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Complex exponential signal angle estimation based on angle invariant combiner

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Abstract: In order to achieve estimation performance limits, we often need to use computationally demanding estimation algorithms and/or signal information of higher order such as cumulants. Our goal is to reduce the computational complexity of angle estimation, and to achieve the Cramer-Rao estimation bound, and the maximum-likelihood signal-to-noise ratio threshold by using iterative estimation where the most computationally demanding processing is done as much as possible in the initialisation step, while in each iteration we require less complex processing. This is achieved by using the angle invariant combinations of signal autocorrelation samples for estimation.

Keywords: signal processing, direction of arrival estimation, frequency estimation, array processing

1 Introduction

Estimation of parameters of a sum of complex exponential signals in additive noise is an important and much studied problem with many practical applications. Various authors have proposed numerous solutions for this problem with the goal of unbiased, and consistent estimation from a finite set of signal samples, that achieves minimum estimation error variance with the lowest computational load, and signal-to-noise ratio (SNR) threshold, [1], [2], [3], [4].

The cost of achieving estimation performance limits is the complexity of the estimation algorithm, and the pre-processing of the signal. In general, performance of the algorithms that are used for parameter estimation of a sum of complex exponential signals depends on the number of signal samples, and the number of snapshots, i.e. the number of signal samples sets taken over a certain observation interval. Maximum likelihood (ML) estimator is asymptotically optimal, and for a finite set of signal samples it has been shown to exhibit the best resolution, accuracy, and the lowest SNR threshold, [2], [5]. However, the ML estimator is also very computationally demanding, especially when the number of signals

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increases. As the number of signal samples increases, in the presence of additive white Gaussian noise (AWGN), the ML estimates are approximately given by the maximum of the less computationally demanding periodogram (PG), [6]. The high resolution subspace algorithms are characterised by high SNR threshold that can be reduced at the price of significantly higher computational complexity, [2], [4], [7]. Computationally attractive iterative estimation algorithms refine suboptimal initial estimates through iterative maximising of the likelihood derived cost function. Note that in this case the SNR threshold for which the estimator performance degrades significantly is in general limited by the SNR threshold of the algorithm that is used for initial estimation, [6]. Estimation based on higher order spectrum or poly-spectra, that are obtained by taking the Fourier transform (FT) of the highly computationally demanding cumulant sequence, can help us reduce the influence of additive Gaussian noise on estimation, [8], [9], [10]. In the presence of AWGN we can use other forms of signal information instead of cumulants to improve estimation performance with lower processing requirements, [11], [12], [13].

In this letter, we use a quasi ML cost function under the assumption of AWGN to introduce two sequences of modified signal information samples that are obtained as angle invariant combinations (AIC) of signal autocorrelation samples. Our goal is to use the proposed AIC sequences, that require less demanding processing than the cumulants, in combination with a novel iterative estimator, to achieve the CRB, and the ML SNR threshold with reduced computational complexity, by having the most demanding processing in the initialisation step, and with much less complex processing in each iteration.

2 System model

We consider a system similar to [5] described by equation:

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{Z} \in \mathbb{C}^{N \times M}, \tag{1}$$

where N denotes the number of signal samples per snapshot, and M denotes the number of snapshots. Matrix of signal observations $\boldsymbol{Y} \in \mathbb{C}^{N \times M}$ is given by:

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{y}_0 & \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_{N-1} \end{bmatrix}^T, \tag{2}$$

where $\boldsymbol{y}_n \in \mathbb{C}^{M \times 1}$, and $(\cdot)^T$ denotes the matrix transpose. Samples of AWGN are given in $\boldsymbol{Z}$ that is equal to:

$$\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \cdots & \boldsymbol{z}_{N-1} \end{bmatrix}^T, \tag{3}$$

where $\boldsymbol{z}_n \in \mathbb{C}^{M \times 1}$. The elements of $\boldsymbol{Z}$ are independent, identically distributed (i.i.d.) zero mean complex Gaussian random variables with variance σ_z^2. Matrix $\boldsymbol{A}$ is equal to:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_{N,1} & \boldsymbol{a}_{N,2} & \cdots & \boldsymbol{a}_{N,K} \end{bmatrix} \in \mathbb{C}^{N \times K}, \tag{4}$$

where

$$\boldsymbol{a}_{N,k} = \begin{bmatrix} 1 & e^{j\varphi_k} & \cdots & e^{j(N-1)\varphi_k} \end{bmatrix}^T \in \mathbb{C}^{N \times 1}, \tag{5}$$

and φ_k is the unknown angle of the k-th complex exponential signal. We assume that for large N, and small n, vectors $\boldsymbol{a}_{N-n,k}$ are orthogonal:

$$\boldsymbol{a}_{N-n,k_1}^H \boldsymbol{a}_{N-n,k_2}/N \cong 0 \Leftrightarrow k_1 \neq k_2. \tag{6}$$

We consider a scenario when signal amplitudes change in each snapshot. In this case, the matrix of signal amplitudes is equal to:

$$\boldsymbol{X} = \boldsymbol{X}_p = \begin{bmatrix} \boldsymbol{x}_{p,1} & \boldsymbol{x}_{p,2} & \cdots & \boldsymbol{x}_{p,K} \end{bmatrix}^T, \tag{7}$$

where $\boldsymbol{x}_{p,k} \in \mathbb{C}^{M\times 1}$ is the k-th signal vector of amplitudes per snapshot. We assume that the average of the signal amplitudes is equal to:

$$\frac{1}{M}\mathbf{1}_M^T \boldsymbol{x}_{p,k} \cong 0, \tag{8}$$

and that the amplitude vectors corresponding to two different complex exponential signals are orthogonal:

$$\frac{1}{M}\boldsymbol{x}_{p,k_1}^H \boldsymbol{x}_{p,k_2} \cong 0 \Leftrightarrow k_1 \neq k_2, \tag{9}$$

as the value of M increases. The matrix conjugate transpose is denoted as $(\cdot)^H$, and $\mathbf{1}_M$ denotes the $M \times 1$ vector with all elements equal to one. The assumptions in (8), and (9) are commonly used in literature to satisfy the persistence excitation condition, [5], which is necessary if the signal correlation matrix is defined as:

$$\boldsymbol{R} = \boldsymbol{Y}\boldsymbol{Y}^H/M. \tag{10}$$

As a consequence, ML and subspace algorithms achieve estimation error bounds only for large M, [5]. Let us introduce the following vector:

$$\boldsymbol{y}_{N_w}^{(l)} = \begin{bmatrix} \boldsymbol{y}_l^T & \boldsymbol{y}_{l+1}^T & \cdots & \boldsymbol{y}_{l+N_w-1}^T \end{bmatrix}^T \in \mathbb{C}^{N_w M\times 1}, \tag{11}$$

where N_w denotes the number of samples that we consider, $0 \leq l \leq N - N_w$, and $1 \leq N_w \leq N$. Then, the samples of the autocorrelation sequence are defined as:

$$r_p(n) = \left(\boldsymbol{y}_{N-n}^{(0)}\right)^H \boldsymbol{y}_{N-n}^{(n)} / \left(M(N-n)\right), \tag{12}$$

for $0 \leq n \leq N-1$, and $r_p(n) = r_p(-n)^*$ for $-(N-1) \leq n < 0$. The complex conjugate is denoted as $(\cdot)^*$.

3 Iterative sample shifted estimator

For $K = 1$, similarly to the generalised weighted linear predictor (GWLP) in [3], we can write:

$$\widetilde{\varphi}^{(i+1)} = \measuredangle\left[\left(\boldsymbol{y}_{N-1}^{(0)}\right)^H \boldsymbol{C}_y^{-1} \boldsymbol{y}_{N-1}^{(1)}\right], \tag{13}$$

where $\measuredangle$ denotes the phase angle of a complex number. The covariance matrix $\boldsymbol{C}_y \in \mathbb{C}^{(N-1)M\times(N-1)M}$ is equal to:

$$\boldsymbol{C}_y = \sigma_z^2 \begin{bmatrix} 2 & -e^{-j\widetilde{\varphi}^{(i)}} & 0 & \cdots \\ -e^{j\widetilde{\varphi}^{(i)}} & 2 & -e^{-j\widetilde{\varphi}^{(i)}} & \cdots \\ 0 & -e^{j\widetilde{\varphi}^{(i)}} & 2 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \otimes \boldsymbol{I}_M, \tag{14}$$

where $\widetilde{\varphi}^{(i)}$ is an angle estimate in the i-th iteration, $\boldsymbol{I}_M$ denotes an $M \times M$ identity matrix, and $\otimes$ denotes the Kronecker product. The GWLP in [3] is initialised by the weighted least squares (LS) solution over the received data sequence:

$$\widetilde{\varphi}^{(0)} = \arg\min_{\varphi} \left\| \operatorname{diag}(\boldsymbol{w})^{-1/2} \left(\boldsymbol{y}_{N-1}^{(1)} - e^{j\varphi} \boldsymbol{y}_{N-1}^{(0)} \right) \right\|^2, \tag{15}$$

which results in, [14]:

$$\widetilde{\varphi}^{(0)} = \measuredangle r_p(1). \tag{16}$$

With $\operatorname{diag}(\boldsymbol{w})$ we denote a diagonal matrix with the elements of vector $\boldsymbol{w}$ on the main diagonal. It is noted in [3] that other algorithms can be used for initialisation of GWLP. Computation of GWLP requires one matrix-vector, and one vector-vector multiplication per iteration. This results in $(M(N-1))^2 + M(N-1)$ complex multiplications per iteration. Computational complexity of (13) can be reduced by setting the amplitude of the elements on each diagonal of matrix $\boldsymbol{C}_y^{-1}$ equal to the maximum absolute value of these elements. Then, the expression in (13) reduces to:

$$\widetilde{\varphi}^{(i+1)} = \measuredangle \left[\sum_{k=-(N-2)}^{N-2} w_{AP}^{(N-1)}(k) r(k+1) e^{-jk\widetilde{\varphi}^{(i)}} \right], \tag{17}$$

where $w_{AP}^{(N)}(k) = w_s(k)/w_s(0)$ is a window function (WF) of length N, and $w_s(k)$ is equal to the maximum absolute value of the elements on the k-th diagonal of $\boldsymbol{C}_y^{-1}$:

$$w_s(k) = \max_{1 \le m \le N-k} \left| [\boldsymbol{C}_y^{-1}]_{m,m+k} \right|, \tag{18}$$

where:

$$w_s(k) = \Delta_N^{-1} \sum_{m=0}^{N_k - 1} (N - k - 2m),\ N_k = \lceil (N-k)/2 \rceil, \tag{19}$$

and Δ_N is a real constant that depends on N, $|k| \le N-1$, and $w_{AP}^{(N)}(-k) = w_{AP}^{(N)}(k)$. With $[\cdot]_{i,j}$ we denote the (i,j)-th element of a matrix, and with $\lceil x \rceil$ we denote integer greater than or equal to x. We will call this WF arithmetic progressive (AP) because its k-th element is proportional to the sum of arithmetic sequence.

The unknown angle is estimated in (17) as an angle of the FT of the weighted autocorrelation function that is shifted by one sampling interval. When the iterative sample shifted estimator (ISSE) in (17) is initialised with $\varphi^{(0)}$ from (16), the resulting estimator has the same performance as the GWLP estimator. However one iteration of ISSE requires only $2(N-2)$ complex multiplications in addition to the processing in (12). In comparison to the GWLP, the estimation complexity of ISSE is reduced by having computationally demanding processing only in the initial step.

The unknown angle is estimated in [2] by using the normalised likelihood function as a probability density function (pdf), to obtain the mean value over all angle values. The algorithm for initial angle estimation needs to have reasonably low computational complexity, and the SNR threshold that is as low as possible. At high SNRs the initial angle estimates need to be accurate enough for the second iterative step to converge. To reduce the processing demands relative to [2], and improve the SNR threshold of the ISSE, we will use PG as a pseudo-pdf for initialisation. If the unitary discrete FT (DFT) of the autocorrelation function is given by:

$$\Phi(n) = \frac{1}{\sqrt{NN_{dft}}} \sum_{k=-(N-1)}^{N-1} w(k)r(k)e^{-j\frac{2\pi}{NN_{dft}}nk}, \tag{20}$$

where $N_{dft} \geq 2$ is the oversampling factor, the average PG (APG) initial estimate is given by:

$$\widetilde{\varphi}^{(0)} = \frac{\sum_{|n-n_{max}|\leq N_{min}/2} |\Phi(n)|n}{\sum_{|n-n_{max}|\leq N_{min}/2} |\Phi(n)|} \cdot \frac{2\pi}{NN_{dft}}, \tag{21}$$

where n_{max} denotes the approximate location of the $|\Phi(n)|$ maximum. With N_{min} we denote the minimum number of samples that correspond to the WF main-lobe width.

4 Angle invariant combiner of autocorrelation samples

By taking either the maximum or the angle of the same sum in (17) we obtain two different estimators, PG or ISSE, respectively. Both PG, and ISSE require exhaustive search of an angle that either maximises the PG or provides the combination of autocoreelation samples with the lowest angle error. We note that from (17) we can write:

$$\begin{aligned} \left| \sum_{k=-(N-2)}^{N-2} w_{AP}^{(N-1)}(k)r(k+1)e^{-jk\varphi} \right| &\leq w_{AP}^{(N-1)}(0)|r(1)| \\ &+ \sum_{k=1}^{N-2} w_{AP}^{(N-1)}(k) \left| r(k+1)e^{-jk\varphi} + r(1-k)e^{jk\varphi} \right|, \end{aligned} \tag{22}$$

from which we will try to replace the optimisation of a sum with respect to one parameter φ, with a set of simpler optimisations per sum components. The expression in (22) corresponds to the sample shift value of $n = 1$. It can be generalised for any $n \neq 1$, and $K = 1$, as follows:

$$q_s(n) = w_{AP}^{(N-n)}(0)r(n) + \sum_{k=1}^{N-n-1} w_{AP}^{(N-n)}(k)\beta_k^{(n)} \left[r(n+k)e^{j\alpha_k^{(n)}} + r(n-k)e^{-j\alpha_k^{(n)}} \right], \tag{23}$$

for $0 \le n \le N-1$, and $q_s(n) = q_s(-n)^*$ for $-N+1 \le n < 0$. Computational complexity of (23) can be further reduced if we use the same correction angle for all n as for $n = 0$, $\alpha_k^{(n)} = \alpha_k^{(0)} = \alpha_k$, and $\beta_k^{(n)} = \beta_k^{(0)} = \beta_k$, $\forall n \neq 0$. In this case, parameters α_k, and β_k are chosen such that:

$$(\alpha_k, \beta_k) = \arg\max_{\alpha \in \mathscr{A}_{N_a}, \beta=\pm 1} \beta \mathrm{Re}\left(r(k)e^{j\alpha}\right), \tag{24}$$

where $\mathscr{A}_{N_a} = \mathscr{A}_{N_a-1} \cup \{\pm\pi/2^{(N_a+1)}\}$, $\mathscr{A}_0 = \{0, \pi/2\}$, and integer $N_a \ge 0$. In this way we can substitute the problem of finding the value of $\widetilde{\varphi}^{(i)}$ in (17) over a continuous interval with the one where we search for the optimum value of multiple parameters α_k over a discrete set of values. The expression in (23) is AIC of autocorrelation samples because the sequence $q_s(n)$ is also complex exponential signal.

Similarly to the estimator in (16), which is based on the generalised weighted autocorrelation function, we can define the AIC estimator as:

$$\widetilde{\varphi} = \measuredangle q_s(1). \tag{25}$$

For $K = 1$ we can use angle invariant property of sequence $q_s(n)$ in optimisation in (15) to define the LS-q estimator as:

$$\widetilde{\varphi} = \measuredangle \sum_{k=0}^{N-2} w(k) q_s(k)^* q_s(k+1), \tag{26}$$

which we can use as an initial angle estimate for the ISSE-q estimator, that is obtained from (17) if we use $q_s(n)$ instead of $r(n)$. In the same way we can obtain the APG-q estimator if in (20), and (21) we use sequence $q_s(n)$.

For any $K \ge 1$ we introduce the following AIC sequence as:

$$q_m(n) = r(n) w_{AP}^{(N-n)}(0) + \sum_{k=1}^{N-n-1} w_{AP}^{(N-n)}(k) \left[r(n+k)r(k)^* + r(n-k)r(k)\right], \tag{27}$$

which is also a sum of complex exponential signals with the same angles as in (4).

5 Successive component suppression

For $K > 1$, in every step we use previous angle estimates to suppress all complex exponential signals except one. Let us introduce a Toeplitz-Hermitian matrix $\boldsymbol{Q} \in \mathbb{C}^{N \times N}$ as:

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{q}_N^{(0)} & \boldsymbol{q}_N^{(-1)} & \cdots & \boldsymbol{q}_N^{(-N+1)} \end{bmatrix}, \tag{28}$$

where

$$\boldsymbol{q}_N^{(l)} = \begin{bmatrix} q_m(l) & \cdots & q_m(l+N-1) \end{bmatrix}^T, \tag{29}$$

and we need $N \ge K$ to be able to resolve different complex exponential signals. Although the ESPRIT algorithm does not achieve neither the CRB nor the ML SNR threshold, we use it in combination with (28) for initial angle estimation because of its stable performance relative to the minimum angle separation, and its relatively low computational complexity in its original form. By using it with $\boldsymbol{Q}$ in (28) we achieve lower computational complexity, because the eigenvalue decomposition (EVD) of $\boldsymbol{Q}$ is proportional to $\mathscr{O}(N^2)$ operations per eigenmode, [15].

After the initial step, the successive component suppression (SCS) is performed in the $(i+1)$-th iteration, and for the k-th signal, from (6), by using:

$$\widetilde{\boldsymbol{Y}}_k^{(i+1)} = \left(\boldsymbol{I}_N - \widetilde{\boldsymbol{A}}_k^{(i)} \left(\widetilde{\boldsymbol{A}}_k^{(i)}\right)^H / N\right) \boldsymbol{Y}, \tag{30}$$

where

$$\widetilde{\boldsymbol{A}}_k^{(i)} = \left[\begin{array}{cccccc} \widetilde{\boldsymbol{a}}_{N,1}^{(i)} & \cdots & \widetilde{\boldsymbol{a}}_{N,k-1}^{(i)} & \widetilde{\boldsymbol{a}}_{N,k+1}^{(i)} & \cdots & \widetilde{\boldsymbol{a}}_{N,K}^{(i)} \end{array}\right], \tag{31}$$

and $\widetilde{\boldsymbol{a}}_{N,k}^{(i)}$ is an estimate of (5) with the estimate of $\widetilde{\varphi}_k^{(i)}$ from previous iteration. The signal observation matrix for the k-th signal $\widetilde{\boldsymbol{Y}}_k^{(i+1)}$ is then used to obtain the sequence $\widetilde{q}_{s,k}^{(i+1)}(n)$, $n = 0, \ldots, N-1$ from (12), and (23). Next, we can use any of the previously proposed algorithms, LS-q, APG-q, and ISSE-q, to estimate $\widetilde{\varphi}_k^{(i+1)}$ in the current iteration.

6 Numerical results

The signal parameters are chosen to be $N = 64$, and $M = 8$. We use Taylor-Villneuve WF with minimum side-lobe attenuation of -40dB, [1]. Signal amplitudes are modelled as random uniformly distributed quadrature amplitude modulated (QAM) signals. We assume that $\varphi_1 < \cdots < \varphi_K$. The minimum angle separation is fixed, and it is defined as a simulation parameter $\Delta\varphi_{min}$. The smallest angle value is uniformly distributed over interval $\varphi_1 \in [-\pi, \pi - (K-1)\Delta\varphi_{min}]$, and $\varphi_k = \varphi_1 + (k-1)\Delta\varphi_{min}$, $k > 1$. ISSE is stopped when either $|\varphi - \widetilde{\varphi}|/\varphi \leq 10^{-5}$ or if the maximum number of iterations $N_{ISSE} = 5$ is reached. We have noticed in simulations that for high SNRs we do not need more than one to two iterations. The maximum number of iterations for SCS is fixed to $N_{SCS} = 2$. The oversampling factor in (20) is set to $N_{dft} = 8$, and $N_{min} = 6$. In our simulations we used between 5000 and 10000 simulation runs. The computation of the sequence in (12) requires approximately $MN^2/2$ of complex multiply, and add operations.

In Fig. 1 we compare performance of various estimators for $K = 1$. The biggest gains for AIC relative to the LS-y estimator are obtained for $N_a = 0$, which also has the lowest computational complexity compared to the case when $N_a > 0$. For $N_a = 0$, the AIC estimator requires $2(N-2)$ complex add operations, and $(N-2)$ decisions based on real numbers, while LS-y requires $(N-2)$ complex add, and multiply operations. By using sequence $q_s(n)$ for $N_a = 0$, both LS-q, and ISSE-q that is initialised with LS-q (LS/ISSE-q) achieve much lower SNR threshold than the GWLP. Calculation of $q_s(n)$ for $N_a = 0$ requires $(N-2)(N-1)$ complex add operations, and $N-2$ tests of real numbers. In total, the LS-q requires $\mathscr{O}(MN^2 + N^2 + N)$ operations, as opposed to the GWLP that requires $\mathscr{O}(N)$ operations for

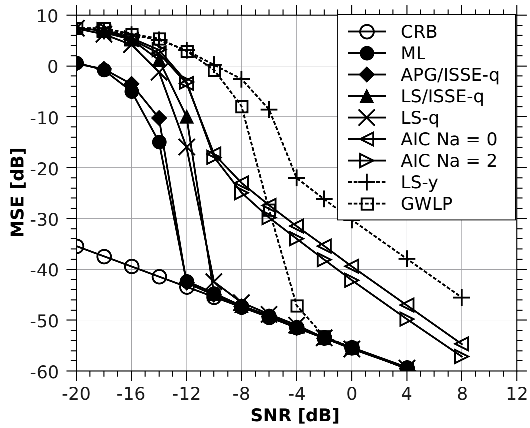

Fig. 1. The MSE as a function of SNR with AWGN when signal amplitude varies over snapshots for $K = 1$, $N = 64$, $M = 8$.

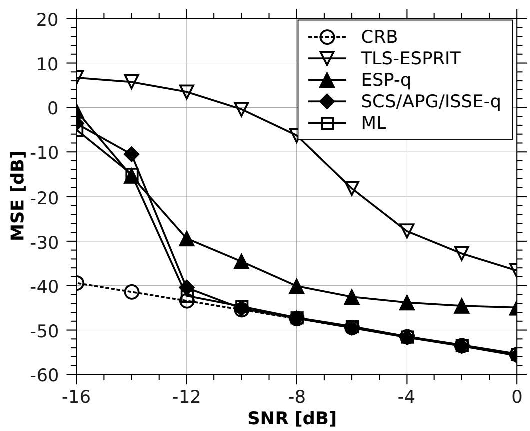

Fig. 2. The MSE as a function of SNR with AWGN for $N = 64$, $M = 8$, $K = 2$, and $\Delta\varphi_{min} = \pi/16$. Signal amplitudes change over snapshots.

initialisation, and $\mathcal{O}(M(N^2+N))$ operations per iteration. By initialising the ISSE-q with APG-q (APG/ISSE-q) we achieve the ML SNR threshold with $\mathcal{O}(MN^2+N^2+N\log N)$ operations for initialisation, and $\mathcal{O}(N)$ operations per iteration.

From Fig. 2 we see that by using the TLS-ESPRIT in combination with the matrix $\boldsymbol{Q}$ in (28) (ESP-q), we achieve the SNR threshold of the ML estimator. The estimates that are obtained by using the ESP-q are used to initialise the SCS. In total, calculation of SCS/APG/ISSE-q requires approximately $\mathcal{O}((M+K+1)N^2)$ operations for initialisation, and $\mathcal{O}(K(M+K+1)N^2)$ operations per iteration which is not excessive compared to either the EPUMA which requires $\mathcal{O}(N^3+MN^2)$ operations for initialisation, and approximately $\mathcal{O}(N^3+K^2N^2)$ operations per iteration [4], or [7] which requires $\mathcal{O}(MN^2+K^2N+K^3)$ operations for initialisation, and $\mathcal{O}(N^3K^3+N^2K^4+K^6)$ operations per iteration. The SCS/APG/ISSE-q achieves the CRB, and the SNR threshold of the ML estimator, and it is less computationally demanding.

7 Conclusions

In this paper, we have introduced two AIC sequences, novel ISSE, and APG for initialisation. They enable us to perform all of the computationally demanding processing only in the initialisation step, while each iteration has only linear complexity. When we have multiple complex exponential signals we successively estimate the angle of each signal separately, with the computational load that is lower than that of the comparable algorithms. At the same time we achieve the CRB, and the ML SNR threshold.

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