NEKI ASPEKTI U ANALIZI POPREČNIH SLOBODNIH VIBRACIJA PRIZMATIČNIH GREDA OPTEREĆENIH AKSIJALNOM SILOM

SOME ASPECTS OF ANALYSIS OF TRANSVERSE FREE VIBRATIONS OF UNIFORM BEAMS LOADED WITH AXIAL FORCE

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1 UVOD

Prave grede su najčešće korišćeni elementi konstrukcija, prije svega zbog dostupnosti, jednostavnosti i višestruke primjenljivosti. Shodno tome, najrazvijenije i najutemeljenije grane mehanike kontinuuma jesu upravo gredne teorije. Iako su mnogi aspekti ponašanja ovih elemenata dobro proučeni, analiza greda i dalje privlači pažnju istraživača čiji je cilj njihovo pojednostavljenje i unapređivanje. Jedna od najinteresantnijih oblasti istraživanja jeste nelinearna dinamička analiza grednih konstrukcija. Odgovarajuće modeliranje ovakvog ponašanja od presudnog je značaja u savremenom građevinarstvu, s obzirom na to što su moderne konstrukcije vitke i osjetljive na dinamičke uticaje. Prilikom analiza koje razmatraju slučajeve s velikim pomjeranjima i deformacijama, nelinearni efekti moraju se uzeti u obzir. Matrica krutosti više nije jednaka samo linearnoj, tj. materijalnoj, već se moraju uzeti u obzir i efekti početnih pomjeranja i početnih napona. Kao posljedica izmjene krutosti, mijenjaju se i svojstvene frekvencije i svojstveni oblici vibracija konstrukcije.

1 INTRODUCTION

Due to their availability, simplicity and versatility, straight beams are the most frequently used structural component. Consequently, beam theories are the most established and recognized branches of continuum mechanics. Although many aspects of behaviour of these elements are well-studied, analysis of beams still receives attention from researchers and refinements and improvements of beam theories are ongoing. One of the most interesting areas of investigation is the nonlinear dynamic analysis of beam structures. Appropriate modelling of this behaviour is of crucial importance in contemporary structural engineering, since modern constructions are slender and therefore sensitive to dynamic excitation. For analyzing cases where displacements and strains are significant, nonlinear effects must be considered. Stiffness matrix is no longer equal to the linear, material one, but the effects of initial displacements and initial stresses must be included also. Due to the change of stiffness, free vibration eigenfrequencies and eigenshapes of structure change accordingly.

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Standardni inženjerski zadatak jeste modeliranje složenog ponašanja konstrukcija s jednostavnim modelom koji je prikladan za dati slučaj. Prema tome, jedan od prvih koraka - koji prethodi opštoj nelinearnoj dinamičkoj analizi - jeste pojednostavljena linearizovana analiza slobodnih poprečnih vibracija greda opterećenih konstantnom aksijalnom silom. Analitička rješenja ovog problema dobro su poznata. Galef je još 1968. godine definisao vezu između svojstvene frekvencije ω_n i nivoa aksijalne sile P za Bernuli-Ojlerovu gredu u sljedećem obliku [1]

$$\Omega_n = \sqrt{1 - \bar{U}_n} \quad (1)$$

gdje su $\Omega_n = \omega_n/\omega_{n0}$ i $\bar{U}_n = P/P_{cr,n}$, dok je ω_{n0} n -ta sopstvena frekvencija neopterećene grede i $P_{cr,n}$ n -ta sila izvijanja. Ovaj izraz pokazuje da svojstvene frekvencije za n -ti mod (oblik) aksijalno pritisnutih greda konvergira ka nuli kako se vrijednost sile P približava vrijednosti kritične sile za n -ti mod izvijanja, a što je i dobro poznata činjenica iz eksperimentalne analize [2].

Treba napomenuti i to da se ovaj rezultat odnosi na idealno ravnu gredu pri kontroli opterećenja. Za realne grede, s početnom zakrivljenošću i pri kontroli pomjeranja, ponašanje se značajno razlikuje, uglavnom zbog različitih graničnih uslova u tački unošenja sile [3].

U radu se razmatraju tri klasične teorije za analizu vibracija greda. Bernuli-Ojlerova pretpostavka o apsolutno krutim presjecima, upravnim na deformisanu osu, potiče iz 18. vijeka i uzima u obzir inerciju isključivo usljed translacije poprečnog presjeka. Efekat rotacione inercije uveo je Rejli 1894, dok je Timošenko 1921. godine predložio svoju teoriju u kojoj je uključen i efekat smicanja, [4]. Detaljan pregled literature u vezi sa analizom slobodnih vibracija neopterećenih greda dat je u [5]. U istom radu prikazano je detaljno izvođenje jednačina poprečnih vibracija prizmatičnih greda primjenom četiri teorije. Četvrta teorija koja se razmatra u [5] naziva se *teorija smicanja* i ona – za razliku od Timošenkovke teorije – zanemaruje efekat rotacione inercije. Analiza teorije smicanja i Rejljeve teorije, kao graničnih slučajeva Timošenkovke teorije, data je u [6]. Autori u [4] zaključuju da Timošenkov model grede dobro aproksimira rješenje dvodimenzionalne teorije za niže modove grede proizvoljne visine poprečnog presjeka, dok je Bernuli-Ojlerova teorija ograničena na analizu vitkih greda.

U radu [2] prikazane su neke od prvih detaljnih analitičkih i eksperimentalnih analiza slobodnih vibracija greda opterećenih aksijalnom silom. Dobijen je isti odnos koji je kasnije formulisao Galef u [1]. Autor je zaključio da je ovaj odnos validan ako je mod slobodnih vibracija identičan modu izvijanja, što je kasnije i potvrdilo mnogo autora. Za sve slučajeve dobijena je nulta frekvencija kada je opterećenje jednako kritičnom opterećenju.

Galef je predstavio svoju čuvenu formulu (1) u [1], s detaljnom analizom rezultata koje je dao Amba-Rao [7]. Detaljna analitička razmatranja svih graničnih uslova data su u [8], gdje su tačno definisane granice primjenljivosti Galefove formule. Ova formula važi za sljedeće granične uslove oslanjanja: obostrano slobodno oslonjena, slobodno oslonjena-poprečno pokretno uklještena i obostrano poprečno pokretno uklještena. Za osnovni mod, ova zavisnost važi i za obostrano uklještenu i uklještenu-poprečno pokretno uklještenu gredu, a približna je za slobodno oslonjenu-uklještenu i

Standard engineering task is to model complex behaviour with simple model which is appropriate for given task. Therefore, one of the first steps preceding the general nonlinear dynamic analysis is a simplified linear analysis of free transverse vibrations of beams loaded with constant axial force. Analytical solutions of this problem are well-known. Galef, in 1968, defined relation between eigenfrequencies ω_n and level of axial force P for Bernoulli-Euler beam in the following form [1]

where are $\Omega_n = \omega_n/\omega_{n0}$ and $\bar{U}_n = P/P_{cr,n}$, while ω_{n0} is n th eigenfrequency of an unloaded beam and $P_{cr,n}$ is n th buckling load.

This expression shows that eigenfrequency of n th mode of axially compressed beam converges to zero as the value of load P approaches value of critical force of n th buckling mode, which is the fact well-known from experimental analysis [2].

It should be noted that this result is related to ideally straight beam with load control. For realistic beams, with initial curvatures and under displacement control, behaviour is significantly different, mainly due to different boundary condition at the point of application [3].

There are three classic theories for vibration analysis of beams. The Bernoulli-Euler assumption of absolutely rigid cross sections perpendicular to deformed axis has its origins in the 18th century and it considers inertia solely due to translation of cross section. Effect of rotary inertia was introduced by Rayleigh in 1894, while Timoshenko proposed his theory where shear is also taken into account in 1921, [4]. Detailed review of literature on free vibration analyses of unloaded beams is given in [5]. In the same paper, the full derivation of four models for the transverse vibrations of uniform beam is presented. Fourth theory considered is named *shear theory* and it, contrary to the Timoshenko one, ignores effect of rotary inertia. Analysis of shear and Rayleigh theories as limit cases of the Timoshenko one is given in [6]. Authors in [4] concludes that Timoshenko beam model closely approximates solution of the two-dimensional theory for lower modes of arbitrarily thick beams, while the Bernoulli-Euler theory is limited to the analysis of thin beams.

One of the first researches which performed thorough analytical and experimental analysis of free vibrations of beams loaded with axial force is [2]. The same relationship formulated later by Galef [1] was obtained. The author concluded that this relationship is valid if the mode of free vibrations is identical to the buckling mode, which is later confirmed by many authors. For all cases, the author obtained zero frequencies when the load is equal to buckling load.

Galef introduced his famous formula (1) in [1] by careful study of results given by Amba-Rao [7]. Detailed analytical considerations of all boundary conditions (BC) is given in [8], where the limits of Galef's formula are exactly defined. This formula is valid for supported-supported, supported-guided and guided-guided beams. For fundamental mode, this dependence is valid for clamped-clamped and clamped-guided beams, and approximate for clamped-supported and supported-free beam. Effect of BCs is significant for few first modes,

slobodno oslonjenu-slobodnu gredu. Uticaj graničnih uslova značajan je za nekoliko prvih modova. Međutim, kako se modalni oblici grede približavaju ka višestrukim sinusnim polutalasima, jednačina (1) daje tačan odnos za sve granične uslove. U radu je ponovo dokazano da Galefova formula daje dobre rezultate za grede za koje su svojstveni oblici vibracija bez aksijalne sile i oblici izvijanja slični.

U [9] naglašeno je da se rješenje zatvorenog oblika može dobiti samo za tri tipa graničnih uslova, i to: obostrano slobodno oslonjena, slobodno oslonjena-poprečno pokretno uklještena i obostrano poprečno pokretno uklještena. Analitička i eksperimentalna analiza vibracija izvijanih greda predstavljena je u [10], gdje je dobijeno odlično poklapanje rezultata za obostrano uklještenu gredu.

U [11] proučavane su slobodne vibracije Rejljeve konzole s proizvoljnim aksijalnim opterećenjem i koncentrisanom masom na kraju, dok je efekat koncentrisanih masa, rotacionih inercija, linearnih opruga, rotacionih opruga i sistema opruga-masa analiziran u [12]. Slobodne vibracije greda opterećenih silom zatezanja razmatrane su u [13] i [14], dok su neke primjene na tankozidne grede date u [15].

Ovaj sažeti pregled literature pokazuje da razmatrani problem i dalje privlači pažnju istraživača. Takođe, uočen je nedostatak detaljne analize uticaja aksijalne sile na poprečne vibracije grede prema sve tri teorije, sistematizovane na jednom mjestu. Stoga, glavni motiv ovog istraživanja jeste poređenje analitičkih rezultata slobodnih poprečnih vibracija prizmatične grede opterećene konstantnom aksijalnom silom pritiska prema navedenim grednim teorijama, kao i razvoj numeričkog modela za analizu greda s različitim uslovima oslanjanja. Da bi se to postiglo, ukratko se izvodi jednačina slobodnih vibracija za Timošenkovu gredu, dok se Bernuli-Ojlerova teorija i Rejljeva teorija razmatraju kao posebni slučajevi. Za numeričku analizu, primijenjen je jednostavni dvodimenzionalni konačni element za ravno stanje napona s *drilling* (*bušećim*) stepenom slobode. Odgovarajući kod je razvijen i verifikovan poređenjem sa Abakusom, komercijalnim softverom na bazi konačnih elemenata (KE), kao i sa odgovarajućim analitičkim rezultatima. Prikazana je detaljna numerička analiza i izvedeni su odgovarajući zaključci.

Sljedeće poglavlje daje kratak prikaz izvođenja analitičkih izraza, dok je opis korištenog konačnog elementa dat u trećem poglavlju. Numerička analiza izvršena je u četvrtom poglavlju, nakon čega su izvedeni zaključci.

2 ANALITIČKA RAZMATRANJA

U ovom dijelu prikazano je izvođenje jednačina kretanja grede, pod dejstvom sile pritiska, poznato iz literature [16]. Razmatrane su tri standardne teorije. Pretpostavka o apsolutno krutim presjecima važi za sve teorije, dok je jedina razlika u tretiranju rotacione inercije i deformacije smicanja. Prva je Bernuli-Ojlerova teorija, koja je najjednostavnija s obzirom na to što zanemaruje deformaciju smicanja i inerciju usljed rotacije poprečnog presjeka. Ako se uvedu i efekti ove inercije, dobija se Rejljeva teorija. Timošenkova teorija je najopštija, jer ne zanemaruje nijedan navedeni efekat. Cilj je dobijanje svojstvenih vibracija pravih prizmatičnih greda opterećenih aksijalnom silom s različitim graničnim uslovima.

and as the mode shape of the beam approaches multi-sine wave, equation (1) gives true variation for all BCs. It is proved again that Galef's formula is valid for beams for which the free vibration eigenshapes without axial force and buckling shapes are similar.

Author in [9] emphasize that closed form solution can be obtained for only three types of BCs, namely supported-supported, supported-guided and guided-guided. Analytical and experimental analysis of vibrations of buckled beams is presented in [10] where the excellent agreement of results for clamped-clamped beam is obtained.

Free vibrations of Rayleigh cantilever with arbitrary axial loading and tip mass are studied in [11] while the effect of point masses, rotary inertias, linear springs, rotational springs and spring-mass systems is analyzed in [12]. Free vibrations of beams loaded with tensile force are considered in [13] and [14], while some applications for thin-walled beams are given in [15].

This brief literature review shows that the considered problem is thoroughly investigated and, yet, it continues to attract attention of researchers. It is also noticed that there is no detail comparison of vibration modes of axially loaded beams using all three theories considered in one paper. Therefore, the driving force of this research is comparison of analytical results for free transverse vibrations of uniform beams loaded with constant compressive axial force using three beam theories, as well as development of a numerical model for the analysis of beams with different BCs. In order to accomplish this, equation for Timoshenko beam is briefly derived and Bernoulli-Euler and Rayleigh theories are considered as special cases. Simple 2D elasticity finite element with drilling degree of freedom (DOF) is utilized for numerical analysis. Appropriate code is developed and verified via comparison with commercial finite element (FE) software Abaqus as well as with analytical results. Detailed numerical study is presented and appropriate conclusions are derived.

Brief derivation of analytical expressions is introduced in the following section, while the used finite element is presented in the third section. Numerical analysis is given in the fourth section which is followed by conclusions.

2 ANALYTICAL CONSIDERATIONS

In this section, well-known derivation of equations of motion of a beam subjected to compressive force is given [16]. Three standard theories are considered. Assumption of absolutely rigid cross sections is valid for all these theories while the only difference is treatment of rotary inertia and shear strain. The first is the Bernoulli-Euler one, which is the simplest one since it ignores shear strain and inertia due to rotation of cross-section. If the effects of this rotary inertia are introduced, the Rayleigh theory is obtained. Finally, the Timoshenko theory does not ignore either. The aim is to obtain vibration eigenmodes of straight uniform beams loaded with axial force with different end BCs.

2.1 Jednačina kretanja

U standardnoj tehničkoj teoriji savijanja grede [17], poprečno pomjeranje ose dobija se kao zbir pomjeranja usljed savijanja i smicanja

$$v = v_b + v_s, \quad (2)$$

koje definiše nagib ose grede u deformisanoj konfiguraciji

$$\frac{\partial v}{\partial x} = \frac{\partial v_b}{\partial x} + \frac{\partial v_s}{\partial x}. \quad (3)$$

Infinitesimalna rotacija poprečnog preseka je

$$\beta = \frac{\partial v_b}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial v_s}{\partial x} = \varphi - \varphi_T, \quad (4)$$

gdje je φ_T smičuća komponenta deformacije. Energija deformacije grede (Π) može se izraziti kao

$$\begin{aligned} \Pi &= \frac{1}{2} \int_V (\sigma_x \varepsilon_x + \tau_{xy} \varphi_T) dV = \frac{1}{2} \int_V [E \varepsilon_x^2 + k G \varphi_T^2] dV = \\ &= \frac{1}{2} \int_0^L \int_A \left[E y^2 \left(\frac{\partial \beta}{\partial x} \right)^2 + k G \left(\frac{\partial v}{\partial x} - \beta \right)^2 \right] dA dx = \\ &= \frac{1}{2} \int_0^L \left[EI \left(\frac{\partial \beta}{\partial x} \right)^2 + k GA \left(\frac{\partial v}{\partial x} - \beta \right)^2 \right] dx, \end{aligned} \quad (5)$$

gdje je $v=v(x,t)$ poprečno pomjeranje tačke ose grede s koordinatom x u vremenskom trenutku t , E je modul elastičnosti, G je modul smicanja, A je površina poprečnog presjeka, I je moment inercije poprečnog presjeka, a k je koeficijent korekcije smicanja. Kinetička energija grede (T) data je izrazom

$$T = \frac{1}{2} \int_V \rho \left[\left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right] dV = \frac{1}{2} \int_0^L \left[\rho A \left(\frac{\partial v}{\partial t} \right)^2 + \rho I \left(\frac{\partial \beta}{\partial t} \right)^2 \right] dx, \quad (6)$$

gdje je $u = -y\beta$ dok ρ predstavlja zapreminsku masu materijala od kog je greda napravljena.

Za izvođenje rada spoljašnjih sila, posmatraćemo Sliku 1 na kojoj je prikazan deformisani položaj prave prizmatične grede opterećene aksijalnom silom pritiska pri slobodnim vibracijama. Primijenimo pretpostavku o vibracijama s malim amplitudama, iz koje slijedi $\cos \varphi \approx 1$, te ćemo pomjeranje presjeka u pravcu nedeformisane ose grede aproksimirati kao

$$du = ds - dx = \left[(dx)^2 + \left(\frac{\partial v}{\partial x} dx \right)^2 \right]^{1/2} - dx = dx \left\{ \left[1 + \left(\frac{\partial v}{\partial x} \right)^2 \right]^{1/2} - 1 \right\}. \quad (7)$$

Ako se dobijeni izraz aproksimira s prva dva člana Tejlorovog reda razvijenog u okolini $\partial v / \partial x = 0$, za ovo podužno pomjeranje dobija se

$$du \approx \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 dx. \quad (8)$$

2.1 Equation of motion

Standard beam theory [17] defines the transverse displacement of the beam axis as the sum of the displacements due to bending and shear

$$v = v_b + v_s, \quad (2)$$

which gives the slope of the beam axis in deformed configuration

$$\frac{\partial v}{\partial x} = \frac{\partial v_b}{\partial x} + \frac{\partial v_s}{\partial x}. \quad (3)$$

Infinitesimal rotation of cross section is

$$\beta = \frac{\partial v_b}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial v_s}{\partial x} = \varphi - \varphi_T, \quad (4)$$

where φ_T is shear strain component. The strain energy of the beam (Π) can be expressed as

$$\begin{aligned} \Pi &= \frac{1}{2} \int_V (\sigma_x \varepsilon_x + \tau_{xy} \varphi_T) dV = \frac{1}{2} \int_V [E \varepsilon_x^2 + k G \varphi_T^2] dV = \\ &= \frac{1}{2} \int_0^L \int_A \left[E y^2 \left(\frac{\partial \beta}{\partial x} \right)^2 + k G \left(\frac{\partial v}{\partial x} - \beta \right)^2 \right] dA dx = \\ &= \frac{1}{2} \int_0^L \left[EI \left(\frac{\partial \beta}{\partial x} \right)^2 + k GA \left(\frac{\partial v}{\partial x} - \beta \right)^2 \right] dx, \end{aligned} \quad (5)$$

where $v=v(x,t)$ is the transverse displacement of a beam cross-section located at the distance x at the time t , E is the modulus of elasticity, G is the shear modulus, A is the cross-sectional area, I is the second moment of area and k is shear correction factor. The kinetic energy of the beam (T) is given by

$$T = \frac{1}{2} \int_V \rho \left[\left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right] dV = \frac{1}{2} \int_0^L \left[\rho A \left(\frac{\partial v}{\partial t} \right)^2 + \rho I \left(\frac{\partial \beta}{\partial t} \right)^2 \right] dx, \quad (6)$$

where $u = -y\beta$ while ρ is mass density of the beam material.

For derivation of external work, Figure 1 which shows free vibrations of uniform straight beam loaded with compressive axial force is observed. Assumption of small-amplitude vibrations is used, which implies $\cos \varphi \approx 1$, and the displacements of cross-section along the direction of reference beam axis is

$$du = ds - dx = \left[(dx)^2 + \left(\frac{\partial v}{\partial x} dx \right)^2 \right]^{1/2} - dx = dx \left\{ \left[1 + \left(\frac{\partial v}{\partial x} \right)^2 \right]^{1/2} - 1 \right\}. \quad (7)$$

If the obtained expression is approximated with first two terms of Taylor series at $\partial v / \partial x = 0$, longitudinal displacement is

$$du \approx \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 dx. \quad (8)$$

Treba primijetiti da je komponenta pomjeranja koja potiče od dilatacije u osi štapa zanemarena, dok se usvaja da je pravac sile nepromjenljiv.

It should be noticed that the component of displacement due to axial stiffness is neglected while the force is considered as non-follower.

Slika 1. Slobodne vibracije prave grede s malim amplitudama pri dejstvu aksijalne sile. Ravnoteža diferencijalnog elementa.

Figure 1. Small free vibrations of uniform straight beam loaded with axial force. Equilibrium of differential element.

Spoljašnja komponenta potencijalne energije sistema jednaka je radu aksijalne sile pritiska na podužnom pomjeranju poprečnog preseka:

Now, the external work done by the compressive axial force against the longitudinal displacement of the cross-section can be expressed as:

$$W = \int_0^L P du = \frac{1}{2} \int_0^L P \left(\frac{\partial v}{\partial x} \right)^2 dx. \quad (9)$$

Primjena generalizovanog Hamiltonovog principa daje:

The application of the generalized Hamilton's principle gives

$$\begin{aligned} & \delta \int_{t_1}^{t_2} (T - \Pi + W) dt = \\ & = \int_{t_1}^{t_2} \int_0^L \left[\rho I \frac{\partial \beta}{\partial t} \delta \left(\frac{\partial \beta}{\partial t} \right) + \rho A \frac{\partial v}{\partial t} \delta \left(\frac{\partial v}{\partial t} \right) \right] dx dt - \\ & - \int_{t_1}^{t_2} \int_0^L \left[EI \frac{\partial \beta}{\partial x} \delta \left(\frac{\partial \beta}{\partial x} \right) + kGA \left(\frac{\partial v}{\partial x} - \beta \right) \delta \left(\frac{\partial v}{\partial x} \right) - kGA \left(\frac{\partial v}{\partial x} - \beta \right) \delta \beta \right] dx dt + \\ & + \int_{t_1}^{t_2} \int_0^L P \frac{\partial v}{\partial x} \delta \left(\frac{\partial v}{\partial x} \right) dx dt = 0. \end{aligned} \quad (10)$$

Koristeći parcijalnu integraciju u odnosu na vremensku i prostornu koordinatu i imajući u vidu definiciju virtuelnih pomjeranja, jednačina (10) daje sistem

Using integration by parts with respect to time t and space coordinate x and having in mind definition of virtual displacements, equation(10) gives the system of

spregnutih diferencijalnih jednačina kretanja prema Timošenkovoј teoriji:

coupled differential equations of motion according to the Timoshenko's theory:

$$\frac{\partial}{\partial x} \left[kGA \left(\beta - \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial x} \left(P \frac{\partial v}{\partial x} \right) + \rho A \frac{\partial^2 v}{\partial t^2} = 0, \quad (11)$$

$$kGA \left(\beta - \frac{\partial v}{\partial x} \right) - \frac{\partial}{\partial x} \left(EI \frac{\partial \beta}{\partial x} \right) + \rho I \frac{\partial^2 \beta}{\partial t^2} = 0, \quad (12)$$

s graničnim uslovima:

with the boundary conditions:

$$EI \frac{\partial \beta}{\partial x} \delta \beta \Big|_0^L = 0, \quad \left[kGA \left(\frac{\partial v}{\partial x} - \beta \right) - P \frac{\partial v}{\partial x} \right] \delta v \Big|_0^L = 0. \quad (13)$$

Pretpostavljajući da je greda prizmatična, građena od homogenog materijala te da je i sila konstantna, jednačine (11) i (12) mogu biti razdvojene, [9]:

Assuming the uniform beam and constant end force, equations (11) and (12) can be decoupled, [9]:

$$EI \left(1 - \frac{P}{kGA} \right) \frac{\partial^4 v}{\partial x^4} + P \frac{\partial^2 v}{\partial x^2} + \rho A \frac{\partial^2 v}{\partial t^2} - \rho I \left(1 + \frac{E}{kG} - \frac{P}{kGA} \right) \frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 v}{\partial t^4} = 0, \quad (14)$$

$$EI \left(1 - \frac{P}{kGA} \right) \frac{\partial^4 \beta}{\partial x^4} + P \frac{\partial^2 \beta}{\partial x^2} + \rho A \frac{\partial^2 \beta}{\partial t^2} - \rho I \left(1 + \frac{E}{kG} - \frac{P}{kGA} \right) \frac{\partial^4 \beta}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 \beta}{\partial t^4} = 0. \quad (15)$$

Interesantno je napomenuti i to da su obe jednačine (14) i (15) istog oblika, s različitim nepoznatim funkcijama. Izostavljanjem sabiraka koji sadrže modul smicanja iz jednačine (14), dobija se Rejljeva formulacija, dok dalje ukidanje i sabiraka koji sadrže ρI rezultuje klasičnom Bernuli-Ojlerovom formulacijom. U slučaju sile zatezanja, treba je uvrstiti s negativnim predznakom.

It is interesting to note that both equations (14) and (15) are the same, with different unknown functions. Exclusion of terms which have shear modulus G in denominator in eq. (14) gives Rayleigh theory while further reduction of term consisting ρI reveals classic BE equation. Also, if the force is tensile, it should be inserted with minus sign.

2.2 Rješenje jednačine kretanja

Pod pretpostavkom sinhronog i harmonijskog kretanja, nepoznate funkcije mogu se prikazati kao

2.2 Solution of equation of motion

Assuming synchronous and harmonic motion, unknown functions can be represented as

$$v(x, t) = \bar{v}(x) T(t), \quad (16)$$

$$\beta(x, t) = \bar{\beta}(x) T(t), \quad (17)$$

$$T(t) = A \cos \omega t + B \sin \omega t, \quad (18)$$

gdje $\bar{v}(x)$ predstavlja oblik poprečnog moda vibracija, $\bar{\beta}(x)$ jeste rotacija ovog modalnog oblika, dok je ω svojstvena frekvencija. Ova multiplikativna dekompozicija dovodi do homogenog sistema jednačina koji ima beskonačan broj netrivialnih rješenja, od kojih svako predstavlja jedan svojstveni mod vibracija (ω_n , v_n).

Uvrštavanjem jednačina (16), (17) i (18) u jednačine (14) i (15) dobija se linearna diferencijalna jednačina s konstantnim koeficijentima:

where $\bar{v}(x)$ denotes the transverse mode shape of vibration, $\bar{\beta}(x)$ is the rotation of this mode shape while ω is its eigenfrequency. This multiplicative decomposition leads to a homogenous system of equations which has infinite number of solutions of which each represents one eigenpairs (ω_n , v_n).

Substituting equations (16), (17) and (18) in equations (14) and (15) yields linear differential equation with constant coefficients:

$$EI \left(1 - \frac{P}{kGA} \right) \frac{d^4 Z}{dx^4} + \left[P + \rho I \omega^2 \left(1 + \frac{E}{kG} - \frac{P}{kGA} \right) \right] \frac{d^2 Z}{dx^2} - \rho A \omega^2 \left(1 - \frac{\rho I \omega^2}{kGA} \right) Z = 0, \quad (19)$$

gdje je $Z(x) = \bar{v}(x)$ ili $Z(x) = \bar{\beta}(x)$. Opšte

where $Z(x) = \bar{v}(x)$ or $Z(x) = \bar{\beta}(x)$. The general

rješenje jednačine (19) zavisi od korjena karakteristične jednačine:

solution of equation (19) depends of the roots of characteristic equation:

$$r_{1,2} = \pm \sqrt{-\frac{P+c\rho I\omega^2}{2a} + \sqrt{\left(\frac{P+c\rho I\omega^2}{2a}\right)^2 + \frac{\omega^2}{a}(d-f\omega^2)}}, \quad (20)$$

$$r_{3,4} = \pm \sqrt{-\frac{P+c\rho I\omega^2}{2a} - \sqrt{\left(\frac{P+c\rho I\omega^2}{2a}\right)^2 + \frac{\omega^2}{a}(d-f\omega^2)}}, \quad (21)$$

gdje je

where

$$\alpha = kGA, \quad a = EI\left(1 - \frac{P}{\alpha}\right), \quad c = 1 + \frac{E}{kG} - \frac{P}{\alpha}, \quad d = \rho A, \quad f = \frac{\rho^2 I}{kG}. \quad (22)$$

Korjeni $r_{3,4}$ uvijek su imaginarni, a korjeni $r_{1,2}$ u zavisnosti od frekvencije ω mogu biti realni ili imaginarni. Realni su kada je frekvencija manja od $\sqrt{kGA/\rho I}$, a imaginarni kada je frekvencija veća od $\sqrt{kGA/\rho I}$. Ovu graničnu vrijednost $\omega_{cr} = \sqrt{kGA/\rho I}$ nazivamo kritična frekvencija.

- Kada je $\omega < \omega_{cr}$:

Roots $r_{3,4}$ are always imaginary, and roots $r_{1,2}$ are either real or imaginary depending on the frequency ω . They are real when the frequency is less than $\sqrt{kGA/\rho I}$ and are imaginary when the frequency is greater than $\sqrt{kGA/\rho I}$. This is called the cut off value of the critical frequency $\omega_{cr} = \sqrt{kGA/\rho I}$.

- When $\omega < \omega_{cr}$:

$$\bar{v}(x) = C_1 \cosh r_1 x + C_2 \sinh r_1 x + C_3 \cos rx + C_4 \sin rx, \quad (23)$$

$$\bar{\beta}(x) = D_1 \cosh r_1 x + D_2 \sinh r_1 x + D_3 \cos rx + D_4 \sin rx, \quad (24)$$

gdje su

where

$$r_1 = \sqrt{-\frac{P+c\rho I\omega^2}{2a} + \sqrt{\left(\frac{P+c\rho I\omega^2}{2a}\right)^2 + \frac{\omega^2}{a}(d-f\omega^2)}}, \quad (25)$$

$$r = \sqrt{\frac{P+c\rho I\omega^2}{2a} + \sqrt{\left(\frac{P+c\rho I\omega^2}{2a}\right)^2 + \frac{\omega^2}{a}(d-f\omega^2)}}. \quad (26)$$

- Kada je $\omega = \omega_{cr}$:

- When $\omega = \omega_{cr}$:

$$\bar{v}(x) = C_1 + C_2 + C_3 \cos rx + C_4 \sin rx, \quad (27)$$

$$\bar{\beta}(x) = D_1 + D_2 + D_3 \cos rx + D_4 \sin rx, \quad (28)$$

parametar r postaje

where parameter r becomes

$$r = \sqrt{\frac{P+c\alpha}{a}}. \quad (29)$$

- Kada je $\omega > \omega_{cr}$:

- When $\omega > \omega_{cr}$:

$$\bar{v}(x) = C_1 \cos r' x + C_2 \sin r' x + C_3 \cos rx + C_4 \sin rx, \quad (30)$$

$$\bar{\beta}(x) = D_1 \cos r' x + D_2 \sin r' x + D_3 \cos rx + D_4 \sin rx, \quad (31)$$

gdje je

where

$$r' = \sqrt{\frac{P + c \rho I \omega^2}{2a}} - \sqrt{\left(\frac{P + c \rho I \omega^2}{2a}\right)^2 + \frac{\omega^2}{a}(d - f \omega^2)}. \quad (32)$$

Veze između koeficijenata u (23) i (24) dobijaju se uvrštavanjem izraza za $\bar{v}(x)$ i $\bar{\beta}(x)$ u jednačine (11) i (12), što rezultuje sljedećim odnosima za $\omega < \omega_{cr}$, [9]:

The relations between coefficients in (23) and (24) are obtained by inserting the expressions for $\bar{v}(x)$ and $\bar{\beta}(x)$ into the equations (11) and (12), resulting with the following relationships for $\omega < \omega_{cr}$, [9]:

$$\begin{aligned} D_1 &= \frac{r_1^2(\alpha - P) + d \omega^2}{r_1 \alpha} C_2, & D_2 &= \frac{r_1^2(\alpha - P) + d \omega^2}{r_1 \alpha} C_1, \\ D_3 &= \frac{r^2(\alpha - P) - d \omega^2}{r \alpha} C_4, & D_4 &= -\frac{r^2(\alpha - P) - d \omega^2}{r \alpha} C_3. \end{aligned} \quad (33)$$

Za $\omega > \omega_{cr}$:

For $\omega > \omega_{cr}$:

$$\begin{aligned} D_1 &= \frac{r'^2(\alpha - P) - d \omega^2}{r' \alpha} C_2, & D_2 &= -\frac{r'^2(\alpha - P) - d \omega^2}{r' \alpha} C_1, \\ D_3 &= \frac{r^2(\alpha - P) - d \omega^2}{r \alpha} C_4, & D_4 &= -\frac{r^2(\alpha - P) - d \omega^2}{r \alpha} C_3. \end{aligned} \quad (34)$$

Parametri C_1 - C_4 određuju se iz graničnih uslova. Njihovim uvođenjem, dolazi se do sistema homogenih jednačina koje imaju netrivialna rešenja za $\det \mathbf{A} = 0$, gdje je $\mathbf{A}$ matrica koeficijenata posmatranog sistema. Ovaj uslov daje transcendentnu jednačinu s beskonačnim brojem rješenja – svojstvenih frekvencija. Uvođenjem ovih sopstvenih frekvencija u sistem homogenih jednačina, dobijamo sopstvene oblike oscilovanja. Granični uslovi za sve teorije dati su u Tabeli 1, [8] i [9].

Važno je primijetiti da za $\omega = \omega_{cr}$, koeficijenti u matrici $\mathbf{A}$ nisu funkcije već konstante, što znači da determinanta neće generisati transcendentnu jednačinu, pa prema tome ova vrijednost frekvencije neće dati ni kretanje.

Parameters C_1 - C_4 are calculated from end BCs. Introduction of BCs leads to a system of homogeneous equations which has non-trivial solution for $\det \mathbf{A} = 0$, where $\mathbf{A}$ is matrix of coefficients of this system. This condition returns transcendent equation with infinite number of solutions – eigenfrequencies. Introduction of this eigenfrequencies in system of homogeneous equations results with eigenshapes. BCs for all theories are given in Table 1, [8] and [9].

It should be noted that for $\omega = \omega_{cr}$, coefficients in matrix $\mathbf{A}$ are not functions but constants, which means that its determinant does not generate transcendent equation and there is no motion associated with this eigenfrequency.

Tabela 1. Granični uslovi za tri teorije.
Table 1. Boundary conditions for three theories.

	Bernuli-Ojlerova i Rejljeva teorija <i>Bernoulli-Euler and Rayleigh theory</i>	Timošenkova teorija <i>Timoshenko theory</i>
slobodno oslonjen kraj <i>simply supported end</i>	$\bar{v} = 0, \quad \bar{v}'' = 0$	$\bar{v} = 0, \quad \bar{\beta}' = 0$
uklješteni kraj <i>clamped end</i>	$\bar{v} = 0, \quad \bar{v}' = 0$	$\bar{v} = 0, \quad \bar{\beta} = 0$
slobodan kraj <i>free end</i>	$\bar{v}'' = 0, \quad \bar{v}'''' + \frac{P}{EI} \bar{v}' = 0$	$\bar{\beta}' = 0, \quad \left(1 - \frac{P}{\alpha}\right) \bar{v}' - \bar{\beta} = 0$
poprečno pokretno uklješten kraj <i>guided end</i>	$\bar{v}' = 0, \quad \bar{v}'''' + \frac{P}{EI} \bar{v}' = 0$	$\bar{\beta} = 0, \quad \left(1 - \frac{P}{\alpha}\right) \bar{v}' - \bar{\beta} = 0$

3 PROGRAMSKI KOD ZA ANALIZU RAVNOG STANJA NAPONA

Da bi se obezbijedio numerički alat za analizu slobodnih vibracija pritisnutih greda, razvijen je jednostavan kod za dvodimenzionalnu analizu ravnog stanja napona. Pretpostavka je da se mehaničko ponašanje greda može precizno opisati koristeći ovaj tip konačnih elemenata. Sabirov pravougaoni konačni element za analizu ravnog stanja napona (SRPS2D) s *drilling* stepenom slobode [18] korišten je za programiranje u softverskom paketu *Wolfram Mathematica*. Geometrija elemenata, stepeni slobode i lokalni koordinatni sistem prikazani su na Slici 2a. Ovaj element je jedan od prvih uspješno izvedenih konačnih elemenata s *drilling* stepenom slobode, i sposoban je da elementa s *drilling* stepenom slobode, i sposoban je da tačno opiše tri moda pomjeranja krutog tijela, što je osnovni zahtjev za konačni element.

Treba napomenuti da implementacija elemenata s *drilling* stepenom slobode predstavlja izuzetno kontroverznu temu u MKE, zbog nulte krutosti koja je povezana s njim. Prvu uspješnu implementaciju trougaonog elementa za analizu problema naprezanja u ravni uveo je Alman u [19]. On je koristio kvadratne funkcije oblika za normalne, i linearne funkcije oblika za tangencijalne komponente pomjeranja duž ivica elementa. Da bi se izbjegao problem preodređenog sistema jednačina, uveden je peti granični uslov koji predstavlja razliku rotacionih parametara, konektora, u uglovima elementa. Ovaj pristup dao je efikasan i konvergentan element s tri stepena slobode po čvoru. Almanov metod su potom proširili Sabir [18] i Kuk [20]. Generalne teorijske postavke pomoću varijacione tehnike kasnije su uradili Hjuž i Brezi [21], a uspješno su je koristili i drugi istraživači [22].

3 PROGRAM CODE FOR PLANE STRESS ANALYSIS

In order to acquire numerical tool for analysis of free vibrations of compressed beams, simple code for 2D elasticity analysis is developed. Assumption is that mechanical behaviour of beams can be accurately described utilizing this type of finite elements. Sabir's rectangular plane-stress (SRPS2D) finite element with drilling DOF [18] is used and coded in *Wolfram Mathematica*. Element geometry, DOFs and coordinate system are given in Figure 2a. This element is the first successfully implemented finite element with drilling DOF, and it is able to accurately represent three rigid body modes, which is standard requirement for finite elements.

It should be noted that implementation of elements with drilling DOFs presents a rather controversial subject in FEM due to zero stiffness associated with it. First successful implementation of plane elasticity triangular element is introduced by Allman in [19]. He used quadratic shape functions for normal, and linear shape function for tangential component of displacement along element side. To avoid problem of overdetermined system of equations he introduced fifth boundary condition as a difference of rotational connectors at element vertices. This approach returned efficient and convergent element with three DOFs per node. His method is extended by Sabir [18] and Cook [20]. General theoretical foundation using variation technique is posed later by Hughes and Brezzi [21] and it was used successfully by other researchers [22].

Slika 2. SRPS2D element (lijevo), drilling stepen slobode (desno).
Figure 2. SRPS2D element (left), Drilling DOF (right).

Drilling stepen slobode može se fizički tumačiti kao stvarna rotacija simetrale ugla između susednih ivica konačnog elementa. Na Slici 2b može se vidjeti da je ova rotacija jednaka polovini razlike gradijenata pomjeranja.

Drilling DOF may be physically interpreted as a true rotation of the bisector of the angle between adjacent edges of the finite element. Figure 2b shows that this rotation is equal to a half of a difference of displacement gradients

$$\Phi = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right). \quad (35)$$

Sabir je izveo interpolacione funkcije analizirajući naponsko stanje [18]. Ove funkcije i njihova izvođenja izostavljeni su zbog sažetosti izlaganja. Grafički prikaz interpolacionih funkcija dat je na Slici 3. Stepenn slobode rotacije krutog tela prikazan je na prvoj skici Slike 3. Ostale tri skice prikazuju deformisanu konfiguraciju elementa s različitim stepenima slobode u čvoru 1, dok su ostali stepeni slobode spriječeni.

Sabir derived interpolation functions using strain analysis[18]. These functions and their derivation are omitted due to brevity but graphic representation is given in Figure 3. Rotational rigid body DOF of element is given in Figure 3. Other three images represent deformed configuration of element for non-zero values of each DOF at node 1, while all the other DOFs are zero.

Slika 3. Kruta rotacija i funkcije oblika za SRPS2D element.
Figure 3. Rigid rotation and shape functions for SRPS2D element.

Problem slobodnih vibracija opterećenih sistema opisan je sledećim standardnim problemom sopstvenih vrednosti

Problem of free vibration of pre-stressed structure is described with following standard eigenvalue problem

$$(\mathbf{K} + P\mathbf{K}_G - \omega^2\mathbf{M})\mathbf{q} = 0, \quad (36)$$

gdje su $\mathbf{K}$, $\mathbf{K}_G$ i $\mathbf{M}$ linearna matrica krutosti, geometrijska matrica krutosti i matrica masa, redom, [12]. Ove matrice su izvedene standardnim MKE pristupom. Matrica masa je konzistentna s funkcijama oblika koje se koriste za definisanje krutosti. Geometrijska matrica krutosti formirana je na osnovu pretpostavke o homogenom stanju napona, pri čemu postoji samo komponenta normalnog napona. Rešavanjem jednačine (36), dobijamo p uređenih parova $(\omega_n, \mathbf{q}_n)$ koji se sastoji od n -te sopstvene frekvencije i sopstvenog vektora, dok je p ukupan broj aktivnih stepeni slobode.

where $\mathbf{K}$, $\mathbf{K}_G$ and $\mathbf{M}$ are linear stiffness matrix, geometric stiffness matrix and mass matrix respectively, [12]. These matrices are derived in standard FE manner. Mass matrix is consistent with shape functions used for determination of stiffness. Geometric stiffness matrix is formed using homogenous stress state where the longitudinal stress component is the only one considered. Solving the equation (36) results with p eigenpairs $(\omega_n, \mathbf{q}_n)$ which consists of n th eigenfrequency and eigenvector while p is the total number of active DOFs.

4 NUMERIČKA ANALIZA I DISKUSIJA

Numerička analiza sprovedena je pomoću SRPS2D koda, analitičkih rešenja i komercijalnog MKE programa Abakus. Opšteprimjenljivi element ljuske S4R5 korišten je za modeliranje u Abakusu, zajedno sa standardnim Bernuli-Ojlerovim i Timošenskovim grednim elementima, B23 i B21. Ovaj element ljuske je četvorostrani sa četiri čvora, sa po pet stepeni slobode u čvoru koji ne sadrži *drilling* stepen slobode, [23]. Iako bi elementi za ravno

4 NUMERICAL ANALYSIS AND DISCUSSION

Numerical analysis is performed using developed SRPS2D code, analytical solutions and commercial FE software Abaqus. General purpose shell element S4R5 is used for modelling in Abaqus, along with standard Bernoulli-Euler and Timoshenko beam elements, B23 and B21. This shell element is four-noded quadrilateral with five DOFs per node, excluding the drilling DOF, [23]. Although some plane stress/strain elements would

stanje naprezanja bili prikladniji za ovu analizu, S4R5 odabran je s ciljem razmatranja primjenljivosti elemenata ljuske za modeliranje ponašanja grede. U nastavku, ovaj model naziva se *2D Abakus*.

Analizirana je greda dužine 0.25 m, visine 0.01 m i širine 0.02 m. Geometrija grede zasnovana je na fizičkom modelu koji će biti eksperimentalno proučavan u daljnjem istraživanju. Osnovno ograničenje s kojim smo se suočili jeste dužina grede – uslovljena maksimalnim otvorom čeljusti prese koju ćemo koristiti. Greda je izrađena od konstrukcionog čelika s modulom elastičnosti $E=200$ GPa, Poasonovim koeficijentom $\nu=0.3$ i zapreminskom masom $\rho=7.8$ t/m³.

Prvi korak jeste analiza konvergencija, koja se koristi za verifikaciju razvijenog koda i poređenje dobijenih rezultata sa analitičkim rješenjima. Ova analiza sprovedena je na slobodno oslonjenoj gredi s različitim brojem elemenata po visini i dužini. Razmatrani su modeli bez sile i s pola prve kritične sile. Na slikama 4 i 5 date su relativne razlike četiri sopstvene frekvencije za 2D elemente, sračunate u SRPS2D i Abakusu, u odnosu na analitička rešenja prema Timošenu. Evidentno je to da je razvijeni kod efikasan i pouzdan za ovu vrstu analiza, dok je za tačnost određenih frekvencija potrebno progustiti mrežu po visini. Ipak, dobijeni rezultati mogu se smatrati dovoljno tačnim za prikazanu analizu, te će se nadalje koristiti mreža 20x160 SRPS2D elemenata. Na osnovu rezultata prikazanih na Slici 5, u Abakusu je usvojena mreža od 60x300 S4R5 elemenata. Karakteristični minimumi koji su vidljivi na Slici 5 zapravo predstavljaju intervale u kojima posmatrane razlike mijenjaju znak, a pojavljuju se jer se u ovoj analizi prikazuju relativne razlike. 1D modeli u Abakusu napravljeni su sa iskonvergiranim mrežama od 150 B23 i B21 elemenata, ali ovi rezultati nisu prikazani zbog sažetosti.

be more appropriate for present analysis, S4R5 is utilized in order to consider the applicability of shell elements for beam behaviour modelling. This model is named as *2D Abaqus* in further text.

Beam with length $L=0.25$ m, height 0.01 m and width 0.02 m is analyzed. The beam geometry is based on the physical model which will be experimentally studied in the further research. Main limit comes from length of the beam, which is defined by vertical daylight of plates of available testing frame. Beam is made of structural steel with elasticity modulus $E=200$ GPa, Poisson coefficient $\nu=0.3$ and mass density $\rho=7.8$ t/m³.

First step is convergence analysis which is used to verify developed code and to examine its relation with analytical solutions. Analysis of convergence was carried out on simply supported beam with different numbers of elements along height and length. This analysis was conducted without force and with half of first critical force. Relative difference of four eigenfrequencies for 2D elements calculated in SRPS2D and Abaqus versus analytical Timoshenko results are given in Figures 4 and 5 for simply supported beam. It is evident that developed code is efficient and reliable for this type of analysis while some frequencies could benefit from denser meshes along height. Nevertheless, obtained results are considered accurate enough for presented analysis, and mesh of 20x160 SRPS2D elements are used for further analysis. Based on the results displayed in Figure 5, the mesh of 60x300 S4R5 elements is adopted in Abaqus. Characteristic minimums that are visible in Figure 5 actually represent the ranges where the observed differences change sign. However, their relative differences are displayed in this analysis. The 1D models in Abaqus are created with converged meshes of 150 B23 and B21 elements, and these convergence results are not presented due to brevity.

Slika 4. Konvergencija relativne razlike za 1, 2, 3. i 5. mod slobodno oslonjene grede za fiksirani broj od 20 elemenata po visini (pune linije) i fiksiranih 160 elemenata po dužini (isprekidane linije) pomoću SRPS2D koda za: $P=0$ (lijevo) i $P=0.5P_{cr1}$ (desno).

Figure 4. Convergence of relative difference for the 1st, the 2nd, the 3rd and the 5th mode for simply supported beam for fixed number of 20 elements by height (solid line) and fixed number of 160 elements along length (dashed line) using SRPS2D code: $P=0$ (left) and $P=0.5P_{cr1}$ (right).

Slika 5. Konvergenција relativne razlike za 1, 2, 3. i 5. mod slobodno oslonjene grede za fiksirani broj od 20 elemenata po visini (pune linije) i fiksiranih 300 elemenata po dužini (isprekidane linije) pomoću Abakusa za: $P=0$ (lijevo) i $P=0.5P_{cr1}$ (desno).

Figure 5. Convergence of relative difference for the 1st, the 2nd, the 3rd and the 5th mode for simply supported beam for fixed number of 20 elements by height (solid line) and fixed number of 300 elements along length (dashed line) using Abaqus: $P=0$ (left) and $P=0.5P_{cr1}$ (right).

Nadalje, kao referentna rješenja usvojeni su analitički rezultati Timošenkovke teorije, kao najtačniji gredni model. Sopstvene frekvencije određene su po svim grednim teorijama, SRPS2D kodom i 1D i 2D modelima u Abakusu. Razmotrene su dvije vrijednosti aksijalne sile s dva tipa graničnih uslova. Rezultati sedam najnižih normiranih sopstvenih frekvencija za slobodno oslonjenu gredu i konzolnu gredu bez aksijalne sile predstavljeni su u Tabeli 2 i Tabeli 4, redom, dok su rezultati sa aksijalnom silom pritiska $P=0.95P_{cr1}$ prikazani u Tabeli 3 i Tabeli 5. Frekvencije su normirane sa $r\sqrt{E/\rho}/L^2$, gdje je r poluprečnik inercije poprečnog presjeka.

Sračunate sopstvene frekvencije, normirane sa analitičkim vrednostima po Timošenkovoj teoriji Ω^* , prikazane su na Slikama 6 i 7. Za model bez sile, može se uočiti da su numerički i analitički rezultati za Bernuli-Ojlerovu gredu u gotovo potpunoj saglasnosti. Međutim, za modele koji uključuju smicanje postoje razlike između analitičkog i numeričkih rezultata, koje su izraženije za konzolnu gredu nego za slobodno oslonjenu gredu. Očigledno je poklapanje svih numeričkih rezultata za modele koji uključuju smicanje.

Za grede sa $P=0.95P_{cr1}$ primjećuju se zanimljivi efekti. Na Slici 6 prikazane su značajne razlike između dobijenih rezultata za prvi mod slobodno oslonjene grede. Poklapanje ovih rezultata bolje je za konzolnu gredu, kao što se može vidjeti na Slici 7. Ove razlike posljedica su graničnih uslova i velikog intenziteta nanijete aksijalne sile. Prije svega, granični uslovi koji bi zadovoljili gredne teorije za slobodno oslonjene grede nezgodni su za definisanje na 2D modelu. Za slobodno oslonjenu gredu, potrebno je obezbijediti granične uslove koji sprječavaju pomjeranje težišta krajnjih preseka, što uzrokuje koncentraciju napona i deformacije u ovim tačkama. Ova pojava ne utiče na konzolni štap, s obzirom na to što su granični uslovi homogeni za sve tačke poprečnog presjeka.

Furthermore, the analytical results by Timoshenko theory, which is the most accurate beam model, are adopted as the reference solutions. The eigenfrequencies are calculated with all beam theories, SRPS2D code and 1D and 2D Abaqus. Two values of axial force and two types of BCs are considered. The results of the lowest seven normed eigenfrequencies for simply supported beam and cantilever beam without axial force are presented in Table 2 and Table 4, respectively, while results with compressive axial force $P=0.95P_{cr1}$ are presented in Table 3 and Table 5. They are normed with $r\sqrt{E/\rho}/L^2$, where r is radius of gyration of cross section.

The calculated eigenfrequencies normed with analytical values from Timoshenko theory Ω^* are presented in Figures 6 and 7. For the model without force, it can be observed that numerical and analytical predictions for BE beam are in almost full compliance. However, for the shear deformable models there are some discrepancies between analytical and numerical results which are more pronounced for cantilever beam, compared to the simply supported one. It is evident that all numerical results for the shear deformable model are in excellent agreement.

For beams with $P=0.95P_{cr1}$ interesting effects are noticed. Figure 6 shows a significant difference between the results obtained for the first mode for simply supported beam. Compliance of these results is better for cantilever beam, as it can be seen in Fig. 7. These differences are consequences of the BCs and high level of axial force. Firstly, simply supported BCs are rather cumbersome to define for the 2D model in order to comply it with the beam theories. For the simply supported beam, it is necessary to impose BCs which prevent displacements of centroids of the end sections which results with stress/strain concentrations at these points. This phenomenon does not affect the cantilever beam, since the homogenous BCs are imposed at the all points of cross section.

Tabela 2. Prikaz prvih sedam normiranih svojstvenih frekvencija slobodno oslonjene grede sa $P=0$,

$$\bar{\omega}_i = \omega_i / \left(r \sqrt{E / \rho} / L^2 \right).$$

Table 2. Comparison of the lowest seven normed eigenfrequencies for simply supported beam for

$$P=0, \bar{\omega}_i = \omega_i / \left(r \sqrt{E / \rho} / L^2 \right).$$

Mode	Analytical results				Numerical results		
	Timoshenko theory	Rayleigh theory	Bernoulli-Euler Beam	SRPS2D	2D Model	Abaqus	
					S4R5 Element	B21 Element	B23 Element
1	1.567	1.570	1.571	1.567	1.567	1.567	1.571
2	6.216	6.267	6.283	6.219	6.218	6.217	6.283
3	13.81	14.05	14.14	13.82	13.82	13.81	14.14
4	24.12	24.87	25.13	24.16	24.15	24.14	25.13
5	36.90	38.64	39.27	36.98	36.98	36.93	39.27
6	51.85	55.26	56.55	52.02	52.00	51.92	56.55
7	68.70	74.60	76.97	68.98	68.96	68.80	76.97

Tabela 3. Prikaz prvih sedam normiranih svojstvenih frekvencija slobodno oslonjene grede sa $P=0.95P_{cr1}$,

$$\bar{\omega}_i = \omega_i / \left(r \sqrt{E / \rho} / L^2 \right).$$

Table 3. Comparison of the lowest seven normed eigenfrequencies for simply supported beam for

$$P=0.95P_{cr1}, \bar{\omega}_i = \omega_i / \left(r \sqrt{E / \rho} / L^2 \right).$$

Mode	Analytical results				Numerical results		
	Timoshenko theory	Rayleigh theory	Bernoulli-Euler Beam	SRPS2D	2D Model	Abaqus	
					S4R5 Element	B21 Element	B23 Element
1	0.336	0.351	0.351	0.333	0.313	0.344	0.347
2	5.414	5.472	5.487	5.413	5.410	5.428	5.486
3	13.03	13.29	13.37	13.03	13.03	13.06	13.37
4	23.35	24.12	24.38	23.37	23.37	23.41	24.37
5	36.12	37.90	38.52	36.18	36.18	36.22	38.52
6	51.06	54.52	55.80	51.20	51.20	51.21	55.80
7	67.89	73.87	76.22	68.14	68.14	68.11	76.22

Nadalje, usklađenost razmatranih modela za sve modove, osim prvog, jeste odlična. To je takođe uzrokovano efektom primijenjene aksijalne sile koja je blizu najniže kritične sile. Numerička i analitička analiza daju različite rezultate u prvom modu zbog različitog modeliranja početnog napona i njegovog efekta na svojstvene vibracije. Naime, uvođenje aksijalne sile je različito za SRPS2D i Abakus 2D modele. U razvijenom kodu, normalni naponi su homogeno raspoređeni putem geometrijske matrice krutosti, dok ta raspodjela u Abakusu nije potpuno homogena, jer je rezultat linearnog statičkog proračuna usled nanesenog linijskog opterećenja. Zahvaljujući ovoj činjenici, rezultati SRPS2D u boljoj su saglasnosti sa analitičkim

Furthermore, compliance of considered models for all modes, except for the first one, is excellent. This is also caused by the effect of applied axial force which is close to the lowest critical one. Numerical and analytical predictions return different results for first mode due to different modelling of initial stress and its effect on eigenmodes. Namely, introduction of axial force is different for SRPS2D and Abakus 2D models. In developed code, normal stress is homogenously distributed via geometric stiffness matrix, while in Abakus it is not completely homogenous since it is a result of linear static step due to applied line load. Owing to this fact, results from SRPS2D are in better compliance with analytical Timoshenko predictions,

rezultatima Timošenkove teorije za slobodno oslonjenu gredu. Za konzolne grede, ta razlika je zanemarljiva. Ovo detaljno poređenje potvrđuje primjenljivost razvijenog SRPS2D koda za sprovedenu vrstu analize.

Trivijalno je primijetiti da su Rejljeva teorija i Bernuli-Ojlerova teorija pogrešne za više modove zbog uvedenih pretpostavki. Hipoteza o upravnosti presjeka na deformisanu osu, za predstavljenu analizu, ima veći uticaj u odnosu na efekat rotacione inercije. Stoga, model Bernuli-Ojlerove grede daje najveće odstupanje jer zanemaruje i efekat rotacione inercije, za razliku od Rejljevog modela.

compared to the Abaqus ones, for simply supported beam. For cantilever beam, this difference is negligible. This thorough comparison verifies applicability of developed SRPS2D code for the present type of analysis.

Trivially, notice that Rayleigh and BE theories are erroneous for higher modes due to introduced simplifications. Shear rigid assumption is more influential for presented analysis, compared to the effect of rotary inertia. Therefore, BE model returns the most significant relative differences since it neglects effect of rotary inertia, contrary to the Rayleigh model.

Tabela 4. Prikaz prvih sedam normiranih svojstvenih frekvencija konzolne grede sa $P=0$,

$$\bar{\omega}_i = \omega_i / \left(r \sqrt{E / \rho} / L^2 \right).$$

Table 4. Comparison of the lowest seven normed eigenfrequencies for cantilever beam for $P=0$,

$$\bar{\omega}_i = \omega_i / \left(r \sqrt{E / \rho} / L^2 \right).$$

Mode	Analytical results				Numerical results		
	Timoshenko theory	Rayleigh theory	Bernoulli-Euler Beam	SRPS2D	2D Model S4R5 Element	Abaqus	
						1D Model B21 Element	B23 Element
1	0.560	0.560	0.560	0.559	0.559	0.559	0.560
2	3.494	3.504	3.507	3.480	3.478	3.476	3.507
3	9.699	9.790	9.819	9.633	9.627	9.620	9.819
4	18.75	19.12	19.24	18.57	18.56	18.54	19.24
5	30.44	31.45	31.81	30.10	30.07	30.03	31.81
6	43.76	46.70	47.52	43.93	43.89	43.81	47.52
7	59.51	64.76	66.37	59.79	59.73	59.59	66.37

Tabela 5. Prikaz prvih sedam normiranih svojstvenih frekvencija konzolne grede sa $P=0.95P_{cr1}$,

$$\bar{\omega}_i = \omega_i / \left(r \sqrt{E / \rho} / L^2 \right).$$

Table 5. Comparison of the lowest seven normed eigenfrequencies for cantilever beam for

$$P=0.95P_{cr1}, \bar{\omega}_i = \omega_i / \left(r \sqrt{E / \rho} / L^2 \right).$$

Mode	Analytical results				Numerical results		
	Timoshenko theory	Rayleigh theory	Bernoulli-Euler Beam	SRPS2D	2D Model S4R5 Element	Abaqus	
						1D Model B21 Element	B23 Element
1	0.129	0.130	0.130	0.130	0.129	0.129	0.130
2	3.189	3.217	3.220	3.192	3.191	3.191	3.220
3	9.382	9.554	9.583	9.395	9.389	9.388	9.583
4	18.31	18.90	19.02	18.35	18.34	18.33	19.02
5	29.79	31.24	31.59	29.88	29.86	29.83	31.59
6	43.55	46.50	47.31	43.71	43.68	43.61	47.31
7	59.30	64.56	66.16	59.57	59.52	59.40	66.16

Slika 6. Normirane frekvencije Ω^* za prvih sedam modova slobodno oslonjene grede: $P=0$ (lijevo), $P=0.95P_{cr1}$ (desno).
Figure 6. Normed frequencies Ω for the lowest seven modes for simply supported beam: $P=0$ (left), $P=0.95P_{cr1}$ (right).

Slika 7. Normirane frekvencije Ω^* za prvih sedam modova za konzolnu gredu: $P=0$ (lijevo), $P=0.95P_{cr1}$ (desno).
Figure 7. Normed frequencies Ω for the lowest seven modes for cantilever beam: $P=0$ (left), $P=0.95P_{cr1}$ (right).

Poređenje analitičkih rezultata prema tri teorije prikazano je na slikama 8 i 9. Rezultati Bernuli-Ojlerove teorije i Rejljeve teorije upoređeni su s Timošenkovom teorijom. Predstavljene su relativne razlike dobijenih frekvencija za prvih sedam modova i prva dva kritična opterećenja. Očekivano, Rejljevi rezultati bliži su Timošenkovoj teoriji. Takođe, razlika se povećava s povećanjem intenziteta opterećenja, što je izraženije pri dejstvu druge kritične sile.

Razmatrana greda je vitka, te se prvih nekoliko modova dobro podudara prema svim teorijama. Međutim, kod viših modova, skraćuje se dužina sinusnih polutalasa, što dovodi do razlika u sopstvenim frekvencijama. Navedeno se jasno uočava na slikama 8. i 9, gdje efekat rotacione inercije i deformacije smicanja postaje izraženiji s redukcijom dužine sinusnog polutalasa za više modove.

Interesantno je primijetiti uticaj nivoa opterećenja na posmatrane relativne razlike za različite modove. Za slobodno oslonjenu gredu, ovaj uticaj nije značajan, osim za modove koji odgovaraju kritičnoj sili, i te funkcije

Next, comparison of analytical solutions using three theories is given in Figures 8 and 9. The results of Bernoulli-Euler and Rayleigh theories are compared with Timoshenko beam theory and relative differences of the obtained frequencies are calculated for first seven modes and first two critical loads. Again, it is clear that Rayleigh results are closer to the Timoshenko ones, as expected. Also, difference increases with increase of load level, which is more pronounced for second critical force.

The selected beam is slender, and the first few modes finely coincide in all theories. However, for higher modes, the length of the sine half-waves of vibration modes shortens, which leads to differences of eigenfrequencies. This is clear from Figures 8 and 9 where the effect of rotary inertia and shear strain becomes more pronounced as the sine half-wave reduces for higher modes.

It is interesting to notice the influence of load level on observed relative differences for different modes. For simply supported beam, there is no significant influence,

su paralelne, Slika 8. S druge strane, za konzolni štap, ove funkcije su takođe skoro paralelne, ali relativna razlika je izraženija za niže modove, a određeni skok za model sa $P=0$ primjećuje se između 5. i 6. moda, Slika 9. Slična situacija može se vidjeti na Slici 7.

Očigledno, razlike u teorijama izraženije su za slobodno oslonjenu gredu, u odnosu na konzolu. Takođe, primjećuje se veća razlika za 2. mod grede opterećene silom blizu vrijednosti druge kritične sile P_{cr2} . Ovo je uzrokovano smanjenjem talasne dužine za drugi mod oscilovanja. To smanjenje izraženije je za slobodno oslonjenu gredu, što direktno utiče na pomenutu razliku.

excluding the modes which correspond to critical force, and these functions are parallel, Figure 8. On the other hand, for the cantilever beam, these functions are also nearly parallel, but the relative difference is more pronounced for lower modes, and the sort of a *jump* of results for $P=0$ is noticed between 5th and 6th mode, Figure 9. Similar observations can be made on Figure 7.

Clearly, differences of theories are more significant for simply supported beam, compared to the cantilever one. Also, notice the increased difference of theories for 2nd mode of the beam loaded with the force close to second critical force P_{cr2} . This is caused by the decrease in the wavelength for 2nd mode of vibrations. Again, this decrease is more pronounced for simply supported beam which directly affects the observed difference.

Slika 8. Relativna razlika najnižih sedam svojstvenih frekvencija slobodno oslonjene grede za Bernuli-Ojlerovu (puna linija) teoriju i Rejljevu (isprekidana linija) teoriju u poređenju s Timošenkovom teorijom, za prvu kritičnu silu (lijevo) i drugu kritičnu silu (desno).

Figure 8. Relative difference of the lowest seven eigenfrequencies for Bernoulli-Euler (solid line) and Rayleigh (dashed line) vs. Timoshenko theory for the first critical load (left) and the second critical load (right) of simply supported beam.

Slika 9. Relativna razlika najnižih sedam svojstvenih frekvencija konzolne grede za Bernuli-Ojlerovu (puna linija) teoriju i Rejljevu (isprekidana linija) teoriju u poređenju s Timošenkovom teorijom, za prvu kritičnu silu (lijevo) i drugu kritičnu silu (desno).

Figure 9. Relative difference of the lowest seven eigenfrequencies for Bernoulli-Euler (solid line) and Rayleigh (dashed line) vs. Timoshenko theory for the first critical load (left) and the second critical load (right) of cantilever beam.

Na kraju, naglašene su prednosti razvoja sopstvenih kodova. SRPS2D iskorišten je za dobijanje zavisnosti normirane frekvencije Ω i normirane aksijalne sile $\bar{U}$. Kod je stavljen u petlju za deset vrijednosti aksijalne sile, i automatski dobijeni rezultati prikazani su u tabelama 6 i 7, i na slikama 10 i 11. Uočeno je očekivano smanjenje frekvencije s porastom aksijalnog opterećenja. Frekvencija moda koji odgovara kritičnom opterećenju smanjuje se kako se sila približava ovoj vrijednosti. Ovo opadanje primjećuje se i kod ostalih razmatranih modova, ali u mnogo manjoj mjeri.

Finally, benefits of developing own codes are illuminated. Namely, developed code is used to obtain dependencies of normed frequency Ω by normed axial force $\bar{U}$. Code is inserted into loop by ten levels of axial force and results are automatically obtained and presented in Tables 6 and 7, and in Figures 10 and 11. Expected decrease in frequencies with increase in axial load is observed. Frequency of the mode which corresponds to critical load is diminished while the force approaches this value. Other observed modes are influenced too, but in much lesser extent.

Tabela 6. 1, 2, 3. i 5. normirana sopstvena frekvencija slobodno oslonjene grede $\Omega_n=\omega_n/\omega_{n0}$ za različite nivoe normirane sile $\bar{U}_n=P/P_{cr,n}$, za prve dvije kritične sile pomoću SRPS2D.

Table 6. The 1st, the 2nd, the 3rd and the 5th normed eigenfrequencies $\Omega_n=\omega_n/\omega_{n0}$ for different values of normed axial force $\bar{U}_n=P/P_{cr,n}$ for the lowest two critical load of simply supported beam using SRPS2D.

$\bar{U}_n$	P_{cr1}				P_{cr2}			
	n=1	n=2	n=3	n=5	n=1	n=2	n=3	n=5
0	1.0002	1.0005	1.0009	1.0023	1.0002	1.0005	1.0009	1.0023
0.1	0.94885	0.98768	0.99512	1.00004	0.77872	0.94914	0.97786	0.99340
0.3	0.83681	0.96161	0.98341	0.99553	0	0.83708	0.93002	0.97537
0.5	0.70724	0.93481	0.97155	0.99099	0	0.70748	0.87956	0.95699
0.7	0.54783	0.90722	0.95955	0.98643	0	0.54802	0.82603	0.93826
0.9	0.31629	0.87877	0.94739	0.98185	0	0.31641	0.76877	0.91914
0.95	0.22365	0.87151	0.94433	0.98070	0	0.22374	0.75378	0.91429
0.99999	$3.1623 \cdot 10^{-3}$	0.86419	0.94126	0.97955	0	$3.1641 \cdot 10^{-3}$	0.73848	0.90943

Slika 10. Normirane sopstvene frekvencije slobodno oslonjene grede u funkciji normiranog opterećenja, za prvu kritičnu silu (lijevo) i drugu kritičnu silu (desno) upotrebom SRPS2D.

Figure 10. Normed eigenfrequencies vs. normed axial force for the first critical load (left) and the second critical load (right) of simply supported beam using SRPS2D.

Tabela 7. 1, 2, 3. i 5. normirana sopstvena frekvencija konzolne grede $\Omega_n = \omega_n / \omega_{n0}$ za različite nivoe normirane sile $\bar{U}_n = P / P_{cr,n}$, za prve dvije kritične sile pomoću SRPS2D.

Table 7. The 1st, the 2nd, the 3rd and the 5th normed eigenfrequencies $\Omega_n = \omega_n / \omega_{n0}$ for different values of normed axial force $\bar{U}_n = P / P_{cr,n}$, for the lowest two critical load of cantilever beam using SRPS2D.

$\bar{U}_n$	P_{cr1}				P_{cr2}			
	n=1	n=2	n=3	n=5	n=1	n=2	n=3	n=5
0	0.99938	0.99594	0.99321	0.98892	0.99938	0.99594	0.99321	0.98892
0.1	0.95160	0.98763	0.99066	0.98816	0.34283	0.91903	0.97023	0.98212
0.3	0.84569	0.97078	0.98553	0.98663	0.00000	0.74244	0.92260	0.96840
0.5	0.72054	0.95359	0.98038	0.98511	0.00000	0.53585	0.87263	0.95447
0.7	0.56289	0.93607	0.97520	0.98358	0.00000	0.33969	0.82017	0.94034
0.9	0.32791	0.91819	0.96999	0.98205	0.00000	0.17016	0.76515	0.92601
0.95	0.23241	0.91366	0.96868	0.98167	0.00000	0.11736	0.75100	0.92239
0.99999	$3.2973 \cdot 10^{-3}$	0.90911	0.96737	0.98129	0.00000	$1.6239 \cdot 10^{-3}$	0.73669	0.91876

Slika 11. Normirane sopstvene frekvencije konzolne grede u funkciji normiranog opterećenja, za prvu kritičnu silu (lijevo) i drugu kritičnu silu (desno) upotrebom SRPS2D.

Figure 11. Normed eigenfrequencies vs. normed axial force for the first critical load (left) and the second critical load (right) of cantilever beam using SRPS2D.

Svojeviti oblici prva dva moda za dva tipa graničnih uslova i različite nivoe aksijalne sile dati su na slikama 12, 13 i 14. Analitička rešenja na slikama 12 i 13 pokazuju male razlike u ovim oblicima s povećanjem opterećenja. Kao što je i očekivano, ova razlika je izraženija za prvi sopstveni mod. Rezultati sličnih analiza pomoću SRPS2D koda prikazani su na Slici 14 za konzolnu gredu i rezultati su u skladu sa analitičkim.

Eigenshapes of first two modes for two types of BCs and different levels of axial force are given in Figures 12,13 and 14. Analytical solutions in Figures 12 and 13 show small differences in these shapes as the load increases. As expected, this difference is more pronounced for first eigenmode. The results of similar analysis using SRPS2D are presented in Figure 14 for cantilever beam and the results resemble the analytical ones.

Slika 12. Svojeviti oblici prva dva moda konzolne grede za različite nivoe aksijalne sile pomoću analitičkog Timošenkog rješenja.

Figure 12. Eigenshapes for the lowest two modes of cantilever beam for different values of the lowest critical force using analytical Timoshenko solution

Slika 13. Svojeviti oblici prva dva moda grede uklještena na jednom kraju i slobodno oslonjena na drugom, za različite nivoe aksijalne sile pomoću analitičkog Timošenkog rješenja.

Figure 13. Eigenshapes for the lowest two modes of beam fully clamped on one and simply supported on the other end for different values of the lowest critical force using analytical Timoshenko solution.

Slika 14. Svojeviti oblici prva dva moda konzolne grede za $P=0$ (zelena boja) i $P=0.995P_{cr1}$ (crvena boja) pomoću SRPS2D.

Figure 14. Eigenshapes for the lowest two modes of cantilever beam for $P=0$ (green) and $P=0.995P_{cr1}$ (red) using SRPS2D.

5 ZAKLJUČCI

Strukturalna analiza grednih sistema od inženjera zahtijeva da izabere teoriju koja je prikladna za posmatrani problem. Iako je Bernuli-Ojlerova teorija idealna za tanke grede, njene rezultate treba uzeti s rezervom, jer je za više modove potrebno koristiti tačnije teorije. Ovaj fenomen je posljedica činjenice da je dužina sinusnih polutalasa pri vibracijama ta koja određuje efektivnu dužinu grede, a samim tim i njenu vitkost. Savremene vitke građevinske konstrukcije neophodno je proračunavati s visokim nivoom tačnosti, te se pri dinamičkoj analizi kao neizbježnost nameće uzimanje u obzir napregnutosti konstrukcije u svim fazama izgradnje i eksploatacije.

Prikazana analiza za model bez sile pokazala je i potvrdila dobro poklapanje rezultata prema svim grednim teorijama za niže modove vitke grede. Za više modove dobijen je značajniji uticaj smičuće deformabilnosti u odnosu na efekat rotacione inercije, koji dovodi do povećanja razlike između analiziranih teorija. Očekivano, model Bernuli-Ojlerove grede pokazuje još veće odstupanje rezultata za više modove u poređenju s Reilijevom teorijom, jer zanemaruje i efekat rotacione inercije.

Model sa silom pokazao je povećanje razlike između sopstvenih frekvencija dobijenih analitički, kao i male promjene u sopstvenim oblicima s povećanjem intenziteta opterećenja. Sprovedena parametarska analiza pokazala je da svojstvena frekvencija za n -ti mod aksijalno pritisnutih greda konvergira ka nuli kako se vrijednost sile P približava vrijednosti kritične sile za n -ti mod izvijanja. Takođe, uočen je i značajan uticaj graničnih uslova na ponašanje grede pri vibraciji.

Razvijeni 2D model, primjenom konačnih elemenata, verifikovan je za analizu greda i, zavisno od razmatranog problema, može se koristiti uporedo sa analitičkim rješenjem. Razvoj sopstvenih kodova značajan je iz mnogo razloga. Mogućnost njihovog modifikovanja i prilagođavanja specifičnim analizama jedna je od najvažnijih mogućnosti. Posebno zanimljiv predmet daljnjeg istraživanja jeste drugi spektar frekvencija opterećene Timoshenkove grede, dok će se razmotriti i grede s funkcionalno promjenljivim materijalnim karakteristikama po visini. Pored toga, razmatrana greda će biti podvrgnuta eksperimentalnoj modalnoj analizi, te će se upoređivati s prikazanim analitičkim i numeričkim modelima.

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5 CONCLUSIONS

Structural analysis of beams requires from engineer to choose the theory which is adequate for considered problem. Although Bernoulli-Euler theory is ideally suited for slender beams, one has to be careful not to use the results blindly, since the higher modes require usage of more sophisticated theories. This is due to the fact that the length of the sine half-waves of vibration modes is the one which determines the effective length of the beam, and therefore its slenderness. Contemporary slender engineering structures need to be calculated with a high level of accuracy, so it is inevitable in dynamic analysis to take into account the real stress/strain state during all phases of construction and exploitation.

The presented analysis for the model without force showed and confirmed fine coincide of the results for lower modes of slender beam in all beam theories. However, for higher modes it was obtained that effect of shear deformability is more significant than the effect of rotary inertia, which leads to an increase in the difference between the analyzed theories. As expected, BE model shows the most significant deviation of results for higher modes since it neglects effect of rotary inertia, contrary to the Rayleigh model.

The model with force showed an increase in difference between the frequencies obtained according to beam theories, as well as small changes in eigen shapes with the increase of load. The performed parametric analysis showed that eigenfrequency of n^{th} mode of axially compressed beam converges to zero as the value of load P approaches value of critical force of n^{th} buckling mode. Also, significant influence of the boundary conditions on behaviour of the beam in vibration was observed.

Additionally, developed 2D finite element model is validated for analysis of beams and it can be used next to the analytical one, depending on the problem considered. The authors believe that the development of codes is significant, due to many reasons, where the possibility to modify and adopt them to suit particular analysis is one of the most important. Especially attractive subject of further research is second spectrum of loaded Timoshenko beam, while the analysis of functionally graded beams could be also considered. Furthermore, considered beam will be subjected to experimental modal analysis and comparison with presented models will be performed.

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REZIME

NEKI ASPEKTI U ANALIZI POPREČNIH SLOBODNIH VIBRACIJA PRIZMATIČNIH GREDA OPTEREĆENIH AKSIJALNOM SILOM

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U radu se analizira uticaj konstantne aksijalne sile pritiska na slobodne poprečne vibracije prizmatičnih greda. Primijenjene su tri tehničke gredne teorije za dobijanje odgovarajućih analitičkih rješenja. Hamiltonov varijacioni princip koristi se za izvođenje dobro poznatih diferencijalnih jednačina kretanja Timošenkove grede, dok se Bernuli-Ojlerova teorija i Rejljeva teorija dobijaju kao specijalni slučajevi. Za numeričku analizu, koristi se jednostavan dvodimenzionalni konačni element za ravno stanje napona s *drilling* stepenom slobode. Odgovarajući programski kod razvijen je i verifikovan poređenjem s komercijalnim softverom Abakus, kao i sa odgovarajućim analitičkim rezultatima. Potom, izvršena je parametarska analiza uticaja aksijalne sile na modalne karakteristike greda, o kojoj je detaljno diskutovano.

Ključne riječi: teorije grede, slobodne vibracije grede opterećene aksijalnom silom, *drilling* stepen slobode

SUMMARY

SOME ASPECTS OF ANALYSIS OF TRANSVERSE FREE VIBRATIONS OF UNIFORM BEAMS LOADED WITH AXIAL FORCE

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Influence of constant compressive axial force on free transverse vibrations of uniform beams is considered. Three technical beam theories are applied and appropriate analytical solutions are obtained. Hamilton's variation principle is utilized for derivation of well-known differential equations of motion of Timoshenko beam, while Bernoulli-Euler and Rayleigh theories are considered as special cases. Furthermore, simple two-dimensional plane stress finite element with drilling degree of freedom is employed for numerical analysis. Appropriate code is developed and verified via comparison with commercial finite element software Abaqus, as well as with analytical results. Parametric analysis of influence of the axial force on the modal characteristics of considered beams is performed and discussed in detail.

Keywords: beam theories, free vibration of beam loaded with axial force, drilling degree of freedom