Selective Properties of Fuzzy 2-Metric Spaces

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Abstract: We introduce and study some selective covering properties in fuzzy 2-metric spaces. These properties are related to the classical covering properties of Menger, Hurewicz and Rothberger which are well known in selection principles theory.

Keywords: F-2-Menger bounded, F-2-Hurewicz bounded, F-2-Rothberger bounded, game theory

1 Introduction

In this paper we study some topological properties of fuzzy 2-metric spaces related to the classical covering properties of Menger, Hurewicz and Rothberger (for these properties see the survey articles [7, 11]). Recall that a topological space has the Menger (resp., Hurewicz) covering property if for each sequence \((\mathcal{F}_n)_{n \in \mathbb{N}}\) of open covers of \(X\) there is a sequence \((\mathcal{V}_n)_{n \in \mathbb{N}}\) such that for each \(n\), \(\mathcal{V}_n\) is a finite subset of \(\mathcal{F}_n\) and \(X = \bigcup_{n \in \mathbb{N}} \bigcup \mathcal{F}_n\) (resp., each \(x \in X\) belongs to \(\bigcup \mathcal{F}_n\) for all but finitely many \(n\)). \(X\) has the Rothberger property if for each sequence \((\mathcal{F}_n)_{n \in \mathbb{N}}\) of open covers of \(X\) there are \(U_n \in \mathcal{F}_n\), \(n \in \mathbb{N}\), such that \(X = \bigcup_{n \in \mathbb{N}} U_n\).

In the 1960s, Gähler introduced the notion of 2-metric space [5, 6]. Let \(X\) be a non-empty set and let \(d : X \times X \times X \to \mathbb{R}\) be a mapping satisfying the following conditions:

1. For every pair of distinct points \(x, y \in X\) there exists a point \(z \in X\) such that \(d(x, y, z) \neq 0\);
2. \(d(x, y, z) = 0\) only if at least two of three points are the same;
3. \(d(x, y, z) = d(x, z, y) = d(y, x, z) = d(z, x, y) = d(z, y, x)\) for all \(x, y, z \in X\) (the symmetry);
4. \(d(x, y, z) \leq d(w, y, z) + d(x, w, z) + d(x, y, w)\) for all \(x, y, z, w \in X\) (the tetrahedral inequality).

Then \(d\) is called a 2-metric on \(X\) and \((X, d)\) is called a 2-metric space.

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In this paper, we assume that all 2-metric spaces have at least three distinct points. Observe also that every 2-metric is non-negative.

A typical example of 2-metric spaces is the Euclidean plane \( \mathbb{R}^2 \) with the 2-metric \( d \) defined as the area of the triangle spanned by points \( x, y, z \in \mathbb{R}^2 \). Another such example is \( \mathbb{R}^3 \) with \( d(x, y, z) = \min\{|x-y|,|y-z|,|z-x|\} \). Nowadays there is the large literature on 2-metric spaces and their modifications, mainly in fixed point theory (see, for example, \([1, 2, 3, 4, 9, 10, 12]\)).

To define fuzzy 2-metric spaces we need the well-known notion of triangular or \( t \)-norm.

**Definition 1.1** ([14]) A binary operation \( * : [0, 1] \times [0, 1] \to [0, 1] \) is a **continuous \( t \)-norm** if the following conditions are satisfied:

1. \( * \) is commutative and associative;
2. \( * \) is continuous;
3. \( a * 1 = a \) for all \( a \in [0, 1] \);
4. \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \), \( (a, b, c, d \in [0, 1]) \).

**Definition 1.2** ([13]) A 3-tuple \( (X, M, *) \) is said to be a **fuzzy 2-metric space** if \( X \) is an arbitrary nonempty set, \( * \) is a continuous \( t \)-norm, and \( M \) is a fuzzy set on \( X^3 \times (0, \infty) \) satisfying \((x, y, z \in X, t, t_1, t_2, t_3 \in (0, \infty)) \) the following conditions:

\( (F2M.1) \) given distinct elements \( x, y \in X \) there is an element \( z \in X \) such that \( M(x, y, z, t) > 0 \) for each \( t > 0 \);

\( (F2M.2) M(x, y, z, t) = 1 \) if at least two of \( x, y, z \) are equal;

\( (F2M.3) M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t) \) for all \( x, y, z \in X \) and all \( t > 0 \);

\( (F2M.4) M(x, y, z, t_1 + t_2 + t_3) \geq M(x, z, w, t_1) * M(x, w, z, t_2) * M(w, y, z, t_3) \);

\( (F2M.5) M(x, y, z, \cdot) : (0, \infty) \to (0, 1] \) is a continuous function.

The pair \((M, *)\) (or only \( M \)) is called a **2-fuzzy metric** on \( X \).

The following is a typical example of a fuzzy 2-metric.

**Example 1.3** Let \((X, d)\) be a 2-metric space. Then the mapping \( M_d : X^3 \times (0, \infty) \to [0, 1] \) defined by

\[
M_d(x, y, z, t) = \frac{t}{t + d(x, y, z)}, \quad (x, y, z \in X, t > 0)
\]

is a fuzzy 2-metric on \( X \) induced by the 2-metric \( d \).

Let \((X, M, *)\) be a fuzzy 2-metric space, \( x \in X, S \subset X, r \in (0, 1), t > 0 \). The set

\[
B(x, r, t) = \{ y \in X : M(x, y, z, t) > 1 - r \text{ for each } z \in X \}
\]

is called the **open ball** with center \( x \) and radius \( r \) with respect to \( t \).

The collection of all open balls with center \( x \), \( x \in X \), is a base for a topology on \((X, M, *)\), denoted by \( \tau_M \).

We also define

\[
B(S, \varepsilon, t) := \bigcup_{x \in S} B(x, \varepsilon, t).
\]
2 Results

In this section we define and study some covering properties of fuzzy 2-metric spaces.

**Definition 2.1** A fuzzy 2-metric space \((X,M,\ast)\) is said to be:

- \(\text{FM}_2\): \(\text{-Menger-bounded}\) (or \(\text{FM}_2\)-bounded);
- \(\text{FR}_2\): \(\text{-Rothberger-bounded}\) (or \(\text{FR}_2\)-bounded);
- \(\text{FH}_2\): \(\text{-Hurewicz-bounded}\) (or \(\text{FH}_2\)-bounded)

if for each sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) of elements of \((0,1)\) and each \(t > 0\) there is a sequence

- \(\text{FM}_2\): \((A_n)_{n \in \mathbb{N}}\) of finite subsets of \(X\) such that \(X = \bigcup_{n \in \mathbb{N}} \bigcup_{a \in A_n} B(a, \varepsilon_n, t)\);
- \(\text{FR}_2\): \((x_n)_{n \in \mathbb{N}}\) of elements of \(X\) such that \(X = \bigcup_{n \in \mathbb{N}} B(x_n, \varepsilon_n, t)\);
- \(\text{FH}_2\): \((A_n)_{n \in \mathbb{N}}\) of finite subsets of \(X\) such that for each \(x \in X\) there is \(n_0 \in \mathbb{N}\) such that \(x \in \bigcup_{a \in A_n} B(a, \varepsilon_n, t)\) for all \(n \geq n_0\).

Recall that a fuzzy 2-metric space is said to be \(\text{fuzzy 2-precompact}\) (respectively, \(\text{fuzzy 2-pre-Lindel"of}\)) if for every \(\varepsilon \in (0,1)\) and every \(t > 0\) there is a finite (respectively, countable) set \(A \subset X\) such that \(X = \bigcup_{a \in A} B(a, \varepsilon, t)\).

Evidently,

\[\text{F-2-precompact} \Rightarrow \text{FH}_2-\text{bounded} \Rightarrow \text{FM}_2-\text{bounded} \Rightarrow \text{F-2-pre-Lindel"of}\]

and

\[\text{FR}_2-\text{bounded} \Rightarrow \text{FM}_2-\text{bounded}.\]

**Example 2.2** Let \((X,d)\) be a 2-metric space with the Menger property (with respect to the topology \(\tau_d\)). Then the induced fuzzy 2-metric space \((X,M_d,\ast)\) with \(\ast = \cdot\) (the product \(t\)-norm) is \(\text{FM}_2\)-bounded.

Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a sequence of positive real numbers and \(t > 0\). Applied the fact that \((X,d)\) has the Menger covering property to the sequence \((\%_n)_{n \in \mathbb{N}}, \%_n = \{B(x, \varepsilon_n) : x \in X\}\), to find a sequence \((A_n)_{n \in \mathbb{N}}\) of finite subsets of \(X\) such that

\[X = \bigcup_{n \in \mathbb{N}} \bigcup_{a \in A_n} B(a, \varepsilon_n).\]

Let \(x \in X\). There is \(k \in \mathbb{N}\) and a point \(a_k \in A_k\) such that \(x \in B(a_k, \varepsilon_k)\), i.e. \(\sup_{z \in X} d(a_k, x, z) < \varepsilon_k\). Then for all \(z \in X\) we have

\[M_d(x,a_k,z,t) = \frac{t}{t + d(x,a_k,z)} > \frac{t}{t + \varepsilon_k} = 1 - \frac{\varepsilon_k}{t + \varepsilon_k} > 1 - \varepsilon_k.\]

Therefore, \(x \in B(a_k, \varepsilon_k, t)\), i.e. \(X = \bigcup_{n \in \mathbb{N}} \bigcup_{a \in A_n} B(a, \varepsilon_n, t)\). This means that \((X,M_d,\ast)\) is \(\text{FM}_2\)-bounded.
Similarly we prove: If a 2-metric space \((X, d)\) has the Hurewicz (Rothberger) property, then the induced fuzzy 2-metric space \((X, M_d, \ast)\) with \(\ast = \cdot\) is FH\(_2\)-bounded (FR\(_2\)-bounded).

**Theorem 2.4** For a fuzzy 2-metric space \((X, M, \ast)\) the following are equivalent:

(a) For each sequence \((\varepsilon_n)_{n \in \mathbb{N}} \subset (0, 1)\) and each \(t > 0\) there is a sequence \((A_n)_{n \in \mathbb{N}}\) of finite subsets of \(X\) such that each finite subset \(F \subset X\) is contained in \(B(A_n, \varepsilon_n, t)\) for some \(A_n\);

(b) For each sequence \((\varepsilon_n)_{n \in \mathbb{N}} \subset (0, 1)\) and each \(t > 0\) there is a sequence \((A_n)_{n \in \mathbb{N}}\) of finite subsets of \(X\) and an increasing sequence \(n_1 < n_2 < \cdots\) of natural numbers such that each finite subset \(F \subset X\) is contained in \(\bigcup_{n_k \leq j < n_{k+1}} B(A_j, \delta_j, t)\) for some \(k \in \mathbb{N}\).

**Proof.** Evidently (a) implies (b). We prove \((b) \Rightarrow (a)\). Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a sequence of elements from \((0, 1)\) and \(t > 0\). For each \(n \in \mathbb{N}\) let \(\mu_n = \min\{\varepsilon_i : i \leq n\}\) and apply (b) to \((\mu_n)_{n \in \mathbb{N}}\) and \(t\). There is an increasing sequence \(n_1 < n_2 < \cdots\) in \(\mathbb{N}\) such that each finite set \(F \subset X\) is contained in \(\bigcup_{n_k \leq j < n_{k+1}} B(A_j, \delta_j, t)\) for some \(k \in \mathbb{N}\). Define now

\[
C_n = \bigcup_{i < n_1} A_i, \text{ for each } n < n_1,
C_n = \bigcup_{i < n_{k+1}} A_i, \text{ for each } n \text{ such that } n_k \leq n < n_{k+1}.
\]

We claim that the sequence \((C_n)_{n \in \mathbb{N}}\) of finite subsets of \(X\) witnesses for \((\varepsilon_n)_{n \in \mathbb{N}}\) and \(t\) that (a) is satisfied.

Let \(F\) be a finite subset of \(X\). Choose \(k \in \mathbb{N}\) such that \(F \subset \bigcup_{n_k \leq j < n_{k+1}} B(A_j, \mu_i, t)\). For (each) \(n\) with \(n_k \leq n < n_{k+1}\) put \(C_n = \bigcup_{i < n_{k+1}} S_i\). We have that for each \(x \in F\) there is \(j, n_k \leq j < n_{k+1},\) and \(y \in A_j\) with \(x \in B(y, \mu_j, t)\). Further, we have \(B(y, \mu_j, t) \subset B(y, \varepsilon_j, t)\) and since \(y \in C_n\) we have \(x \in B(C_j, \varepsilon_j, t)\), and thus \(F \subset B(C_j, \varepsilon_j, t)\). \(\square\)

With a small modification in the previous proof one can prove the following.

**Theorem 2.5** For a fuzzy 2-metric space \((X, M, \ast)\) the following are equivalent:

(a) For each sequence \((\varepsilon_n)_{n \in \mathbb{N}} \subset (0, 1)\) and each \(t > 0\) there is a sequence \((A_n)_{n \in \mathbb{N}}\) of finite subsets of \(X\) such that each finite subset \(F \subset X\) is contained in \(B(A_n, \varepsilon_n, t)\) for all but finitely many \(n\);

(b) For each sequence \((\varepsilon_n)_{n \in \mathbb{N}} \subset (0, 1)\) and each \(t > 0\) there is a sequence \((A_n)_{n \in \mathbb{N}}\) of finite subsets of \(X\) and an increasing sequence \(n_1 < n_2 < \cdots\) of natural numbers such that each finite subset \(F \subset X\) is contained in \(\bigcup_{n_k \leq j < n_{k+1}} B(A_j, \varepsilon_j, t)\) for all but finitely many \(k \in \mathbb{N}\).

**Definition 2.6** Let \((X, M, \ast)\) be a fuzzy 2-metric space and \(Y \subset X\). Then the mapping \(M_Y = M \upharpoonright Y^3 \times (0, \infty) : Y^3 \times (0, \infty) \to [0, 1]\) satisfying \((\text{F2M.1})\) is also a fuzzy 2-metric on \(Y\), and \((Y, M_Y, \ast)\) is called the fuzzy 2-metric subspace of \((X, M, \ast)\).

**Theorem 2.7** Every fuzzy 2-metric subspace of an FM\(_2\)-bounded fuzzy 2-metric space \((X, M, \ast)\) is also FM\(_2\)-bounded.
Proof. Let \((Y, M_Y, *)\) be a fuzzy 2-metric subspace of \((X, M, \ast)\) and let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a sequence of elements of \((0, 1)\) and \(t > 0\). Since the \(t\)-norm \(\ast\) is continuous, for each \(n \in \mathbb{N}\) there is \(\eta_n \in (0, 1)\) such that \((1 - \eta_n) \ast (1 - \eta_n) > 1 - \varepsilon_n\). By assumption on \((X, M, \ast)\) (applied to the sequence \((\eta_n)_{n \in \mathbb{N}}\) and \(1/2\)) there is a sequence \((A_n)_{n \in \mathbb{N}}\) of finite subsets of \(X\) such that

\[
X = \bigcup_{n \in \mathbb{N}} \bigcup_{a \in A_n} B(a, \eta_n, t/3).
\]

For each \(n \in \mathbb{N}\) let

\[
C_n = \{c \in A_n : \exists y \in Y \text{ with } y \in B(c, \eta_n, t/3)\}.
\]

Further, for each \(c \in C_n\) pick an element \(y_c \in Y\) such that \(y_c \in B(c, \eta_n, t/3)\) and set

\[
D_n = \{y_c : c \in C_n\}.
\]

Let us show that the sequence \((D_n)_{n \in \mathbb{N}}\) of finite subsets of \(Y\) witnesses for \((\varepsilon_n)_{n \in \mathbb{N}}\) and \(t > 0\) that \((Y, M_Y, \ast)\) is FM\(_2\)-bounded.

Let \(y\) be an arbitrary element of \(Y\). There exist \(n \in \mathbb{N}\) and \(a \in A_n\) such that \(y \in B(a, \eta_n, t/3)\), and from the definition of \(C_n\) it follows \(a \in C_n\). Therefore, there exists \(y_a \in D_n\) such that \(y_a \in B(a, \eta_n, t/3)\), hence \(a \in B(y_a, \eta_n, t/3)\). So, we have

\[
M(a, y, \varepsilon, t/3) > 1 - \eta_n \quad \text{and} \quad M(a, y_a, \varepsilon, t/3) > 1 - \eta_n.
\]

Applying the tetrahedral inequality (F2M.4), we have

\[
M(y, y_a, \varepsilon, t/3) \geq M(y, y_a, a, t/3) \ast M(y, a, \varepsilon, t/3) \ast M(a, y_a, \varepsilon, t/3)
\]

\[
> (1 - s_n) \ast (1 - s_n) \ast (1 - s_n) > 1 - \varepsilon_n,
\]

which means \(y \in B(y_a, \varepsilon_n, t)\). As \(y \in Y\) was arbitrary we conclude

\[
Y = \bigcup_{n \in \mathbb{N}} \bigcup_{y \in D_n} B(y, \varepsilon_n, t),
\]

i.e. \((Y, M_Y, \ast)\) is FM\(_2\)-bounded. \(\Box\)

The proof of the following theorem is similar to the proof of Theorem 2.7 and thus it is omitted.

Theorem 2.8 Every fuzzy 2-metric subspace of an FH\(_2\)-bounded space \((X, M, \ast)\) is also FH\(_2\)-bounded.

Let \((X, M_X, \ast)\) and \((Y, M_Y, \ast)\) be fuzzy 2-metric spaces and let \(Z = X \times Y\). Then the mapping \(M_Z : Z^3 \times (0, \infty) \to [0, 1]\) defined by

\[
M_Z(z_1, z_2, z_3, t) = M_X(x_1, x_2, x_3, t) \ast M_Y(y_1, y_2, y_3, t)
\]
for all \( z_i = (x_i, y_i) \in \mathbb{Z} \), \( i = 1, 2, 3 \), and all \( t > 0 \) is a fuzzy 2-metric on \( Z \), and the triple \((Z, M_Z, *)\) is called the product 2-metric space of \( X \) and \( Y \).

**Theorem 2.9** The product \((Z, M_Z, *)\) of two \( \mathsf{FH}_2 \)-bounded spaces \((X, M_X, *)\) and \((Y, M_Y, *)\) is also \( \mathsf{FH}_2 \)-bounded.

**Proof.** Let a sequence \((\varepsilon_n)_{n \in \mathbb{N}} \subset (0, 1)\) and \( t > 0 \) be given. By continuity of \(*\), choose for each \( n \in \mathbb{N} \) an element \( \eta_n \) in \((0, 1)\) such that \((1 - \eta_n) * (1 - \eta_n) > 1 - \varepsilon_n\). By assumption on \( X \) and \( Y \) there are sequences \((F_n)_{n \in \mathbb{N}}\) and \((H_n)_{n \in \mathbb{N}}\) of finite sets of \( X \) and \( Y \), respectively and natural numbers \( n_1 \) and \( n_2 \) such that each \( x \in X \) belongs to \( \bigcup_{n \in F_n} B(a, \eta_n, t/2) \) for all \( n \geq n_1 \), and each \( y \in Y \) belongs to \( \bigcup_{n \in H_n} B(c, \eta_n, t/2) \) for all \( n \geq n_2 \). We claim that the sequence \((F_n \times H_n)_{n \in \mathbb{N}}\) of finite subsets of \( Z \) witnesses for \((\varepsilon_n)_{n \in \mathbb{N}}\) and \( t \) that \((Z, M_Z, *)\) is \( \mathsf{FH}_2 \)-bounded.

Let \( z = (x, y) \in Z \) and \( n_0 = \max \{n_1, n_2\} \). Then for each \( n \geq n_0 \)

\[
x \in B(a_n, \eta_n, t/2) \text{ for some } a_n \in F_n
\]

and

\[
y \in B(c_n, \eta_n, t/2) \text{ for some } c_n \in H_n.
\]

Therefore, for all \( n \geq n_0 \) and \( z_n = (a_n, c_n) \in F_n \times H_n \) we have

\[
M_Z(z_n, w, t) = M_X(x, a_n, w, t) * M_Y(y, c_n, w, t) > (1 - \eta_n) * (1 - \eta_n) > 1 - \varepsilon_n.
\]

This means that \( z \in B(z_n, \varepsilon_n, t) \) and one concludes that \((Z, M_Z, *)\) is \( \mathsf{FH}_2 \)-bounded. \( \Box \)

In a similar way, with small necessary changes, one can prove the following.

**Theorem 2.10** The product \((Z, M_Z, *)\) of an \( \mathsf{FM}_2 \)-bounded fuzzy 2-metric space \((X, M_X, *)\) and a fuzzy 2-precompact fuzzy 2-metric space \((Y, M_Y, *)\) is \( \mathsf{FM}_2 \)-bounded.

We end the paper by two open questions.

There are infinitely long two-person games associated to \( \mathsf{FM}_2 \)-boundedness, \( \mathsf{FH}_2 \)-boundedness, and \( \mathsf{FR}_2 \)-boundedness. We describe the game associated to the \( \mathsf{FR}_2 \)-boundedness; it is clear how to define games related to the other two properties.

The game \( G_{\mathsf{FR}_2} \) on a fuzzy 2-metric space \((X, M, *)\) is defined in the following way. Let \( t > 0 \) be fixed. Two players, I and II, play a round for each positive integer \( n \). In the \( n \)-th round I takes \( \varepsilon_n \in (0, 1) \), and II responds by choosing an element \( a_n \in X \). A play \( \varepsilon_1, a_1; \varepsilon_2, a_2; \cdots; \varepsilon_n, a_n; \cdots \) is won by II if and only if \( X = \bigcup_{n \in \mathbb{N}} B(a_n, \varepsilon_n, t) \).

Evidently, if the player II has a winning strategy (or weaker, if I does not have a winning strategy) in the game \( G_{\mathsf{FR}_2} \), then \((X, M, *)\) is \( \mathsf{FR}_2 \)-bounded.

Call a fuzzy 2-metric space \( X \) strongly \( \mathsf{FR}_2 \)-bounded if II has a winning strategy in the game \( G_{\mathsf{FR}_2} \). Similarly we define strongly \( \mathsf{FM}_2 \)-boundedness and strongly \( \mathsf{FH}_2 \)-boundedness.

**Problem 2.11** Find fuzzy 2-metric spaces which are \( \mathsf{FR}_2 \)-bounded (respectively, \( \mathsf{FM}_2 \)-bounded, \( \mathsf{FH}_2 \)-bounded), but not strongly \( \mathsf{FR}_2 \)-bounded (respectively, strongly \( \mathsf{FM}_2 \)-bounded, strongly \( \mathsf{FH}_2 \)-bounded).

**Problem 2.12** Characterize strongly \( \mathsf{FR}_2 \)-bounded, strongly \( \mathsf{FM}_2 \)-bounded and strongly \( \mathsf{FH}_2 \)-bounded fuzzy 2-metric spaces.
References


