Coupled fixed point theorems on complex partial metric space using different type of contractive conditions

M. Gunaseelan, L. N. Mishra

Abstract: In this paper, we obtain coupled fixed point theorems on complex partial metric space using different type of contractive conditions. An example to support our result is presented.

Keywords: Coupled fixed point; complex partial metric space.

1 Introduction

In many branches of science, economics, computer science, engineering and the development of nonlinear analysis, the fixed point theory is one of the most important tool. In 1989, Backhtin [2] introduced the concept of b-metric space. In 1993, Czerwik [3] extended the results of b-metric spaces. Azam et al.[4] introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. P. Dhivya and M. Marudai [5] introduced new spaces called complex partial metric space and established the existence of common fixed point theorems under the contraction condition of rational expression. Bhaskar and Lakshmikantham [7] introduced the concept of coupled fixed point. Ćirić and Lakshmikantham [8] investigated some more coupled fixed point theorems in partially ordered sets. Hassen Aydi [1] introduced coupled fixed point results on partial metric spaces. In this paper, we introduced coupled fixed point results on complex partial metric spaces under the contractive condition.

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $c_1, c_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

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M. Gunaseelan is with the Department of Mathematics, Sri Sankara Arts and Science College, Enathur, Kanchipuram, India; L. N. Mishra is with the Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India.

$c_1 \preceq c_2$ if and only if $Re(c_1) \leq Re(c_2)$ and $Im(c_1) \leq Im(c_2)$.

Consequently, one can infer that $c_1 \preceq c_2$ if one of the following conditions is satisfied:

- (i) $Re(c_1) = Re(c_2), Im(c_1) < Im(c_2)$,
- (ii) $Re(c_1) < Re(c_2), Im(c_1) = Im(c_2)$,
- (iii) $Re(c_1) < Re(c_2), Im(c_1) < Im(c_2)$,
- (iv) $Re(c_1) = Re(c_2), Im(c_1) = Im(c_2)$.

In particular, we write $c_1 \not\preceq c_2$ if $c_1 \neq c_2$ and one of (i), (ii) and (iii) is satisfied and we write $c_1 \prec c_2$ if only (iii) is satisfied. Notice that

- (a) If $0 \preceq c_1 \not\preceq c_2$, then $|c_1| < |c_2|$,
- (b) If $c_1 \preceq c_2$ and $c_2 \prec c_3$ then $c_1 \prec c_3$,
- (c) If $a, b \in \mathbb{R}$ and $a \leq b$ then $ac \preceq bc$ for all $c \in \mathbb{C}$.

Definition 2.1. [5] A complex partial metric on a non-empty set U is a function $\xi_c : U \times U \rightarrow \mathbb{C}^+$ such that for all $p, r, s \in U$:

- (i) $0 \preceq \xi_c(p, p) \preceq \xi_c(p, r)$ (small self-distances)
- (ii) $\xi_c(p, r) = \xi_c(r, p)$ (symmetry)
- (iii) $\xi_c(p, p) = \xi_c(p, r) = \xi_c(r, r)$ if and only if $p = r$ (equality)
- (iv) $\xi_c(p, r) \preceq \xi_c(p, s) + \xi_c(s, r) - \xi_c(s, s)$ (triangularity).

A complex partial metric space is a pair (U, ξ_c) such that U is a non empty set and ξ_c is complex partial metric on U .

For the complex partial metric ξ_c on U , the function $d_{\xi_c} : U \times U \rightarrow \mathbb{C}^+$ given by $\xi_c^t = 2\xi_c(p, r) - \xi_c(p, p) - \xi_c(r, r)$ is a (usual) metric on U . Each complex partial metric ξ_c on U generates a topology τ_{ξ_c} on U with the base family of open ξ_c -balls $\{B_{\xi_c}(p, \varepsilon) : p \in U, \varepsilon > 0\}$, where $B_{\xi_c}(p, \varepsilon) = \{r \in U : \xi_c(p, r) < \xi_c(p, p) + \varepsilon\}$ for all $p \in U$ and $0 < \varepsilon \in \mathbb{C}^+$.

Definition 2.2. [5] Let (U, ξ_c) be a complex partial metric space (CPMS). A sequence (p_n) in a CPMS (U, ξ_c) is converges to $p \in U$, if for every $0 \prec \varepsilon \in \mathbb{C}^+$ there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we get $p_n \in B_{\xi_c}(p, \varepsilon)$

Definition 2.3. [5] Let (U, ξ_c) be a complex partial metric space. A sequence (p_n) in a CPMS (U, ξ_c) is called Cauchy if there is $a \in \mathbb{C}^+$ such that for every $\varepsilon \prec 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$ $|\xi_c(p_n, p_m) - a| < \varepsilon$.

Definition 2.4. [5] Let (U, ξ_c) be a complex partial metric space (CPMS).

- (1) A CPMS (U, ξ_c) is said to be complete if a Cauchy sequence (p_n) in U converges, with respect to τ_{ξ_c} , to a point $p \in U$ such that $\xi_c(p, p) = \lim_{n, m \rightarrow \infty} \xi_c(p_n, p_m)$.
- (2) A mapping $H : U \rightarrow U$ is said to be continuous at $p_0 \in U$ if for every $\varepsilon \prec 0$, there exist $\delta > 0$ such that $H(B_{\xi_c}(p_0, \delta)) \subset B_{\xi_c}(H(p_0, \varepsilon))$.

Lemma 2.1. [5] Let (U, ξ_c) be a complex partial metric space. A sequence $\{y_n\}$ is Cauchy sequence in the CPMS (U, ξ_c) then $\{y_n\}$ is Cauchy in a metric space (U, ξ_c^t) .

Definition 2.5. Let (U, ξ_c) be a complex partial metric space (CPMS). Then an element $(p, r) \in U \times U$ is said to be a coupled fixed point of the mapping $F : U \times U \rightarrow U$ if $F(p, r) = p$ and $F(r, p) = r$.

Theorem 2.2. Let (U, ξ_c) be a complete complex partial metric space. Suppose that the mapping $\phi : U \times U \rightarrow U$ satisfies the following contractive condition for all $\alpha, \beta, \gamma, \delta \in U$

$$\xi_c(\phi(\alpha, \beta), \phi(\gamma, \delta)) \preceq k\xi_c(\phi(\alpha, \beta), \alpha) + l\xi_c(\phi(\gamma, \delta), \gamma),$$

where k, l are nonnegative constants with $k + l < 1$. Then, ϕ has a unique coupled fixed point.

Proof. Choose $u_0, v_0 \in U$ and set $u_1 = \phi(u_0, v_0)$ and $v_1 = \phi(v_0, u_0)$. Continuing this process, set $u_{n+1} = \phi(u_n, v_n)$ and $v_{n+1} = \phi(v_n, u_n)$.

Then,

$$\begin{aligned} \xi_c(u_n, u_{n+1}) &= \xi_c(\phi(u_{n-1}, v_{n-1}), \phi(u_n, v_n)) \\ &\preceq k\xi_c(\phi(u_{n-1}, v_{n-1}), u_{n-1}) + l\xi_c(\phi(u_n, v_n), u_n) \\ &= k\xi_c(u_n, u_{n-1}) + l\xi_c(u_{n+1}, u_n) \\ \xi_c(u_n, u_{n+1}) &\preceq \frac{k}{1-l}\xi_c(u_n, u_{n-1}) \end{aligned}$$

which implies that

$$|\xi_c(u_n, u_{n+1})| \leq p|\xi_c(u_{n-1}, u_n)| \quad (1)$$

where $p = \frac{k}{1-l} < 1$. Similarly, one can prove that

$$|\xi_c(v_n, v_{n+1})| \leq p|\xi_c(v_{n-1}, v_n)| \quad (2)$$

From (1) and (2), we get

$$|\xi_c(u_n, u_{n+1})| + |\xi_c(v_n, v_{n+1})| \leq p(|\xi_c(u_{n-1}, u_n)| + |\xi_c(v_{n-1}, v_n)|)$$

where $p < 1$.

Also,

$$|\xi_c(u_{n+1}, v_{n+2})| \leq p|\xi_c(u_n, u_{n+1})| \quad (3)$$

$$|\xi_c(v_{n+1}, v_{n+2})| \leq p|\xi_c(v_n, v_{n+1})| \quad (4)$$

From (3) and (4), we get

$$|\xi_c(u_{n+1}, v_{n+2})| + |\xi_c(v_{n+1}, v_{n+2})| \leq p(|\xi_c(u_n, u_{n+1})| + |\xi_c(v_n, v_{n+1})|)$$

Repeating this way, we get

$$\begin{aligned} |\xi_c(u_n, v_{n+1})| + |\xi_c(v_n, v_{n+1})| &\leq p(|\xi_c(v_{n-1}, v_n)| + |\xi_c(u_{n-1}, u_n)|) \\ &\leq p^2(|\xi_c(v_{n-2}, v_{n-1})| + |\xi_c(u_{n-2}, u_{n-1})|) \\ &\leq \cdots \leq p^n(|\xi_c(v_0, v_1)| + |\xi_c(u_0, u_1)|) \end{aligned}$$

Now, if $|\xi_c(u_n, v_{n+1})| + |\xi_c(v_n, v_{n+1})| = t_n$, then

$$t_n \leq p t_{n-1} \leq \cdots \leq p^n t_0 \quad (5)$$

If $t_0 = 0$ then $|\xi_c(u_0, u_1)| + |\xi_c(v_0, v_1)| = 0$. Hence $u_0 = u_1 = \phi(u_0, v_0)$ and $v_0 = v_1 = \phi(v_0, v_0)$, which implies that (u_0, v_0) is a coupled fixed point of ϕ .

Let $t_0 > 0$. For each $n \geq m$, we have

$$\begin{aligned} \xi_c(u_n, u_m) &\preceq \xi_c(u_n, u_{n-1}) + \xi_c(u_{n-1}, u_{n-2}) - \xi_c(u_{n-1}, u_{n-1}) \\ &\quad + \xi_c(u_{n-2}, u_{n-3}) + \xi_c(u_{n-3}, u_{n-4}) - \xi_c(u_{n-3}, u_{n-3}) \\ &\quad + \cdots + \xi_c(u_{m+2}, u_{m+1}) + \xi_c(u_{m+1}, u_m) - \xi_c(u_{m+1}, u_{m+1}) \\ &\preceq \xi_c(u_n, u_{n-1}) + \xi_c(u_{n-1}, u_{n-2}) + \cdots + \xi_c(u_{m+1}, u_m) \end{aligned}$$

which implies that

$$|\xi_c(u_n, u_m)| \leq |\xi_c(u_n, u_{n-1})| + |\xi_c(u_{n-1}, u_{n-2})| + \cdots + |\xi_c(u_{m+1}, u_m)|.$$

Similarly, one can prove that

$$|\xi_c(v_n, v_m)| \leq |\xi_c(v_n, v_{n-1})| + |\xi_c(v_{n-1}, v_{n-2})| + \cdots + |\xi_c(v_{m+1}, v_m)|.$$

Thus,

$$\begin{aligned} |\xi_c(u_n, u_m)| + |\xi_c(v_n, v_m)| &\leq t_{n-1} + t_{n-2} + t_{n-3} + \cdots + t_m \\ &\leq (p^{n-1} + p^{n-2} + \cdots + p^m) t_0 \\ &\leq \frac{p^m}{1-p} t_0 \rightarrow 0 \quad n \rightarrow \infty. \end{aligned}$$

which implies that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequence in (U, ξ_c) . Since the partial metric space (U, ξ_c) is complete, there exists $u, v \in U$ such that $\{u_n\} \rightarrow u$ and $v_n \rightarrow v$ as $n \rightarrow \infty$ and $\xi_c(u, u) = \lim_{n \rightarrow \infty} \xi_c(u, u_n) = \lim_{n, m \rightarrow \infty} \xi_c(u_n, u_m) = 0$, $\xi_c(u, u) = \lim_{n \rightarrow \infty} \xi_c(v, v_n) = \lim_{n, m \rightarrow \infty} \xi_c(v_n, v_m) = 0$. We now show that $u = \phi(p, q)$. We suppose on the contrary that

$u \neq \phi(u, v)$ and $v \neq \phi(v, u)$ so that $0 \prec \xi_c(u, \phi(u, v)) = l_1$ and $0 \prec \xi_c(v, \phi(v, u)) = l_2$ then

$$\begin{aligned} l_1 &= \xi_c(u, \phi(u, v)) \preceq \xi_c(u, u_{n+1}) + \xi_c(u_{n+1}, \phi(u, v)) \\ &= \xi_c(u, u_{n+1}) + \xi_c(\phi(u_n, v_n), \phi(u, v)) \\ &\preceq \xi_c(u, u_{n+1}) + k\xi_c(u_{n-1}, u_n) + l\xi_c(\phi(u, v), u) \\ &\preceq \frac{1}{1-l}\xi_c(u, u_{n+1}) + \frac{k}{1-l}\xi(u_{n-1}, u_n) \end{aligned}$$

which implies that

$$|l_1| \leq \frac{1}{1-l}|\xi_c(u, u_{n+1})| + \frac{k}{1-l}|\xi(u_{n-1}, u_n)|$$

As $n \rightarrow \infty$, $|l_1| \leq 0$. Which is a contradiction, therefore $|\xi_c(u, \phi(u, v))| = 0$ implies $u = \phi(u, v)$. Similarly we can prove that $v = \phi(v, u)$. Thus (u, v) is a coupled fixed point of ϕ . Now, if (g, h) is another coupled fixed point of ϕ , then

$$\begin{aligned} \xi_c(u, g) &= \xi_c(\phi(u, v), \phi(g, h)) \preceq k\xi_c(\phi(u, v), u) + l\xi_c(\phi(g, h), g) \\ &= k\xi_c(u, u) + l\xi_c(g, g) = 0 \end{aligned}$$

Thus, we have $g = u$. Similarly, we get $h = v$. Therefore ϕ has a unique coupled fixed point. $\square$

Corollary 2.3. Let (U, ξ_c) be a complete complex partial metric space. Suppose that the mapping $\phi : U \times U \rightarrow U$ satisfies the following contractive condition for all $\alpha, \beta, \gamma, \delta \in U$

$$\xi_c(\phi(\alpha, \beta), \phi(\gamma, \delta)) \preceq \frac{k}{2}(\xi_c(\phi(\alpha, \beta), \alpha) + \xi_c(\phi(\gamma, \delta), \gamma)), \quad (6)$$

where $0 \leq k < 1$. Then, ϕ has a unique coupled fixed point.

Example 2.4. Let $U = [0, \infty)$ endowed with the usual complex partial metric $\xi_c : U \times U \rightarrow [0, \infty)$ defined by $\xi_c(p, q) = \max\{p, q\}(1 + i)$. The complex partial metric space (U, ξ_c) is complete because (U, ξ_c^t) is complete. Indeed, for any $p, q \in U$,

$$\begin{aligned} \xi_c^t &= 2\xi_c(p, r) - \xi_c(p, p) - \xi_c(r, r) \\ &= 2\max\{p, q\}(1 + i) - (p + ip) - (q + iq) \\ &= |p - q| + i|p - q|. \end{aligned}$$

Thus, (U, ξ_c) is the Euclidean complex metric space which is complete. Consider the mapping $\phi: U \times U \rightarrow U$ defined by $\phi(p, q) = \frac{p+q}{12}$. For any $p, q, g, h \in U$, we have

$$\begin{aligned}\xi_c(\phi(p, q), \phi(g, h)) &= \frac{1}{12} \max\{p + g, \phi(p, q) + \phi(g, h)\}(1 + i) \\ &\leq \frac{1}{12} [\max\{\phi(p, q), p\} + \max\{\phi(g, h), g\}](1 + i) \\ &= \frac{1}{12} [\xi_c(\phi(p, q), p) + \xi_c(\phi(g, h), g)].\end{aligned}$$

which is the contractive condition (6) for $k = \frac{1}{6}$. Therefore, by Corollary 2.3, and hence ψ has a unique coupled fixed point, which is $(0, 0)$. Note that if the mapping $\phi: U \times U \rightarrow U$ is given by $\phi(p, q) = \frac{p+q}{2}$, then ϕ satisfies the contractive condition (6) for $k = 1$, that is,

$$\begin{aligned}\xi_c(\phi(p, q), \phi(g, h)) &= \frac{1}{2} \max\{p + g, \phi(p, q) + \phi(g, h)\}(1 + i) \\ &\leq [\max\{\phi(p, q), p\} + \max\{\phi(g, h), g\}](1 + i) \\ &= \frac{1}{2} [\xi_c(\phi(p, q), p) + \xi_c(\phi(g, h), g)].\end{aligned}$$

In this case, $(0, 0)$ and $(1, 1)$ are both coupled fixed points of ϕ , and, hence, the coupled fixed point of ϕ is not unique. This shows that the condition $k < 1$ in Corollary 2.3, and hence $k + l < 1$ in Theorem 2.2 cannot be omitted in the statement of the aforesaid results.

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