SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 12, 1 (2020), 29-35.

Some remarks on general sum–connectivity coindex

M. Matejić, E. Milovanović, I. Milovanović

Abstract: Let $G = (V,E)$, $V = \{v_1, v_2, \dots, v_n\}$ be a simple connected graph with n vertices, m edges and a sequence of vertex degrees $d_1 \ge d_2 \ge \cdots \ge d_n > 0$, $d_i = d(v_i)$. The general sum–connectivity coindex is defined as $\overline{H}_\alpha(G) = \sum_{i \nsim j}(d_i + d_j)^\alpha$, while multiplicative first Zagreb coindex is defined as $\overline{\Pi}_1(G) = \prod_{i \nsim j}(d_i + d_j)$. Here α is an arbitrary real number, and $i \nsim j$ denotes that vertices i and j are not adjacent. Some relations between $\overline{H}_\alpha(G)$ and $\overline{\Pi}_1(G)$ are obtained.

Keywords: Topological indices and coindices, sum–connectivity coindex, multiplicative Zagreb coindex.

1 Introduction

Let $G = (V,E)$, $V = \{v_1, v_2, \dots, v_n\}$, $E = \{e_1, e_2, \dots, e_m\}$, be a simple connected graph with $n = |V|$ vertices and $m = |E|$ edges. With $d_1 \ge d_2 \ge \cdots \ge d_n > 0$, $d_i = d(v_i)$, a sequence of vertex degrees of G is designated. If vertices v_i and v_j are adjacent, we write $i \sim j$, otherwise we write $i \nsim j$. We define values $\overline{\Delta}_e$ and $\overline{\delta}_e$ as

$$\overline{\Delta}_e = \max_{i \nsim j}\{d_i + d_j\} \quad \text{and} \quad \overline{\delta}_e = \min_{i \nsim j}\{d_i + d_j\}.$$

A topological index of a graph is a numerical quantity which is invariant under automorphisms of the graph.

Two vertex-degree based topological indices, the first and the second Zagreb index, M_1 and M_2, are defined as [7, 8]

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2 = M_2(G) = \sum_{i \sim j} d_i d_j.$$

Manuscript received December 21, 2019; accepted February 14, 2020.

M. M. Matejić, E. I. Milovanović, I. Ž. Milovanović are with the Faculty of Electronic Engineering, Niš, Serbia

As shown in [12], the first Zagreb index can be also expressed as

$$M_1 = \sum_{i\sim j}(d_i + d_j)\,.$$

A so-called forgotten topological index, F, is defined as [6]

$$F = F(G) = \sum_{i=1}^{n} d_i^3\,.$$

By analogy to M_1, the invariant F can be written in the following way

$$F = \sum_{i\sim j}(d_i^2 + d_j^2)\,.$$

The general sum–connectivity index was conceived in [17] as

$$H_\alpha(G) = \sum_{i\sim j}(d_i + d_j)^\alpha\,,$$

where α is an arbitrary real number. Some special cases of this index are the first Zagreb index $M_1(G) = H_1(G)$, the harmonic index $H(G) = 2H_{-1}(G)$ [5], the sum–connectivity index $SC(G) = H_{-1/2}(G)$ [18], and hyper–Zagreb index $HM(G) = H_2(G)$ [13]. It is not difficult to see that

$$HM(G) = \sum_{i\sim j}(d_i + d_j)^2 = F(G) + 2M_2(G)\,.$$

In [4] a concept of coindices was introduced. In this case the sum runs over the edges of the complement of G. Thus, the first and the second Zagreb coindices are defined as [4]

$$\overline{M}_1(G) = \sum_{i\nsim j}(d_i + d_j) \quad \text{and} \quad \overline{M}_2(G) = \sum_{i\nsim j} d_i d_j\,,$$

and the forgotten Zagreb coindex as [3] (see also [10]) as

$$\overline{F}(G) = \sum_{i\nsim j}(d_i^2 + d_j^2)\,.$$

The general sum–connectivity coindex was defined in [14] as

$$\overline{H}_\alpha(G) = \sum_{i\nsim j}(d_i + d_j)^\alpha\,,$$

where α is an arbitrary real number. Again, some special cases of $\overline{H}_\alpha(G)$ are apart from $\overline{M}_1(G)$, the sum–connectivity coindex $\overline{SC}(G) = \overline{H}_{-1/2}(G)$, the harmonic coindex $\overline{H}(G) =$

$2\overline{H}_{-1}(G)$, the hyper Zagreb coindex $\overline{HM}(G) = \overline{H}_2(G)$ [15]. It is not difficult to see that the following identity holds

$$\overline{HM}(G) = \overline{F}(G) + 2\overline{M}_2(G)\,.$$

The multiplicative first Zagreb coindex was defined in [16] as

$$\overline{\Pi}_1(G) = \prod_{i \nsim j}(d_i + d_j)\,.$$

In this paper we determine the bound for the difference

$$\overline{H}_\alpha(G) - \bar{m}(\overline{\Pi}_1(G))^{\alpha/\overline{m}}\,,$$

where $\bar{m} = \frac{n(n-1)}{2} - m$.

2 Preliminaries

In this section we recall some analytical inequalities for the real number sequences that will be used in the subsequent considerations.

Let $a = (a_i)$ and $b = (b_i)$, $i = 1,2,\ldots,n$, be positive real number sequences with the properties

$$0 < r_1 \le a_i \le R_1 < +\infty \quad \text{and} \quad 0 < r_2 \le b_i \le R_2 < +\infty\,.$$

In [1] (see also [11]) the following inequality was proven

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \le n^2\gamma(n)(R_1 - r_1)(R_2 - r_2)\,, \tag{1}$$

where

$$\gamma(n) = \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor\right) = \frac{1}{4}\left(1 - \frac{(-1)^{n+1}+1}{2n^2}\right).$$

Equality holds if and only if $R_1 = a_1 = \cdots = a_n = r_1$ or $R_2 = b_1 = \cdots = b_n = r_2$.

For the positive real number sequence $a = (a_i)$, $i = 1,2,\ldots,n$, the following inequality was proven in [9]

$$\left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 \le (n-1)\sum_{i=1}^{n} a_i + n\left(\prod_{i=1}^{n} a_i\right)^{1/n}\,, \tag{2}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

For the positive real number sequence $a = (a_i)$, $i = 1,2,\ldots,n$, with the property $a_1 \ge a_2 \ge \cdots \ge a_n > 0$, in [2] the following inequality was proven

$$\sum_{i=1}^{n} a_i - n\left(\prod_{i=1}^{n} a_i\right)^{1/n} \ge \left(\sqrt{a_1} - \sqrt{a_n}\right)^2\,. \tag{3}$$

Equality holds if $a_2 = a_3 = \cdots = a_{n-1} = \sqrt{a_1 a_n}$.

3 Main results

In the next theorem we establish lower and upper bounds for the difference $\overline{H}_\alpha(G) - \overline{m}(\overline{\Pi}_1(G))^{\alpha/\overline{m}}$ depending on the parameters α, $\overline{m}$, $\overline{\Delta}_e$ and $\overline{\delta}_e$.

Theorem 1. *Let G be a simple graph with $m \geq 2$ edges. If $\alpha \geq 0$, then*

$$\left(\overline{\Delta}_e^{\frac{\alpha}{2}} - \overline{\delta}_e^{\frac{\alpha}{2}}\right)^2 \leq \overline{H}_\alpha(G) - \overline{m}\left(\overline{\Pi}_1(G)\right)^{\alpha/\overline{m}} \leq \overline{m}^2\gamma(\overline{m})\left(\overline{\Delta}_e^{\frac{\alpha}{2}} - \overline{\delta}_e^{\frac{\alpha}{2}}\right)^2 . \tag{4}$$

If $\alpha \leq 0$, $G \ncong K_n$, then

$$\left(\overline{\delta}_e^{\frac{\alpha}{2}} - \overline{\Delta}_e^{\frac{\alpha}{2}}\right)^2 \leq \overline{H}_\alpha(G) - \overline{m}\left(\overline{\Pi}_1(G)\right)^{\alpha/\overline{m}} \leq \overline{m}^2\gamma(\overline{m})\left(\overline{\delta}_e^{\frac{\alpha}{2}} - \overline{\Delta}_e^{\frac{\alpha}{2}}\right)^2 .$$

Equality on the left–hand side holds if $\alpha = 0$, or $d_i + d_j = \sqrt{\overline{\Delta}_e\overline{\delta}_e}$, for any pair of nonadjacent vertices of G. Equality on the right–hand side holds if and only if $\alpha = 0$ or $d_i + d_j$ is a constant for any pair of non adjacent vertices of G.

Proof. For $\alpha \geq 0$, $n := \overline{m}$, $a_i = b_i := (d_i + d_j)^{\frac{\alpha}{2}}$, $R_1 = R_2 = \overline{\Delta}_e^{\frac{\alpha}{2}}$, $r_1 = r_2 = \overline{\delta}^{\frac{\alpha}{2}}$, with summation performed over all non adjacent vertices of G, the inequality (1) becomes

$$\overline{m}\sum_{i\nsim j}(d_i + d_j)^\alpha - \left(\sum_{i\nsim j}(d_i + d_j)^{\frac{\alpha}{2}}\right)^2 \leq \overline{m}^2\gamma(\overline{m})\left(\overline{\Delta}_e^{\frac{\alpha}{2}} - \overline{\delta}_e^{\frac{\alpha}{2}}\right)^2 ,$$

that is

$$\overline{m}\overline{H}_\alpha(G) - \left(\sum_{i\nsim j}(d_i + d_j)^{\frac{\alpha}{2}}\right)^2 \leq \overline{m}^2\gamma(\overline{m})\left(\overline{\Delta}_e^{\frac{\alpha}{2}} - \overline{\delta}_e^{\frac{\alpha}{2}}\right)^2 . \tag{5}$$

For $\alpha \geq 0$, $n := \overline{m}$, $a_i := (d_i + d_j)^\alpha$, where summation is performed over all pairs of non adjacent vertices of G, the inequality (2) transforms into

$$\left(\sum_{i\nsim j}(d_i + d_j)^{\frac{\alpha}{2}}\right)^2 \leq (\overline{m} - 1)\sum_{i\nsim j}(d_i + d_j)^\alpha + \overline{m}\left(\prod_{i\nsim j}(d_i + d_j)^\alpha\right)^{1/\overline{m}} ,$$

that is

$$\left(\sum_{i\nsim j}(d_i + d_j)^{\frac{\alpha}{2}}\right)^2 \leq (\overline{m} - 1)\overline{H}_\alpha(G) + \overline{m}\left(\overline{\Pi}_1(G)\right)^{\alpha/\overline{m}} . \tag{6}$$

Now from (5) and (6) we obtain right-hand side of (4). Equalities in (5) and (6), and consequently in the right-hand side of (4), hold if and only if $\alpha = 0$ or $d_i + d_j$ is a constant for any pair of non adjacent vertices of G.

For $\alpha \geq 0$, $n := \overline{m}$, $a_i := (d_i + d_j)^\alpha$, $a_1 := \overline{\Delta}_e^\alpha$, $a_n := \overline{\delta}_e^\alpha$, with summation performed over all pairs of non adjacent vertices, the inequality (3) becomes

$$\sum_{i \nsim j} (d_i + d_j)^\alpha - \overline{m} \left(\prod_{i \nsim j} (d_i + d_j)^\alpha \right)^{1/\overline{m}} \geq \left(\overline{\Delta}_e^{\frac{\alpha}{2}} - \overline{\delta}_e^{\frac{\alpha}{2}} \right)^2 , \tag{7}$$

from which left–hand part of (4) is obtained. Equality in (7), and consequently in (4), holds if $\alpha = 0$ or $d_i + d_j = \sqrt{\overline{\Delta}_e \overline{\delta}_e}$ for any pair of non adjacent vertices of G.

The case $\alpha \leq 0$ is proved analogously, thus omitted. □

Since for any $\overline{m}$ holds $\gamma(\overline{m}) \leq \frac{1}{4}$, we have the next corollary of Theorem 1.

Corollary 1. *Let G be a simple graph with $m \geq 2$ edges. If $\alpha \geq 0$, then*

$$\overline{H}_\alpha(G) - \overline{m} \left(\overline{\Pi}_1(G) \right)^{\alpha/\overline{m}} \leq \frac{\overline{m}^2}{4} \left(\overline{\Delta}_e^{\frac{\alpha}{2}} - \overline{\delta}_e^{\frac{\alpha}{2}} \right)^2 .$$

If $\alpha \leq 0$ and $G \ncong K_n$, then

$$\overline{H}_\alpha(G) - \overline{m} \left(\overline{\Pi}_1(G) \right)^{\alpha/\overline{m}} \leq \frac{\overline{m}^2}{4} \left(\overline{\delta}_e^{\frac{\alpha}{2}} - \overline{\Delta}_e^{\frac{\alpha}{2}} \right)^2 .$$

Equalities hold if and only if $\alpha = 0$, or $d_i + d_j$ is a constant for any pair of non adjacent vertices of G.

For some specific values of parameter α the following inequalities are obtained.

Corollary 2. *Let G, $G \ncong K_n$, be a simple graph with $m \geq 2$ edges. Then we have*

$$\begin{aligned}
\left(\frac{\sqrt{\overline{\Delta}_e} - \sqrt{\overline{\delta}_e}}{\sqrt{\overline{\Delta}_e \overline{\delta}_e}} \right)^2 &\leq \frac{1}{2} \overline{H}(G) - \overline{m} \left(\overline{\Pi}_1(G) \right)^{-1/\overline{m}} \leq \overline{m}^2 \gamma(\overline{m}) \left(\frac{\sqrt{\overline{\Delta}_e} - \sqrt{\overline{\delta}_e}}{\sqrt{\overline{\Delta}_e \overline{\delta}_e}} \right)^2 \leq \\
&\leq \frac{\overline{m}^2}{4} \left(\frac{\sqrt{\overline{\Delta}_e} - \sqrt{\overline{\delta}_e}}{\sqrt{\overline{\Delta}_e \overline{\delta}_e}} \right)^2 , \\
\left(\frac{\sqrt[4]{\overline{\Delta}_e} - \sqrt[4]{\overline{\delta}_e}}{\sqrt[4]{\overline{\Delta}_e \overline{\delta}_e}} \right)^2 &\leq \overline{SC}(G) - \overline{m} \left(\overline{\Pi}_1(G) \right)^{-1/(2\overline{m})} \leq \overline{m}^2 \gamma(\overline{m}) \left(\frac{\sqrt[4]{\overline{\Delta}_e} - \sqrt[4]{\overline{\delta}_e}}{\sqrt[4]{\overline{\Delta}_e \overline{\delta}_e}} \right)^2 \leq \\
&\leq \frac{\overline{m}^2}{4} \left(\frac{\sqrt[4]{\overline{\Delta}_e} - \sqrt[4]{\overline{\delta}_e}}{\sqrt[4]{\overline{\Delta}_e \overline{\delta}_e}} \right)^2 , \\
\left(\sqrt{\overline{\Delta}_e} - \sqrt{\overline{\delta}_e} \right)^2 &\leq \overline{M}_1(G) - \overline{m} \left(\overline{\Pi}_1(G) \right)^{1/\overline{m}} \leq \overline{m}^2 \gamma(\overline{m}) \left(\sqrt{\overline{\Delta}_e} - \sqrt{\overline{\delta}_e} \right)^2 \leq \\
&\leq \frac{\overline{m}^2}{4} \left(\sqrt{\overline{\Delta}_e} - \sqrt{\overline{\delta}_e} \right)^2 ,
\end{aligned}$$

$$\left(\overline{\Delta}_e-\overline{\delta}_e\right)^2 \leq \overline{HM}(G)-\overline{m}\left(\overline{\Pi}_1(G)\right)^{2/\overline{m}} \leq \overline{m}^2\gamma(\overline{m})\left(\overline{\Delta}_e-\overline{\delta}_e\right)^2 \leq \frac{\overline{m}^2}{4}\left(\overline{\Delta}_e-\overline{\delta}_e\right)^2 .$$

Equalities in the left-hand sides of the above inequalities hold if $d_i+d_j=\sqrt{\overline{\Delta}_e\overline{\delta}_e}$ for any pair of non–adjacent vertices v_i and v_j of G. Equalities in the right–hand sides of the above inequalities hold if and only if d_i+d_j is constant for any pair of non–adjacent vertices v_i and v_j of G.

Since $2\overline{F}(G)\geq\overline{HM}(G)=\overline{F}(G)+2\overline{M}_2(G)\geq 4\overline{M}_2(G)$, the following is valid.

Corollary 3. *Let G be a simple graph with $m\geq 2$ edges. Then*

$$4\overline{M}_2(G)-\overline{m}\left(\overline{\Pi}_1(G)\right)^{2/\overline{m}} \leq \overline{m}^2\gamma(\overline{m})\left(\overline{\Delta}_e-\overline{\delta}_e\right)^2 \leq \frac{\overline{m}^2}{4}\left(\overline{\Delta}_e-\overline{\delta}_e\right)^2 ,$$

$$2\overline{F}(G)-\overline{m}\left(\overline{\Pi}_1(G)\right)^{2/\overline{m}} \geq \left(\overline{\Delta}_e-\overline{\delta}_e\right)^2 .$$

Equalities hold if and only if $d_i=d_j$ for any pair of non adjacent vertices of G.

References

[1] M. BIERNACKI, H. PIDEK, C. RYLL–NARDZEWSKI, *Sur une inequality des integralles*, UNIV. MARIE CURIE–SKLODOWSKA, A4 (1950) 1–4.

[2] V. CIRTOAJE, *The best lower bound depended on two fixed variables for Jensen's inequality with order variables*, J. INEQ. APPL. 2010 (2010)# 12858.

[3] N. DE, S. M. A. NAYEEM, A. PAL, *The F–coindex of some graph operations*, SPRINGERPLUS, 5 (2016) ARTICLE 221.

[4] T. DOŠLIĆ, *Vertex–weighted Wiener polynomials for composite graphs*, ARS MATH. CONTEMP. 1 (2008) 66–80.

[5] S. FAJTLOWICZ, *On conjectures of Graffiti-II*, CONGR. NUMER. **60** (1987) 187–197.

[6] B. FURTULA, I. GUTMAN, *A forgotten topological index*, J. MATH. CHEM. 53 (2015) 1184–1190.

[7] I. GUTMAN, N. TRINAJSTIĆ, *Graph theory and molecular orbitals. Total π–electron energy of alternant hydrocarbons*, CHEM. PHYS. LETT. 17 (1972) 535–538.

[8] I. GUTMAN, B. RUŠČIĆ, N. TRINAJSTIĆ, C. F. WILCOX, *Graph theory and molecular orbitals. XII. Acyclic polyenes*, J. CHEM. PHYS. 62 (1975) 3399–3405.

[9] H. KOBER, *On the arithmetic and geometric means and on Hölder's inequality*, PROC. AMER. MATH. SOC. 9 (1958) 452–459.

[10] T. MANSOUR, C. SONG, *The a and (a,b)–analogs of Zagreb indices and coindices of graphs*, INT. J. COMBINATRORICS 2012 (2012) ARTICLE ID909285.

[11] D. S. MITRINOVIĆ, P. M. VASIĆ, *Analytic inequalities*, SPRINGER VERLAG, BERLIN–HEIDELBERG–NEW YORK, 1970.

[12] S. Nikolić, G. Kovačević, A. Milićević, N. Trinajstić, *The Zagreb indices 30 years after*, Croat. Chem. Acta 76 (2003) 113–124.

[13] G. H. Shirdel, H. Rezapour, A. M. Sayad, *The hyper – Zagreb index of graph operations*, Iran. J. Math. Chem. 4 (2013) 213–220.

[14] G. Su, L. Xu, *On the general sum–connectivity co–index of graphs*, Iran. J. Math. Chem. 2 (1) (2011) 89–98.

[15] M. Veylaki, M. J. Nikmehr, *The third and hyper–Zagreb coindex of some graph operations*, J. Appl. Math. Comput. 50 (2016) 315–325.

[16] K. Xu, K. C. Das, K. Tang, *On the multiplicative Zagreb coindex of graphs*, Opuscula Math. 33 (1) (2013) 191–204.

[17] B. Zhou, N. Trinajstić, *On general sum–connectivity index*, J. Math. Chem. 47, (2010) 210–218.

[18] B. Zhou, N. Trinajstić, *On a novel connectivity index*, J. Math. Chem. 46 (2009) 1252–1270.