A Survey on Randić (Normalized) Incidence Energy of Graphs

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Abstract: For a graph $G$ of order $n$ with normalized signless Laplacian eigenvalues $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$, the Randić (normalized) incidence energy is defined as $IR\text{E}(G) = \sum_{i=1}^{n} \sqrt{\gamma_i^+}$. In this paper, we present a survey on the results of $IR\text{E}(G)$, especially with emphasis on the properties, bounds and Coulson integral formula of $IR\text{E}(G)$.

1 Introduction

All graphs considered in this paper are simple finite undirected graphs. The terminology and notation not defined here can be found in [11].

Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. The vertex set and edge set of $G$ are, respectively, denoted by $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Let $d_i$ be the degree of the vertex $v_i \in V$, $i = 1, 2, \ldots, n$. Denote by $\delta$ and $\Delta$ the minimum degree and the maximum degree of $G$, respectively. If $v_i$ and $v_j$ are two adjacent vertices of $G$, then it is written as $i \sim j$. Let $A(G) = (a_{ij})$ be the $(0, 1)$-adjacency matrix of the graph $G$. It is defined by $a_{ij} = 1$ if $i \sim j$ and 0 otherwise. The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of $A(G)$ are the (ordinary) eigenvalues of $G$ [11].

For a graph $G$, the (ordinary) graph energy was introduced as the sum of absolute values of its eigenvalues [15]. It is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|. \quad (1.1)$$

For details on the theory of $E(G)$, see [21] and the references cited therein.

As the generalization of the graph energy concept, the energy of a real matrix (not necessarily square) $M$, denoted by $E(M)$, is defined by Nikiforov [29] as the sum of its singular values that are equal to the square roots of the eigenvalues of $MM^T$, where $M^T$ is the transpose of $M$. Especially, for a graph $G$, $E(G) = E(A(G))$.

Denote by $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ the Laplacian matrix and the signless Laplacian matrix of $G$, respectively [26]. Here, $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$

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is the diagonal degree matrix of $G$. For a graph $G$ without isolated vertices, the matrix $D(G)^{-1/2}$ is well defined. Then, the normalized Laplacian matrix is defined as

$$\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2} = I_n - R(G)$$

and the normalized signless Laplacian matrix as [9]

$$\mathcal{L}^+(G) = D(G)^{-1/2}Q(G)D(G)^{-1/2} = I_n + R(G)$$

where $I_n$ is the $n \times n$ unit matrix and $R(G)$ is the Randić matrix. Throughout this paper, the eigenvalues of $R(G)$, $\mathcal{L}(G)$ and $\mathcal{L}^+(G)$ (or Randić, normalized Laplacian and normalized signless Laplacian eigenvalues of $G$) will be denoted by $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0$, respectively. Details on these eigenvalues can be found in [2, 9, 11].

Let $I(G)$ be the vertex-edge incidence matrix of the graph $G$. The $ij$-entry of $I(G)$ is 1 if $v_i$ is incident to $e_j$ and 0 otherwise. The incidence energy of $G$, denoted by $IE(G)$, is defined as the energy of its incidence matrix [18]. Since $Q(G) = I(G)I(G)^T$, Gutman et al. also discovered that [17]

$$IE(G) = \sum_{i=1}^{n} \sqrt{q_i};$$

where $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$ are the eigenvalues of $Q(G)$ [12]. For the basic properties and several lower and upper bounds of $IE(G)$, see [4, 13, 18, 23].

Gu et al. [14] and Cheng and Liu [8] independently introduced the Randić (normalized) incidence matrix of $G$ as $I_R(G) = D(G)^{-1/2}I(G)$ and referred to its energy as the Randić (normalized) incidence energy $I_R E(G)$ of $G$. Since $\mathcal{L}^+(G) = I_R(G)I_R(G)^T$, in full analogous manner with the incidence energy, it was also pointed out that [8, 14]

$$I_R E(G) = \sum_{i=1}^{n} \sqrt{\gamma_i^+}. \quad (1.2)$$

For the recent results on $I_R E(G)$, see [8, 14, 22, 30].

This survey is organized in the following way. In Section 2, we recall some known results regarding the normalized signless Laplacian eigenvalues. In Section 3, we deal with a few elementary properties of $I_R E(G)$. In Sections 4 and 5, some lower and upper bounds for $I_R E(G)$ are given. In Section 6, the results on the Coulson integral formula of $I_R E(G)$ are presented.

## 2 Some Known Results

In this section, we recall some known results associated with the normalized signless Laplacian eigenvalues of graphs.

**Lemma 2.1.** [14] Let $G$ be a graph of order $n$ with no isolated vertices. Then $\gamma_i^+ = 1 + \rho_i$, for $i = 1, 2, \ldots, n$. 

Lemma 2.2. [6] If $G$ is a connected non-bipartite graph of order $n$, then $\gamma_i^+ > 0$, for $i = 1, 2, \ldots, n$.

Lemma 2.3. [14] For a graph $G$ with no isolated vertices, the largest normalized signless Laplacian eigenvalue $\gamma_1^+ = 2$.

Let $\tilde{G}$ denote the subdivision graph of a graph $G$ obtained by inserting additional vertex into the each edge of $G$. If $G$ is a graph with $n$ vertices and $m$ edges, then its subdivision graph $\tilde{G}$ possesses $n+m$ vertices and $2m$ edges.

Lemma 2.4. [8] Let $G$ be a graph with $n$ vertices and $m$ edges and let $\tilde{G}$ be its subdivision graph. If $\gamma_i^+$ are the non-zero normalized signless Laplacian eigenvalues of $G$, then the Randić eigenvalues of $\tilde{G}$ consist of the number $\pm \sqrt{\gamma_i^+/2}$ $i = 1, 2, \ldots, h$, and $n+m−2h$ zeros.

Lemma 2.5. [8] Let $G$ be a connected graph with diameter $d$ and $s$ distinct normalized signless Laplacian eigenvalues. Then, $d \leq s−1$.

Lemma 2.6. [14] Let $G$ be a graph of order $n \geq 2$ with no isolated vertices. Then $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+ = \frac{n−2}{n−1}$ if and only if $G \cong K_n$.

Lemma 2.7. [8] Suppose that the $n$-vertex connected graph $G$ is not a complete graph. If the normalized signless Laplacian eigenvalues are ordered as $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+$, then $\gamma_2^+ \geq 1$.

Recall that the general Randić index of a graph $G$ is defined as $R_{-1}(G) = \sum_{i \sim j} \frac{1}{d_i d_j}$ [7]. The following lower bound on the second largest normalized signless Laplacian eigenvalue involving the parameter $n$ and $R_{-1}(G)$ can be found in [24].

Lemma 2.8. [24] Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then

$$\gamma_2^+ \geq \frac{n + 2R_{-1}(G) - 4}{n - 2}.$$ 

Equality holds if and only if $G \cong K_n$.

Lemma 2.9. [6] Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then

$$\gamma_n^+ \leq \frac{n - 2R_{-1}(G)}{n} \leq \frac{\Delta - 1}{\Delta} \leq \frac{n - 2}{n - 1},$$

with equalities if and only if $G \cong K_n$.

Lemma 2.10. [8] Let $G$ be a graph of order $n$ with no isolated vertices. Then

$$\sum_{i=1}^{n} \gamma_i^+ = n \text{ and } \sum_{i=1}^{n} (\gamma_i^+)^2 = n + 2R_{-1}(G).$$
Lemma 2.11. [5] If $G$ is a bipartite graph, then the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}^+(G)$ coincide.

Let $t(G)$ be the total number of spanning trees of $G$. Denote by $G_1 \times G_2$ the Cartesian product of the graphs $G_1$ and $G_2$.

Lemma 2.12. [5, 11] Let $G$ be a connected graph with $n$ vertices, $m$ edges and $t(G)$ spanning trees. If $G$ is bipartite, then

$$\prod_{i=1}^{n-1} \gamma_i = \prod_{i=1}^{n-1} \gamma_i^+ = \frac{2mt(G)}{\prod_{i=1}^{n} d_i}.$$ 

If $G$ is non–bipartite, then

$$\prod_{i=1}^{n} \gamma_i^+ = \frac{2t(G \times K_2)}{t(G) \prod_{i=1}^{n} d_i}.$$ 

3 Elementary Properties of $I_{RE}(G)$

The empty graph of order $n$ is the graph with $n$ isolated vertices and no edges. The Randić (normalized) incidence energy $I_{RE}(G)$ has similar basic properties as graph energy.

Theorem 3.1. [8, 16] Let $G$ be a graph of order $n$. Then

(a) $E(G) \geq 0$, $I_{RE}(G) \geq 0$ with equalities if and only if $G$ is an empty graph.

(b) If the graph $G$ consists of two connected components $G_1$ and $G_2$, then $E(G) = E(G_1) + E(G_2)$ and $I_{RE}(G) = I_{RE}(G_1) + I_{RE}(G_2)$.

(c) If one component of the graph $G$ is $G_1$ and all other components are isolated vertices, then $E(G) = E(G_1)$ and $I_{RE}(G) = I_{RE}(G_1)$.

By full analogy with the graph energy given by (1.1), the Randić energy of a graph $G$ was defined as follows [2]

$$RE(G) = \sum_{i=1}^{n} |\rho_i|.$$ 

Considering the result in Lemma 2.4, Cheng and Liu [8] obtained the following relation between the Randić (normalized) incidence energy of a graph and Randić energy of its subdivision graph.

Theorem 3.2. [8] Let $G$ be a graph with $n$ vertices and $m$ edges and let $\tilde{G}$ be its subdivision graph. Then, $I_{RE}(G) = \frac{\sqrt{2}}{2} RE\left(\tilde{G}\right)$.

Let $\alpha$ be a real number. The sum of the $\alpha$th powers of the non-zero normalized Laplacian eigenvalues of a connected graph $G$ was defined as [3]

$$s_\alpha(G) = \sum_{i=1}^{n-1} \gamma_i^\alpha.$$
This sum generalizes various graph invariants. For more details, see [1, 19]. For $\alpha = 1/2$, $s_{1/2} (G)$ is equal to the Laplacian incidence energy of $G$, defined by [31] (see also, [27,28])

$$LIE (G) = \sum_{i=1}^{n-1} \sqrt{\gamma_i}.$$ 

Recently, the sum of the $\alpha$th powers of the normalized signless Laplacian eigenvalues of $G$ was put forward as [5]

$$\sigma_\alpha (G) = \sum_{i=1}^{n} (\gamma_i^+)^\alpha .$$

Note that $\sigma_{1/2} (G) = I_{RE} (G)$, defined by (1.2). Recall from Lemma 2.11 that the normalized Laplacian and the normalized signless Laplacian eigenvalues coincide. By the fact that, the following result was given in [5].

**Theorem 3.3.** [5] If $G$ is a bipartite graph, then $\sigma_\alpha (G)$ coincide with $s_\alpha (G)$. In particular, for bipartite graphs, $I_{RE} (G) = LIE (G)$.

4 Lower Bounds for $I_{RE} (G)$

Let $K_n$ and $K_{p,q}$ ($p + q = n$) be the complete graph and the complete bipartite graph of order $n$, respectively. We now give some lower bounds on $I_{RE} (G)$.

**Theorem 4.1.** [8,14] Let $G$ be a graph of order $n$ with no isolated vertices. Then, $I_{RE} (G) \geq \sqrt{n}$ with equality if and only if $G \cong K_2$.

**Theorem 4.2.** [8] Let $G$ be a graph of order $n$ with no isolated vertices. Then,

$$I_{RE} (G) \geq \sqrt{\frac{2n^3}{4n - 1 + (-1)^n}}.$$ 

Equality holds if and only if $n$ is even and $G$ is disjoint union of $\frac{n}{2}$ paths of length 1.

**Corollary 4.1.** [8] Let $G$ be a graph of order $n$ with no isolated vertices. Then

$$I_{RE} (G) \geq \frac{n}{\sqrt{2}} .$$

Equality holds if and only if $n$ is even and $G$ is disjoint union of $\frac{n}{2}$ paths of length 1.

For a subset $E' \subseteq E$, the subgraph of $G$ obtained by deleting the edges in $E'$ is denoted by $G - E'$. If $E'$ contains only one edge $e$, then $G - E'$ is denoted by $G - e$.

**Theorem 4.3.** [14] Let $G$ be a graph and $E'$ be a nonempty subset of $E$. Then

$$I_{RE} (G) > I_{RE} (G - E').$$
When the edge subset $E'$ contains exactly one edge, Gu et al. [14] established the following result.

**Theorem 4.4.** [14] Let $G$ be a connected graph, $e = v_i v_j$ be an edge of $G$. Then

$$I_{RE}(G) \geq \sqrt{\frac{1}{d_i} + \frac{1}{d_j} + [I_{RE}(G-e)]^2}.$$  

Equality holds if and only if $G \cong K_2$.

The clique number $\omega = \omega(G)$ of a graph $G$ is equal to the number of vertices in a maximum clique. By the fact that $I_{RE}(K_n) = \sqrt{2 + \sqrt{(n-1)(n-2)}}$, the following result was presented in [14].

**Corollary 4.2.** [14] Let $G$ be a non-empty graph with clique number $\omega$. Then,

$$I_{RE}(G) \geq \sqrt{2 + \sqrt{(\omega - 1)(\omega - 2)}}.$$  

In particular, if $G$ has at least one edge then $I_{RE}(G) \geq \sqrt{2}$.

The following relation exists between $I_{RE}(G)$ and Laplacian incidence energy.

**Theorem 4.5.** [31] Let $G$ be a connected graph of order $n$. Then

$$I_{RE}(G) \geq LIE(G).$$  

Equality holds if $G$ is a bipartite graph [5].

**Corollary 4.3.** [31] Let $G$ be a connected graph of order $n$ with minimum degree $\delta$ and $\gamma_1$ the spectral radius of $L^e(G)$. Then

$$I_{RE}(G) \geq n \max \left\{ \gamma_1^{-1/2}, \sqrt{\frac{\delta}{\delta + 1}} \right\}.$$  

(4.2)

**Remark 4.1.** [31] The lower bound (4.2) is stronger than the lower bound (4.1).

**Theorem 4.6.** [24] Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then, for any $\alpha$, $\gamma_2^+ \geq \alpha \geq 1$, holds

$$I_{RE}(G) > \sqrt{2 + \sqrt{\alpha + n - 2} + \frac{1}{2} \ln \left( \frac{t(G \times K_2)}{\alpha t(G) \prod_{i=1}^{n} d_i} \right)}.$$  

**Corollary 4.4.** [24] Let $G$, $G \not\cong K_n$, be a connected non–bipartite graph with $n \geq 3$ vertices. Then,

$$I_{RE}(G) > \sqrt{2 + n - 1 + \frac{1}{2} \ln \left( \frac{t(G \times K_2)}{t(G) \prod_{i=1}^{n} d_i} \right)}.$$
Theorem 4.7. [24] Let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $m$ edges and $t(G)$ spanning trees. Then, for any $\alpha, \gamma_2^+ = \gamma_2 \geq \alpha \geq 1$, holds

$$I_{RE}(G) = LIE(G) \geq \sqrt{2} + \sqrt{\alpha} + n - 3 + \frac{1}{2} \ln \left( \frac{mt(G)}{\alpha \prod_{i=1}^{n} d_i} \right).$$

Equality holds if and only if $\alpha = 1$ and $G \cong K_{p,q}$, $p + q = n$.

Corollary 4.5. [24] Let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $m$ edges and $t(G)$ spanning trees. Then,

$$I_{RE}(G) = LIE(G) \geq \sqrt{2} + n - 2 + \frac{1}{2} \ln \left( \frac{mt(G)}{\prod_{i=1}^{n} d_i} \right).$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

It should be noted that the remaining lower bounds of this section were actually obtained in more general forms in [5, 6, 19].

Theorem 4.8. [5] Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then,

$$I_{RE}(G) = LIE(G) \geq \sqrt{2} + \left( n - 2 + (n-1)(n-2) \right) \left( \frac{t(G \times K_2)}{t(G) \prod_{i=1}^{n} d_i} \right)^{1/(n-1)}.$$

Equality holds if and only if $G \cong K_n$.

Theorem 4.9. [5] Let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $m$ edges and $t(G)$ spanning trees. Then

$$I_{RE}(G) = LIE(G) \geq \sqrt{2} + \left( n - 2 + (n-2)(n-3) \right) \left( \frac{mt(G)}{\prod_{i=1}^{n} d_i} \right)^{1/(n-2)}.$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

Recall that every tree is bipartite. Furthermore, for a tree $T$, $m = n - 1$ and $t(T) = 1$. Then, from Theorem 4.9, it can be deduced that:

Corollary 4.6. [5] Let $T$ be a tree with $n \geq 3$ vertices. Then

$$I_{RE}(T) = LIE(T) \geq \sqrt{2} + \left( n - 2 + (n-2)(n-3) \right) \left( \frac{n-1}{\prod_{i=1}^{n} d_i} \right)^{1/(n-2)}.$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Theorem 4.10. [5] Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then, there exists a real number $\varepsilon \geq 0$ such that

$$I_{RE}(G) \geq \sqrt{2} + (n-1) \left( \frac{t(G \times K_2)}{t(G) \prod_{i=1}^{n} d_i} \right)^{1/2(n-1)} + \varepsilon.$$
Theorem 4.11. [19] Let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $m$ edges and $t(G)$ spanning trees. Then, there exists a real number $\varepsilon \geq 0$ such that

$$I_{RE}(G) = LI_{E}(G) \geq \sqrt{2} + (n-2) \left( \frac{mt(G)}{\prod_{i=1}^{n} d_i} \right)^{1/2(n-2)} + \varepsilon.$$  

Theorem 4.12. [6] Let $G$ be a connected graph with $n \geq 3$ vertices. Then

$$I_{RE}(G) \geq \sqrt{2} + (n-2) \sqrt{\frac{n-2}{n+2R_{-1}(G)-4}}. \quad (4.3)$$

Equality holds if and only if $G \cong K_n$.

Remark 4.2. It is worth noting here that the lower bound (4.3) is stronger than the lower bound (4.1) for connected graphs. As can be seen in the inequality below

$$\sqrt{2} + (n-2) \sqrt{\frac{n-2}{n+2R_{-1}(G)-4}} \geq \frac{n}{\sqrt{2}}$$

that is,

$$\sqrt{\frac{2(n-2)}{n+2R_{-1}(G)-4}} \geq 1$$

this implies that

$$R_{-1}(G) \leq \frac{n}{2}$$

which is true for the general Randić index $R_{-1}(G)$, see [20].

Corollary 4.7. [6] Let $G$ be a connected graph with $n \geq 3$ vertices and minimum degree $\delta$. Then

$$I_{RE}(G) \geq \sqrt{2} + (n-2) \sqrt{\frac{n-2}{n \left(1 + \frac{1}{\delta} \right)^{-} - 4}}.$$  

Equality holds if and only if $G \cong K_n$.

5 Upper Bounds for $I_{RE}(G)$

In this section, we present some upper bounds on $I_{RE}(G)$ involving various structural graph parameters.

Theorem 5.1. [8,14] Let $G$ be a connected graph of order $n \geq 2$. Then

$$I_{RE}(G) \leq \sqrt{2} + \sqrt{(n-1)(n-2)}. \quad (5.1)$$

Equality holds if and only if $G \cong K_n$. 

Remark 5.1. [31] Among all graphs of order \( n \), the empty graph has the minimum \( I_{RE}(G) \) while the complete graph \( K_n \) reaches the maximum.

Theorem 5.2. [8] Let \( G, G \not\cong K_n \), be a connected graph of order \( n \geq 2 \). Then
\[
I_{RE}(G) \leq \sqrt{2} + 1 + \sqrt{(n-2)(n-3)}.
\] (5.2)
Equality holds if and only if \( \gamma_1^+ = 2, \gamma_2^+ = 1 \) and \( \gamma_i^+ = \frac{n-3}{n-2} \), for \( i = 3, \ldots, n \).

Theorem 5.3. [14] Let \( G \) be a bipartite graph of order \( n \) with no isolated vertices. Then
\[
I_{RE}(G) = LIE(G) \leq \sqrt{2} + n - 2.
\] (5.3)
Equality holds if and only if \( G \) is a complete bipartite graph.

Remark 5.2. [14] Since every tree is a bipartite graph, for any tree \( T \)
\[
I_{RE}(T) = LIE(T) \leq \sqrt{2} + n - 2
\]
with equality if and only if \( T \cong K_{1,n-1} \). Furthermore, among all trees with \( n \) vertices, the star graph \( K_{1,n-1} \) is the unique graph with maximum Randić incidence energy.

Theorem 5.4. [24] Let \( G \) be a connected non–bipartite graph with \( n \geq 3 \) vertices. Then
\[
I_{RE}(G) \leq \sqrt{2} + \sqrt{n-2} - \left( \frac{n+2R_{-1}(G)-4}{n-2} \right)^2.
\] (5.4)
Equality holds if and only if \( G \cong K_n \).

Theorem 5.5. [24] Let \( G \) be a connected non–bipartite graph with \( n \geq 3 \) vertices. Then, for any \( \alpha, \gamma_2^+ \geq \alpha \geq \sqrt{\frac{n+2R_{-1}(G)-4}{n-1}} \), holds
\[
I_{RE}(G) \leq \sqrt{2} + \sqrt{\alpha} + \left( (n-2)^3(n+2R_{-1}(G)-4) - \alpha^2 \right)^{1/4}.
\] (5.5)
Equality holds if and only if \( \alpha = \frac{n-2}{n-1} \) and \( G \cong K_n \).

Corollary 5.1. [24] Let \( G \) be a connected non–bipartite graph with \( n \geq 3 \) vertices. Then
\[
I_{RE}(G) \leq \sqrt{2} + ((n-1)^3(n+2R_{-1}(G)-4))^{1/4}.
\] (5.6)
Equality holds if and only if \( G \cong K_n \).

Corollary 5.2. [24] Let \( G \) be a connected non–bipartite graph with \( n \geq 3 \) vertices and minimum degree \( \delta \). Then
\[
I_{RE}(G) \leq \sqrt{2} + \left( (n-1)^3 \left( \frac{n(1+\delta)}{\delta} - 4 \right) \right)^{1/4}.
\] (5.7)
Equality holds if and only if \( G \cong K_n \).
**Theorem 5.6.** [24] Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then, for any $\alpha$, $\gamma_2^+ = \gamma_2 \geq \alpha \geq \sqrt{\frac{n+2R_1(G)-4}{n-2}}$, holds

$$I_{R}(G) = LIE(G) \leq \sqrt{2} + \sqrt{\alpha} + \left( (n-3)^3(n+2R_1(G)-4-\alpha^2) \right)^{1/4}. $$

Equality holds if and only if $\alpha = 1$ and $G \cong K_{p,q}$, $p+q = n$.

**Corollary 5.3.** [24] Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then

$$I_{R}(G) = LIE(G) \leq \sqrt{2} + \left( (n-2)^3(n+2R_1(G)-4) \right)^{1/4}. $$

Equality holds if and only if $G \cong K_n$.

**Theorem 5.7.** [24] Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then, for any $\alpha$, $\gamma_2^+ = \gamma_2 \geq \frac{n-2}{n-1}$, holds

$$I_{R}(G) = LIE(G) \leq \sqrt{2} + \sqrt{\alpha} + \sqrt{(n-2)(n-2-\alpha)}. $$

(5.4)

Equality holds if and only if $\alpha = \frac{n-2}{n-1}$ and $G \cong K_n$.

**Remark 5.3.** [24] The upper bounds (5.1) and (5.2) are, respectively, obtained from (5.4) for $\alpha = \frac{n-2}{n-1}$ and $\alpha = 1$.

**Theorem 5.8.** [24] Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then, for any $\alpha$, $\gamma_2^+ = \gamma_2 \geq \alpha \geq 1$, holds

$$I_{R}(G) = LIE(G) \leq \sqrt{2} + \sqrt{\alpha} + \sqrt{(n-3)(n-2-\alpha)}. $$

(5.5)

Equality holds if and only if $\alpha = 1$ and $G \cong K_{p,q}$, $p+q = n$.

**Remark 5.4.** [24] Note that the inequality (5.3) is obtained from (5.5) for $\alpha = 1$.

It is worth mentioning that the remaining upper bounds of this section were established in more general forms in [5, 6].

**Theorem 5.9.** [5] Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then

$$I_{R}(G) \leq \sqrt{2} + \sqrt{(n-2)^2 + (n-1) \left( \frac{t(G \times K_2)}{r(G) \prod_{i=1}^n d_i} \right)^{1/(n-1)}}. $$

(5.6)

Equality holds if and only if $G \cong K_n$.

**Remark 5.5.** [5] By using arithmetic-geometric mean inequality, it can be easily shown that the upper bound (5.6) is better than the upper bound (5.1) for connected non-bipartite graphs.
**Theorem 5.10.** [5] Let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $m$ edges and $t(G)$ spanning trees. Then

$$I_{RE}(G) = LIE(G) \leq \sqrt{2} + \sqrt{(n-2)(n-3) + (n-2) \left( \frac{mt(G)}{\prod_{i=1}^{n} d_i} \right)^{1/(n-2)}}. \quad (5.7)$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

**Remark 5.6.** [5] From arithmetic-geometric mean inequality, it can be seen that the upper bound (5.7) is stronger than the upper bound (5.3) for connected bipartite graphs. Moreover, the upper bound (5.3) was obtained in more general form in Theorem 3.7 of [3].

For a tree $T$, $m = n - 1$ and $t(T) = 1$. The following result is obvious from (5.7).

**Corollary 5.4.** [5] Let $T$ be a tree with $n \geq 3$ vertices. Then

$$I_{RE}(T) = LIE(T) \leq \sqrt{2} + \sqrt{(n-2)(n-3) + (n-2) \left( \frac{n-1}{\prod_{i=1}^{n} d_i} \right)^{1/(n-2)}}. \quad (5.7)$$

Equality holds if and only if $T \cong K_{1,n-1}$.

**Theorem 5.11.** [6] Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then

$$I_{RE}(G) \leq \sqrt{2} + \sqrt{\frac{2R_{-1}(G)}{n}} + \sqrt{(n-2) \left( n-3 + \frac{2R_{-1}(G)}{n} \right)}. \quad (5.8)$$

Equality holds if and only if $G \cong K_n$.

**Theorem 5.12.** [6] Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices and maximum degree $\Delta$. Then

$$I_{RE}(G) \leq \sqrt{2} + \sqrt{\frac{1-1}{\Delta}} + \sqrt{(n-2) \left( n-3 + \frac{1}{\Delta} \right)}. \quad (5.9)$$

Equality holds if and only if $G \cong K_n$.

**Remark 5.7.** [6] The upper bounds (5.8) and (5.9) are better than the upper bound (5.1) for connected non-bipartite graphs. Furthermore, (5.8) is the best for $I_{RE}(G)$ among the mentioned upper bounds.

**Theorem 5.13.** [6] Let $G$ be a connected bipartite graph with bipartition $V = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$. If $G \cong K_{p,q}$, then $I_{RE}(G) = LIE(G) = \sqrt{2} + n - 2$ [14]. Otherwise,

$$I_{RE}(G) = LIE(G) \leq \sqrt{2} + \sqrt{1 + \frac{1}{\sqrt{pq}}} + \sqrt{1 - \frac{1}{\sqrt{pq}}} + n - 4. \quad (5.10)$$

Equality holds if and only if $G \cong K_{p,q} - e$ ($e$ is any edge in $K_{p,q}$).
Remark 5.8. [6] Notice that the upper bound (5.10) is better than the upper bound (5.3) for any connected bipartite graph $G \neq K_{p,q}$ with bipartition $V = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$.

Remark 5.9. [6] Among all connected bipartite graphs except complete bipartite graph, $K_{p,q} - e$ has the maximum Randić (normalized) incidence energy or Laplacian incidence energy.

6 Coulson Integral Formula of $I_{RE}(G)$

As early as the 1940s [10], Coulson obtained the following integral formula for graph energy

$$E(G) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[ n - \frac{ix\phi'(A(G),ix)}{\phi(A(G),ix)} \right] dx$$

where $\phi(A(G),x)$ is the characteristic polynomial of the adjacency matrix $A(G)$ of the $n$-vertex graph $G$. This formula is known as Coulson integral formula in the literature. Its generalization [25] directly implies the following integral formula for Randić (normalized) incidence energy [8].

Theorem 6.1. [8] Let $G$ be a graph of order $n$ and $\phi(\mathcal{L}^+(G),x)$ be the characteristic polynomial of its normalized signless Laplacian matrix $\mathcal{L}^+(G)$. Then

$$I_{RE}(G) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ 2n - \frac{ixf'(ix)}{f(ix)} \right] dx$$

where $f(x) = \phi(\mathcal{L}^+(G),x^2)$.

The coefficient form of the characteristic polynomial of the normalized signless Laplacian matrix $\mathcal{L}^+(G)$ can be expressed as [8]

$$\phi(\mathcal{L}^+(G),x) = \sum_{k=0}^{n} (-1)^k b_k(G)x^{n-k}. \quad (6.1)$$

The another way to write the Coulson integral formula was presented in [8] as follows:

Theorem 6.2. [8] Let $G$ be a graph of order $n$ and let $\phi(\mathcal{L}^+(G),x)$ be of the form given by (6.1). Then

$$I_{RE}(G) = \frac{1}{\pi} \int_{0}^{+\infty} \ln \left( \sum_{k=0}^{n} b_k(G) x^{2k} \right) \frac{dx}{x^2}.$$  

In [8], It was also pointed out that the above result makes it possible to compare the Randić (normalized) incidence energies of two graphs.

Corollary 6.1. [8] Let $G_1$ and $G_2$ be two $n$-vertex graphs. If $b_k(G_1) \leq b_k(G_2)$ for $0 \leq k \leq n$, then $I_{RE}(G_1) \leq I_{RE}(G_2)$. Moreover, if a strict inequality $b_k(G_1) < b_k(G_2)$ holds for some $0 \leq k \leq n$, then $I_{RE}(G_1) < I_{RE}(G_2)$. 
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References


