Remark on the upper bounds for arithmetic–geometric
topological index of graphs

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Abstract: In this paper two new inequalities involving upper bounds for arithmetic–geometric
topological index are obtained.

Keywords: Topological indices, arithmetic–geometric index.

1 Introduction

Let $G = (V,E)$, $V = \{v_1,v_2,\ldots,v_n\}$, $E = \{e_1,e_2,\ldots,e_m\}$, be a simple connected graph of
order $n$ and size $m$. Denote by $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$, and $\Delta_e =
d(e_1) + 2 \geq d(e_2) + 2 \geq \cdots \geq d(e_m) + 2 = \delta_e > 0$ the sequences of its vertex and edge
degrees, respectively, given in an nonincreasing order. If vertices $v_i$ and $v_j$ are adjacent in $G$, we write $i \sim j$.
As usual, $L(G)$ denotes a line graph of graph $G$.

Let $v_j$ and $v_k$ be two nonadjacent vertices in $G$ that are both adjacent to $v_i$. Denote by
$\Gamma(G)$ a class of connected graphs with the property $d_i = \sqrt{d_j d_k}$, or $L(G)$ is bidegreed.

A topological index, or a graph invariant, is a numerical quantity which is invariant
under automorphisms of the graph. Many of them are defined as simple functions of the
degrees of the vertices and edges, distances between vertices, or various graph spectra. Here
we recall definitions of some vertex–degree-based indices that are of interest for the present
paper.

The first Zagreb index, $M_1(G)$, introduced in [6], is defined as

$$M_1(G) = \sum_{i=1}^{n} d_i^2.$$ 

In [4] it is proven that the first Zagreb index can also be expressed as

$$M_1 = \sum_{i<j} (d_i + d_j).$$
The second Zagreb index, $M_2(G)$, is defined as [7]

$$M_2(G) = \sum_{i \sim j} d_id_j,$$

and the so called general Randić index, $R_{-1}(G)$, as [2]

$$R_{-1}(G) = \sum_{i \sim j} \frac{1}{d_id_j}.$$

The geometric–arithmetic index, $GA(G)$ index for short, proposed in [28], is defined to be

$$GA(G) = \sum_{i \sim j} \frac{2\sqrt{d_id_j}}{d_i + d_j}.$$

By analogy with $GA$ index, the arithmetic–geometric index, $AG(G)$, is defined as [25]

$$AG(G) = \sum_{i \sim j} \frac{d_i + d_j}{2\sqrt{d_id_j}}.$$

More on this topological index can be found, for example, in [12, 17, 21, 24, 27].

In this paper we prove two inequalities involving upper bounds for $AG(G)$ index, and obtain many upper bounds for $AG(G)$ reported in the literature as corollaries.

2 Preliminaries

In this section we state one analytical inequality for real number sequences which will be used in the rest of the paper.

**Lemma 2.1.** [23] Let $x = (x_i), i = 1, 2, \ldots, n$, be a sequence of nonnegative real numbers and $a = (a_i), i = 1, 2, \ldots, n$, be positive real number sequence. Then for any $r \geq 0$ holds

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \left( \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} a_i} \right)^{r+1}. \tag{2.1}$$

Equality holds if and only if $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

3 Main results

In the next theorem we establish relation between $AG(G)$ and $R_{-1}(G)$. 
Theorem 3.1. Let $G$ be a simple connected graph with $n \geq 3$ vertices and $m$ edges. Then
\[
AG(G) \leq \frac{1}{2} \sqrt{m(n(\Delta_e + \delta_e) - \Delta_e \delta_e \Gamma(G))}. \tag{3.1}
\]
Equality holds if and only if $G$ is a regular or semiregular bipartite graph or $G \in \Gamma(G)$.

Proof. For any two adjacent vertices $v_i$ and $v_j$ in $G$ we have that
\[
(d_i + d_j - \Delta_e)(d_i + d_j - \delta_e) \leq 0,
\]
i.e.
\[
(d_i + d_j)^2 + \delta_e \Delta_e \leq (\Delta_e + \delta_e)(d_i + d_j). \tag{3.2}
\]
After dividing the above inequality by $4d_id_j$ and summing over all edges in $G$, we get
\[
\sum_{i \sim j} \frac{(d_i + d_j)^2}{4d_id_j} + \frac{\Delta_e \delta_e}{4} \sum_{i \sim j} \frac{1}{d_id_j} \leq \frac{\Delta_e + \delta_e}{4} \sum_{i \sim j} d_i + d_j,
\]
that is
\[
\sum_{i \sim j} \frac{(d_i + d_j)^2}{4d_id_j} \leq \frac{1}{4} \left(n(\Delta_e + \delta_e) - \Delta_e \delta_e \Gamma(G) \right). \tag{3.3}
\]
On the other hand, for $r = 1, x_i := \frac{d_i + d_j}{2\sqrt{d_id_j}}, a_i := 1$, with summation performed over all edges in graph $G$, the inequality (2.1) becomes
\[
\sum_{i \sim j} \frac{(d_i + d_j)^2}{4d_id_j} \geq \frac{\left(\sum_{i \sim j} \frac{d_i + d_j}{2\sqrt{d_id_j}}\right)^2}{\sum_{i \sim j} 1},
\]
that is
\[
\sum_{i \sim j} \frac{(d_i + d_j)^2}{4d_id_j} \geq \frac{AG(G)^2}{m}. \tag{3.4}
\]
Now, from (3.3) and (3.4) it follows
\[
\frac{AG(G)^2}{m} \leq \frac{1}{4} \left(n(\Delta_e + \delta_e) - \Delta_e \delta_e \Gamma(G) \right),
\]
from which (3.1) is obtained.

Equality in (3.2) holds if and only if $d_i + d_j \in \{\Delta_e, \delta_e\}$ for any pair of adjacent vertices $v_i$ and $v_j$ in $G$. This means that equality in (3.2) holds if and only if $L(G)$ is regular or bidegreed graph.

Equality in (3.3) holds if and only if $\frac{d_i + d_j}{2\sqrt{d_id_j}}$ is constant for any pair of adjacent vertices $v_i$ and $v_j$ in $G$. Let $v_j$ and $v_k$ be two vertices adjacent to $v_i$. Then
\[
\frac{d_i + d_j}{\sqrt{d_id_j}} = \frac{d_i + d_k}{\sqrt{d_id_k}},
\]
i.e.
\[
\left( \sqrt{d_k} - \sqrt{d_j} \right) \left( d_i - \sqrt{d_jd_k} \right) = 0.
\]

If \( d_k = d_j \), then equality in (3.3) holds if and only if \( G \) is regular or semiregular bipartite graph. Suppose that \( d_k \neq d_j \). Then equality in (3.3) holds if and only if \( d_i = \sqrt{d_jd_k} \). This means that equality in (3.3), and consequently in (3.1), holds if and only if \( G \) is a regular or semiregular bipartite graph or \( G \in \Gamma(G) \).

**Corollary 3.1.** Let \( G \) be a simple connected graph with \( n \geq 3 \) vertices and \( m \) edges. Then

\[
AG(G) \leq \frac{n}{4} \left( \sqrt{\frac{\Delta}{\delta_e}} + \sqrt{\frac{\delta_e}{\Delta_e}} \right) \sqrt{\frac{m}{R_1(G)}}.
\]

Equality holds if and only if \( G \) is regular or semiregular bipartite graph or \( G \in \Gamma(G) \) with even number of vertices.

**Proof.** Based on the arithmetic–geometric mean inequality, AM–GM (see for example [20]), according to (3.1) we have that

\[
2\sqrt{4m\Delta_e\delta_eR_1(G)}AG(G)^2 \leq 4AG(G)^2 + m\Delta_e\delta_eR_1(G) \leq mn(\Delta_e + \delta_e),
\]

from which (3.5) is obtained. \( \square \)

Since

\[
2\delta \leq \delta_e \leq d_i + d_j \leq \Delta_e \leq 2\Delta,
\]

for every edge in \( G \), with equalities holding if and only if \( G \) is regular, we have the following corollaries of Theorem 3.1.

**Corollary 3.2.** Let \( G \) be a simple connected graph with \( n \geq 3 \) vertices and \( m \) edges. Then

\[
AG(G) \leq \frac{\sqrt{2}}{2} \sqrt{m(n(\Delta + \delta) - 2\Delta\delta R_1(G))}.
\]

Equality holds if and only if \( G \) is a regular graph.

**Corollary 3.3.** Let \( G \) be a simple connected graph with \( n \geq 3 \) vertices and \( m \) edges. Then

\[
AG(G) \leq \frac{n}{4} \left( \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}} \right) \sqrt{\frac{m}{R_1(G)}}.
\]

Equality holds if and only if \( G \) is a regular graph.

**Corollary 3.4.** Let \( G \) be a simple connected graph with \( n \geq 3 \) vertices and \( m \) edges. Then

\[
AG(G) \leq \frac{n}{4} \left( \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}} \right) \sqrt{\frac{M_2(G)}{m}}.
\]

Equality holds if and only if \( G \) is regular.
Proof. From the arithmetic–harmonic mean inequality, AM–HM (see for example [20]), we get
\[ \sum_{i \sim j} d_id_j \sum_{i \sim j} \frac{1}{d_id_j} \geq m^2, \]
i.e.
\[ \frac{m}{R_{-1}(G)} \leq \frac{M_2(G)}{m}. \] (3.8)
From the above and (3.6) we get (3.7).

In the following theorem we prove the inequality involving an upper bound for \( AG(G) \) in terms of \( M_1(G) \) and parameter \( n \).

**Theorem 3.2.** Let \( G \) be a simple connected graph with \( n \geq 2 \) vertices. Then
\[ AG(G) \leq \frac{1}{2} \sqrt{nM_1(G)}. \] (3.9)
Equality holds if and only if \( G \) is a regular or semiregular bipartite graph.

**Proof.** For \( r = 1 \), \( x_i := d_i + d_j \sqrt{d_id_j}, \) \( a_i := d_i + d_j \), with summation performed over all edges in graph \( G \), the inequality (2.1) transforms into
\[ \sum_{i \sim j} (d_i + d_j)^2 d_id_j(d_i + d_j) = \sum_{i \sim j} \left( \frac{d_i + d_j}{2\sqrt{d_id_j}} \right)^2 \geq \frac{\left( \sum_{i \sim j} \frac{d_i + d_j}{2\sqrt{d_id_j}} \right)^2}{\sum_{i \sim j} \frac{1}{4}(d_i + d_j)^2}, \]
that is
\[ \sum_{i \sim j} \frac{d_i + d_j}{d_id_j} \geq \frac{AG(G)^2}{\frac{4}{3}M_1(G)}. \] (3.10)
Since
\[ \sum_{i \sim j} \frac{d_i + d_j}{d_id_j} = \sum_{i \sim j} \left( \frac{1}{d_i} + \frac{1}{d_j} \right) = \sum_{i = 1}^n \frac{1}{d_i} = n, \]
from (3.10) it follows
\[ n \geq \frac{AG(G)^2}{\frac{4}{3}M_1(G)}, \]
from which (3.9) is obtained.

Equality in (3.10) holds if and only if \( \frac{1}{\sqrt{d_id_j}} \) is constant for any pair of adjacent vertices \( v_i \) and \( v_j \) in \( G \). Let \( v_j \) and \( v_k \) be two vertices adjacent to \( v_i \). Then
\[ \frac{1}{\sqrt{d_id_j}} = \frac{1}{\sqrt{d_id_k}}, \]
i.e. \( d_j = d_k \). This means that equality in (3.10), and therefore in (3.9), holds if and only if \( G \) is a regular or semiregular bipartite graph. \( \square \)
As mentioned, upper bound for $AG(G)$ given by (3.9) depends on $M_1(G)$. Since the first Zagreb index is for sure the most examined topological index in the literature (see, for example, [1, 8, 9, 22] and the literature cited therein), the inequality (3.9) enables to determine upper bounds for $AG(G)$ in terms of many different graph parameters. In the following corollaries we give some of them.

**Corollary 3.5.** Let $G$ be a simple connected graph with $n$ vertices and $m$ edges. Then

$$AG(G) \leq \frac{1}{2} \sqrt{n(2m(\Delta + \delta) - n\Delta\delta)}.$$  \hspace{1cm} (3.11)

Equality holds if and only if $G$ is regular or semiregular bipartite graph.

**Proof.** In [3] (see also [10, 11, 13, 14, 19]) it was proven that

$$M_1(G) \leq 2m(\Delta + \delta) - n\Delta\delta,$$

with equality holding if and only if $d_i \in \{\Delta, \delta\}$ for every $i = 1, 2, \ldots, n$. From the above and (3.9) we arrive at (3.11). \hfill \Box

**Corollary 3.6.** Let $G$ be a simple connected graph with $m \geq 1$ edges. Then

$$AG(G) \leq \frac{m}{2} \left( \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}} \right).$$  \hspace{1cm} (3.12)

Equality holds if and only if $G$ is a regular or biregular graph.

**Proof.** In [15] (see also [5, 11, 26]) it was proven that

$$M_1(G) \leq \frac{m^2}{n} \left( \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}} \right)^2.$$

From the above and (3.9) we arrive at (3.12). \hfill \Box

The inequality (3.12) was proven in [17] (see also [24]).

Since

$$M_1(G) \leq m\Delta_e \leq 2m\Delta \leq n\Delta^2 \leq n(n-1)^2,$$

we have the following corollary of Theorem 3.2.

**Corollary 3.7.** Let $G$ be a simple connected graph with $n \geq 2$ vertices and $m$ edges. Then

$$AG(G) \leq \frac{1}{2} \sqrt{nm\Delta_e} \leq \frac{1}{2} \sqrt{2nm\Delta} \leq \frac{n\Delta}{2} \leq \frac{n(n-1)}{2}.$$  \hspace{1cm} (3.13)

The last two inequalities in (3.13) were proven in [27].
Corollary 3.8. Let $G$ be a simple connected graph with $n \geq 2$ vertices and $m$ edges. Then
\[
AG(G) \leq \frac{1}{2} \sqrt{4m^2 + n^2 \alpha(n)(\Delta - \delta)^2},
\]
(3.14)
where
\[
\alpha(n) = \frac{1}{4} \left( 1 + \frac{(-1)^{n+1}}{2n^2} \right).
\]
Equality holds if and only if $G$ is a regular graph.

Proof. In [18] the following inequality was proven
\[
M_1(G) \leq \frac{4m^2}{n} + n\alpha(n)(\Delta - \delta)^2.
\]
From the above and (3.9) we obtain (3.14).

Remark 3.1. Since $\alpha(n) \leq \frac{1}{4}$ for every $n$, we have that
\[
M_1(G) \leq \frac{4m^2}{n} + \frac{n}{4}(\Delta - \delta)^2,
\]
which was proven in [5]. According to this, it follows
\[
AG(G) \leq \frac{1}{2} \sqrt{4m^2 + \frac{n^2}{4}(\Delta - \delta)^2}.
\]

Corollary 3.9. Let $T$ be a tree with $n \geq 2$ vertices. Then
\[
AG(T) \leq \frac{1}{2} \sqrt{n(2(n-1) + (n-2)\Delta)}.
\]
(3.15)
Equality holds if and only if $T \cong K_{1,n-1}$.

Proof. In [16] the following inequality was proven
\[
M_1(T) \leq 2(n - 1) + (n - 2)\Delta.
\]
From the above and (3.9) we arrive at (3.15).

Since $\Delta \leq n - 1$, we have the following corollary.

Corollary 3.10. Let $T$ be a tree with $n \geq 2$ vertices. Then
\[
AG(T) \leq \frac{n\sqrt{n - 1}}{2}.
\]
(3.16)
Equality holds if and only if $T \cong K_{1,n-1}$.

The inequality (3.16) was proven in [27].
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References

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