n-1}+n-2\right)\,,$$

with equality if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$ [32]. From the above and inequality (9) we obtain (11). □

Based on (11) we obtain the following result.

Corollary 4. *Let T be a tree with $n \geq 2$ vertices. Then*

$$irr(T) \leq (n-1)(n-2)\,. \tag{12}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Remark 3. *The inequality* (12) *was proven in* [33].

In the next theorem we establish an upper bound for $irr(G)$ in terms of parameter m and invariants $M_2(G)$ and $SDD(G)$.

Theorem 2. *Let G be a simple connected graph with m edges. Then*

$$irr(G) \leq \sqrt{M_2(G)(SDD(G)-2m)}\,. \tag{13}$$

Equality holds if and only if $\frac{|d_i-d_j|}{d_id_j}$ constant for each edge of G.

Proof. For $r=1$, $x_i := |d_i-d_j|$ and $a_i := d_id_j$, where summation is performed over all edges of G, the inequality (6) transforms into

$$\sum_{i\sim j}\frac{(d_i-d_j)^2}{d_id_j} \geq \frac{\left(\sum_{i\sim j}|d_i-d_j|\right)^2}{\sum_{i\sim j}d_id_j} = \frac{irr(G)^2}{M_2(G)}\,. \tag{14}$$

Since

$$0 \leq \sum_{i\sim j}\frac{(d_i-d_j)^2}{d_id_j} = \sum_{i\sim j}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right) - 2\sum_{i\sim j}\frac{d_id_j}{d_id_j} = SDD-2m\,,$$

from the above and inequality (14) we obtain (13).

For $r=1$, the equality in (6) holds if and only if $\frac{x_1}{a_1}=\frac{x_2}{a_2}=\cdots=\frac{x_m}{a_m}$. Therefore we conclude that equality in (14), i.e. in (13), holds if and only if $\frac{|d_i-d_j|}{d_id_j}$ is constant for each edge of G.

□

Remark 4. *Equality in* (13) *holds, for example, if G is regular or semiregular bipartite graph.*

Remark 5. *The inequalities* (7) *and* (13) *are, mainly, incomparable with* (1), (2), (3), (4) *and* (5). *Thus, for example, for $G = C_{n-1}+e$, the inequality* (7) *is stronger than* (1), (2), (3), (4) *and* (5), *and* (13) *from* (1), (2), (3) *and* (5). *For $G = K_n - e$, inequalities* (1), (2) *and* (4) *are stronger than* (7) *and* (13). *By testing for large n we didn't find a graph for which* (13) *is stronger than* (4) *and* (7), *as well as a graph for which* (5) *is stronger than* (4) *and* (7).

Corollary 5. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$irr(G) \leq \frac{\Delta - \delta}{\sqrt{\Delta\delta}} \sqrt{mM_2(G)}\,. \tag{15}$$

Equality holds if and only if G is regular or semiregular bipartite graph.

Proof. The function $f(x) = x + \frac{1}{x}$ is monotone increasing for $x \geq 1$. On the other hand, for every $d_i \geq d_j$ it holds $\frac{\Delta}{\delta} \geq \frac{d_i}{d_j} \geq 1$, so we have

$$\sum_{i\sim j} \frac{(d_i - d_j)^2}{d_i d_j} = \sum_{i\sim j} \left(\sqrt{\frac{d_i}{d_j}} - \sqrt{\frac{d_j}{d_i}} \right)^2 \leq m \left(\sqrt{\frac{\Delta}{\delta}} - \sqrt{\frac{\delta}{\Delta}} \right)^2 = \frac{m(\Delta - \delta)^2}{\Delta\delta}\,.$$

Now, from the above and inequality (14) we obtain (15). □

Based to the (15) we have the next result.

Corollary 6. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$irr(G) \leq \frac{n-2}{\sqrt{n-1}} \sqrt{mM_2(G)}\,. \tag{16}$$

Equality holds if and only if $G \cong K_{1,n-1}$.

Remark 6. *In [34] it was proven than for any tree T holds*

$$SDD(T) \leq (n-1)^2 + 1\,,$$

with equality if and only if $T \cong K_{1,n-1}$.

In [35] (see also [36]) it was proven that

$$M_2(T) \leq (n-1)^2\,, \tag{17}$$

with equality if and only if $T \cong K_{1,n-1}$. It can be easily verified that inequality (12) *can be obtained from the above inequalities and* (13)*, as well as and from* (15)*,* (16) *and* (17)*.*

Remark 7. *The irregularity measure similar to the Albertson irregularity measure, named the sigma index was defined in [37] as*

$$\sigma(G) = \sum_{i\sim j} (d_i - d_j)^2\,.$$

It is not difficult to see that for these two topological indices the following relation is valid

$$\sqrt{\sigma(G)} \leq irr(G) \leq \sqrt{m\sigma(G)}\,,$$

with equalities if and only if G is regular.

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