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On the Degree Kirchhoff Index of Bipartite Graphs

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Abstract: Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be a connected graph of order n and size m. Denote by $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n-1} > \gamma_n = 0$ the normalized Laplacian eigenvalues of G. The degree Kirchhoff index is defined as $Kf^*(G) = 2m\sum_{i=1}^{n-1}\frac{1}{\gamma_i}$. In this paper, we obtain some improved lower bounds on the degree Kirchhoff index of bipartite graphs.

Keywords: Topological indices, degree Kirchhoff index.

1 Introduction

Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be an undirected connected graph with n vertices and m edges and vertex degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$. Denote by $A(G)$ and $D(G) = \mathrm{diag}(d_1,d_2,\dots,d_n)$ the adjacency and the diagonal degree matrix of G, respectively. The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix of G. Since the graph G is considered as connected, the matrix $D(G)^{-1/2}$ is well-defined. Then, the normalized Laplacian is defined as $\mathscr{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$. Its eigenvalues $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n-1} \geq \gamma_n = 0$ represent the normalized Laplacian eigenvalues of G. The following inequalities are valid for γ_i, $i=1,2,\dots,n-1$ [6]:

$$\sum_{i=1}^{n-1}\gamma_i = n \quad \text{and} \quad \sum_{i=1}^{n-1}\gamma_i^2 = n + 2R_{-1}(G)\,,$$

where

$$R_{-1}(G) = \sum_{i\sim j}\frac{1}{d_i d_j}\,,$$

is the general Randić index R_{-1} (also called branching index) introduced in [18] (see also [4]).

In the case of bipartite graphs, which are considered in this paper, we have that $\gamma_1 = 2$ [6], and

$$\sum_{i=2}^{n-1}\gamma_i = n-2 \quad \text{and} \quad \sum_{i=2}^{n-1}\gamma_i^2 = n + 2R_{-1}(G) - 4\,. \tag{1.1}$$

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The degree Kirchhoff index of G is expressed in terms of normalized Laplacian eigenvalues as

$$Kf^*(G) = 2m\sum_{i=1}^{n-1}\frac{1}{\gamma_i}. \tag{1.2}$$

More on its mathematical properties and its lower and upper bounds can be found in [3,5,10–17,20]. In this paper, we obtain an improved lower bound on the degree Kirchhoff index of bipartite graphs.

2 Preliminaries

In this section we recall some analytical inequalities and list some results from spectral graph theory that will be used later the paper.

Lemma 2.1. [7] *Let $a=(a_i)$, $i=1,2,\dots,n$, be a real number sequence with the property $a_1 \geq a_2 \geq \cdots \geq a_n > 0$. Then*

$$\sum_{i+1}^{n} a_i \geq n\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + (\sqrt{a_1}-\sqrt{a_n})^2 \tag{2.1}$$

and

$$\sum_{i=1}^{n} a_i \geq n\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}\left(\frac{1}{2}\left(\sqrt{\frac{a_1}{a_n}}+\sqrt{\frac{a_n}{a_1}}\right)\right)^{\frac{2}{n}}. \tag{2.2}$$

Equality in (2.1) *holds if $a_2 = a_3 = \cdots = a_n = \sqrt{a_1 a_n}$. Equality in* (2.2) *holds if $a_2 = a_3 = \cdots = a_{n-1} = \frac{a_1+a_n}{2}$.*

Lemma 2.2. [1] *Let $a=(a_i)$ and $b=(b_i)$, $i=1,2,\dots,n$, be two positive real numbers with the properties $0 < r_1 \leq a_i \leq R_1 < +\infty$ and $0 < r_2 \leq b_i \leq R_2 < +\infty$. Then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \frac{n^2}{4}(R_1-r_1)(R_2-r_2). \tag{2.3}$$

Equality holds if and only if $r_1 = a_1 = a_2 = \cdots = a_n = R_1$, or $r_2 = b_1 = b_2 = \cdots = b_n = R_2$.

Lemma 2.3. [6] *Let G be a connected graph with $n \geq 2$ vertices. Then*

(i) $\gamma_1 \leq 2$, with equality holding if and only if G is a bipartite graph.

(ii) $\gamma_n = 0$ and $\gamma_{n-1} \neq 0$.

Lemma 2.4. [11] *Let G be a connected graph of order n. Then $\gamma_2 \geq 1$. Equality holds if and only if G is a complete bipartite graph.*

Lemma 2.5. [6] *Let G be a bipartite graph of order n. Then*

$$\gamma_i + \gamma_{n-i+1} = 2\,, \tag{2.4}$$

for $i = 1, 2, \ldots, n$.

Lemma 2.6. [8] *Let G be a connected graph with n vertices, m edges and $t(G)$ spanning trees. Then*

$$\prod_{i=1}^{n-1} \gamma_i = \frac{2mt(G)}{\prod_{i=1}^{n} d_i}\,. \tag{2.5}$$

In [20], Zhou and Trinajstić obtained a lower bound in terms of the number of vertices and edges as:

Lemma 2.7. [20] *Let G be a connected bipartite graph with $n \geq 2$ vertices and m edges. Then*

$$Kf^*(G) \geq m(2n-3)\,. \tag{2.6}$$

Equality holds if and only if G is a complete bipartite graph.

The following lower bound, involving the number of vertices, edges and spanning trees, was presented in [3].

Lemma 2.8. [3] *Let G be a connected bipartite graph with $n \geq 3$ vertices and m edges. Then*

$$Kf^*(G) \geq m + 2m(n-2)\left(\frac{\prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-2}}, \tag{2.7}$$

with equality if and only if G is a complete bipartite graph.

3 Main Results

We now give the main results of this paper.

Theorem 3.1. *Let G be a connected bipartite graph with $n \geq 3$ vertices, m edges and $t(G)$ spanning trees. Then, for any α, $\gamma_2 \geq \alpha \geq 1$, holds*

$$Kf^*(G) \geq m + \frac{2m}{\alpha} + 2m(n-3)\left(\frac{\alpha \prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-3}} + \frac{2m\left(\sqrt{\gamma_3} - \sqrt{\gamma_{n-1}}\right)^2}{\gamma_3 \gamma_{n-1}} \tag{3.1}$$

with equality if $\alpha = 1$ and G is a complete bipartite graph.

Proof. The inequality (2.1) can be considered in the following form

$$\sum_{i=3}^{n-1} a_i \geq (n-3)\left(\prod_{i=3}^{n-1} a_i\right)^{\frac{1}{n-3}} + \left(\sqrt{a_3} - \sqrt{a_{n-1}}\right)^2.$$

For $a_i = \frac{1}{\gamma_{n-i+2}}$, $i = 3,4,\dots,n-1$, the above inequality becomes

$$\sum_{i=3}^{n-1} \frac{1}{\gamma_i} \geq (n-3)\left(\prod_{i=3}^{n-1} \frac{1}{\gamma_i}\right)^{\frac{1}{n-3}} + \left(\sqrt{\frac{1}{\gamma_{n-1}}} - \sqrt{\frac{1}{\gamma_3}}\right)^2$$

that is

$$\sum_{i=1}^{n-1} \frac{1}{\gamma_i} - \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \geq (n-3)\left(\prod_{i=1}^{n-1} \frac{\gamma_1\gamma_2}{\gamma_i}\right)^{\frac{1}{n-3}} + \frac{\left(\sqrt{\gamma_3} - \sqrt{\gamma_{n-1}}\right)^2}{\gamma_3\gamma_{n-1}}.$$

From the above inequality and Lemmas 2.3 and 2.6 we get

$$\sum_{i=1}^{n-1} \frac{1}{\gamma_i} \geq \frac{1}{2} + \frac{1}{\gamma_2} + (n-3)\left(\frac{\gamma_2 \prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-3}} + \frac{\left(\sqrt{\gamma_3} - \sqrt{\gamma_{n-1}}\right)^2}{\gamma_3\gamma_{n-1}}. \tag{3.2}$$

For $x > 0$, consider the auxiliary function defined by

$$f(x) = \frac{1}{x} + (n-3)\left(\frac{x \prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-3}}.$$

Observe that

$$f'(x) = \frac{1}{x^2}\left(\left(\frac{x^{n-2} \prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-3}} - 1\right).$$

Thus, f is a increasing function for $x \geq \left(\frac{mt(G)}{\prod_{i=1}^{n} d_i}\right)^{\frac{1}{n-2}}$. Therefore, for any α, $\gamma_2 \geq \alpha \geq 1$, we have

$$\gamma_2 \geq \alpha \geq 1 = \frac{\sum_{i=2}^{n-1} \gamma_i}{n-2} \geq \left(\prod_{i=2}^{n-1} \gamma_i\right)^{\frac{1}{n-2}} = \left(\frac{mt(G)}{\prod_{i=1}^{n} d_i}\right)^{\frac{1}{n-2}}.$$

Hence

$$f(\gamma_2) \geq f(\alpha) = \frac{1}{\alpha} + (n-3)\left(\frac{\alpha \prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-3}}.$$

Combining this with (3.2) we obtain

$$\sum_{i=1}^{n-1} \frac{1}{\gamma_i} \geq \frac{1}{2} + \frac{1}{\alpha} + (n-3)\left(\frac{\alpha \prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-3}} + \frac{\left(\sqrt{\gamma_3} - \sqrt{\gamma_{n-1}}\right)^2}{\gamma_3\gamma_{n-1}},$$

from which the inequality (3.1) is obtained.

The equality in (3.2) holds if

$$\gamma_2 = \alpha \text{ and } \frac{1}{\gamma_4} = \cdots = \frac{1}{\gamma_{n-3}} = \frac{1}{\gamma_{n-2}} = \sqrt{\frac{1}{\gamma_3\gamma_{n-1}}}.$$

If $\alpha = 1$, then by Lemma 2.4, we get that G is a complete bipartite graph. From the above and Lemma 2.5, we also get

$$\gamma_3 = \gamma_4 = \cdots = \gamma_{n-1} = 1$$

This also confirm that the graph G is complete bipartite. Thus, we conclude that the equality in (3.1) holds if $\alpha = 1$ and G is a complete bipartite graph. □

Remark 3.1. *For a connected bipartite graph G with $n \geq 3$ vertices, it was proved that* [2]

$$t(G) \leq \frac{\prod_{i=1}^{n} d_i}{m}$$

i.e.,

$$\frac{\prod_{i=1}^{n} d_i}{mt(G)} \geq 1$$

with equality if and only if $G \cong K_{p,q}$. Furthermore, $\frac{\left(\sqrt{\gamma_3}-\sqrt{\gamma_{n-1}}\right)^2}{\gamma_3\gamma_{n-1}} \geq 0$ with equality if and only if G is a complete bipartite graph. Then, considering these facts with Theorem 3.1, we have

$$\begin{aligned} Kf^*(G) &\geq m + \frac{2m}{\alpha} + 2m(n-3)\left(\frac{\alpha\prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-3}} + \frac{2m\left(\sqrt{\gamma_3}-\sqrt{\gamma_{n-1}}\right)^2}{\gamma_3\gamma_{n-1}} \\ &\geq 3m + 2m(n-3)\left(\frac{\prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-3}} + \frac{2m\left(\sqrt{\gamma_3}-\sqrt{\gamma_{n-1}}\right)^2}{\gamma_3\gamma_{n-1}} \\ &\geq 3m + 2m(n-3) + \frac{2m\left(\sqrt{\gamma_3}-\sqrt{\gamma_{n-1}}\right)^2}{\gamma_3\gamma_{n-1}} \\ &\geq m(2n-3) \end{aligned}$$

This implies that the lower bound (3.1) *is stronger than the lower bound* (2.6).

Remark 3.2. *Note that the lower bounds* (2.7) *and* (3.1) *are incomparable.*

Since $\frac{\left(\sqrt{\gamma_3}-\sqrt{\gamma_{n-1}}\right)^2}{\gamma_3\gamma_{n-1}} \geq 0$, we have the following corollary of Theorem 3.1.

Corollary 3.1. *Let G be a connected bipartite graph with $n \geq 3$ vertices, m edgse and $t(G)$ spanning trees. Then, for any α, $\gamma_2 \geq \alpha \geq 1$, holds*

$$Kf^*(G) \geq m + \frac{2m}{\alpha} + 2m(n-3)\left(\frac{\alpha\prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-3}},$$

with equality holding if $\alpha = 1$ and G is a complete bipartite graph.

Theorem 3.2. *Let G be a connected bipartite graph with $n \geq 3$ vertices, m edges and $t(G)$ spanning trees. Then*

$$Kf^*(G) \geq m + 2m(n-2)\left(\frac{(n-2)\prod_{i=1}^{n} d_i}{mt(G)(n-2R_{-1}(G))}\right)^{\frac{1}{n-2}}. \tag{3.3}$$

Equality holds if G is a complete bipartite graph.

Proof. The inequality (2.2) can be considered as

$$\sum_{i=2}^{n-1} a_i \geq (n-2)\left(\prod_{i=2}^{n-1} a_i\right)^{\frac{1}{n-2}} \left(\frac{1}{2}\left(\sqrt{\frac{a_2}{a_{n-1}}} + \sqrt{\frac{a_{n-1}}{a_2}}\right)\right)^{\frac{2}{n-2}}.$$

For $a_i = \frac{1}{\gamma_{n-i+1}}$, $a_2 = \frac{1}{\gamma_{n-1}}$, $a_{n-1} = \frac{1}{\gamma_2}$, $i = 2, 3, \ldots, n-1$, the above inequality transforms into

$$\sum_{i=2}^{n-1} \frac{1}{\gamma_i} \geq (n-2)\left(\prod_{i=2}^{n-1} \frac{1}{\gamma_i}\right)^{\frac{1}{n-2}} \left(\frac{1}{2}\left(\sqrt{\frac{\gamma_2}{\gamma_{n-1}}} + \sqrt{\frac{\gamma_{n-1}}{\gamma_2}}\right)\right)^{\frac{2}{n-2}},$$

that is

$$\sum_{i=2}^{n-1} \frac{1}{\gamma_i} \geq (n-2)\left(2\prod_{i=1}^{n-1} \frac{1}{\gamma_i}\right)^{\frac{1}{n-2}} \left(\frac{1}{4}\left(\frac{\gamma_2}{\gamma_{n-1}} + \frac{\gamma_{n-1}}{\gamma_2} + 2\right)\right)^{\frac{1}{n-2}}. \tag{3.4}$$

On the other hand, for $a_i = b_i$, $i = 2, \ldots, n-1$, the inequality (2.3), can be considered as

$$(n-2)\sum_{i=2}^{n-1} a_i^2 - \left(\sum_{i=2}^{n-1} a_i\right)^2 \leq \frac{(n-2)^2}{4}(R_1 - r_1)^2.$$

For $a_i = \gamma_i$, $R_1 = \gamma_2$, $r_1 = \gamma_{n-1}$, $i = 2, 3, \ldots, n-1$, the above inequality becomes

$$(n-2)\sum_{i=2}^{n-1} \gamma_i^2 - \left(\sum_{i=2}^{n-1} \gamma_i\right)^2 \leq \frac{(n-2)^2}{4}(\gamma_2 - \gamma_{n-1})^2.$$

From the above and (1.1) we obtain

$$2(n-2)(R_{-1}(G) - 1) \leq \frac{(n-2)^2}{4}(\gamma_2 - \gamma_{n-1})^2,$$

from which we get

$$\gamma_2 - \gamma_{n-1} \geq 2\sqrt{\frac{2(R_{-1}(G) - 1)}{n-2}}. \tag{3.5}$$

On the other hand, from (2.4) we have

$$\gamma_2 + \gamma_{n-1} = 2.$$

Now, from the above and (3.5) we obtain

$$\gamma_2 \geq 1+\sqrt{\frac{2(R_{-1}(G)-1)}{n-2}} \quad \text{and} \quad \gamma_{n-1} \leq 1-\sqrt{\frac{2(R_{-1}(G)-1}{n-2}}\,.$$

Thus, we have that

$$\frac{1}{4}\left(\frac{\gamma_2}{\gamma_{n-1}}+\frac{\gamma_{n-1}}{\gamma_2}+2\right) \geq \frac{n-2}{n-2R_{-1}(G)}\,. \tag{3.6}$$

Finally, from (3.4), (3.6) and Lemma 2.6, we arrive at (3.3).

Equality in (3.4) holds if $\gamma_2 = \gamma_3 = \cdots = \gamma_{n-1}$. Since $2+(n-2)\gamma_2 = n$, it follows that $\gamma_2 = \gamma_3 = \cdots = \gamma_{n-1} = 1$, which implies that equality in (3.3) holds if G is a complete bipartite graph. □

Corollary 3.2. *Let T be a tree with $n \geq 3$ vertices. Then*

$$Kf^*(T) \geq (n-1)\left(1+2(n-2)\left(\frac{(n-2)\prod_{i=1}^n d_i}{(n-1)(n-2R_{-1}(G))}\right)^{\frac{1}{n-2}}\right).$$

Equality holds if $T \cong K_{1,n-1}$.

In [19] it was proven that

$$R_{-1}(G) \geq \frac{n}{2\Delta}\,,$$

with equality if and only if G is a regular graph. Thus, we have the following corollary of Theorem 3.2

Corollary 3.3. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$Kf^*(G) \geq m\left(1+2(n-2)\left(\frac{\Delta(n-2)\prod_{i=1}^n d_i}{mn(\Delta-1)t(G)}\right)^{\frac{1}{n-2}}\right). \tag{3.7}$$

Equality holds if $G \cong K_{\frac{n}{2},\frac{n}{2}}$, where n is even.

Remark 3.3. *Since for bipartite graphs hold*

$$\frac{n-2}{n-2R_{-1}(G)} \geq 1\,,$$

the inequality (3.3) *is stronger than* (2.7).

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