FORMULARS AND SYMBOLS

\( V \) = volume

\( M \) = mass

\( \sigma \) = density

\( m \) = mass differential

\( dV \) = volume differential

\( \Delta \) = increment

\( f(M) \) = function of point \( M \)

\( f(\mu) \) = function of point \( \mu \)

\( EG \) = first fundamental form

\( F \) = second fundamental form

\( A \) = surface of the body

\( L \) = length in centimeters

\( [\sigma] = ML^{-3} \)

\( [\sigma_1] = ML^{-2} \)

\( [\sigma_2] = ML^{-1} \)

\( m = \iiint_v \sigma(x,y,z) dxdydz \)

\( m = \iiint_v \sigma dV = \iiint_v \sigma(x,y,z) dxdydz \)

\( m = \int_A \sigma_1(u,v) \sqrt{EG - F^2} dudv = \int_A \sigma_1(x,y) dxdy \)

\( m = \int_A \sigma_2(\ell) d\ell \)

\( m = \lim_{\Delta \rightarrow 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV} \)

\( \sigma_1 = \lim_{\Delta A \rightarrow 0} \frac{\Delta m}{\Delta A} = \frac{dm}{dA} \)

\( \sigma_2 = \lim_{\Delta \ell \rightarrow 0} \frac{\Delta m}{\Delta \ell} = \frac{dm}{d\ell} \)

\( \sigma = \lim_{\Delta V \rightarrow 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV} \)

\( \sigma = f(M) = f(\mu) \)

\( m = \int_A \sigma_1(x,y) dxdy \)

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where \( dl \) is the element of length extended to the entire length of the curve. For a real wire it is

\[
m = \int \sigma(x) \, dx.
\]

If \( \sigma, \sigma_1, \sigma_2 \) are constant values, the matter of the body is heterogeneous and to calculate body mass it is sufficient to know the volume i.e. the surface or the length of the body, because in these cases

\[
m = \sigma V = \sigma_1 A = \sigma_2 L.
\]

In the cases of non-homogeneous (heterogeneous) matter, body mass is not proportional to the density.

**CENTER OF MASSES OR CENTER OF INERTIA**

If related to the point \( M \) is the mass \( m \) and \( \overrightarrow{AM} \) is the vector of position of the point \( M \) in respect to point \( A \), the result \( m \overrightarrow{AM} \) is called the position vector burdened by mass \( m \) (of the point \( M \) in respect to point \( A \))(Bessonov & Song, 2001). Suppose that we have a set of \( n \) points of \( M_1, M_2, \ldots, M_n \) with masses \( m_1, m_2, \ldots, m_n \) and one particular point of space \( A \). The position vector for each point of this set may be constructed burdened by mass relative to point \( A \) and we can make a summation of all constructed vectors \( m_1 \overrightarrow{AM_1} + \cdots + m_n \overrightarrow{AM_n} \). The result of this sum is a vector starting at point \( A \). If this vector is divided by the entire mass \( \sum_{j=1}^{n} m_j \), we will get a new vector, whose the end we will mark with \( A \) and one particular point of space \( M \). If this manner, the vector equation

\[
\overrightarrow{AC} = \sum_{j=1}^{n} m_j \overrightarrow{AM_j}
\]

determines the point \( C \). Thus defined point \( C \) is called the center (of inertia of the masses).

In following, we shall examine some typical properties of the center of inertia. In this sense, we shall prove two simple propositions that justify the meaning that this point was called the center.

**Theorem 1.** The position of point \( C \) of a given material system do not depends on the choice of point \( A \), i.e. the beginning of the position vector of all constructed points.

**Proof.** Let us take as the starting point, besides \( A \), some other an arbitrary point \( A_1 \neq A \). If we denote with \( C_1 \) the appropriate center of inertia of the point \( A_1 \), we obtain the new vector equation

\[
m \overrightarrow{AC_1} = \sum_{j=1}^{n} m_j \overrightarrow{AM_j}
\]

Notice that \( \overrightarrow{AC_1} = \overrightarrow{A_1A} + \overrightarrow{AC} + \overrightarrow{CC_1} \) and \( \overrightarrow{AM_j} = \overrightarrow{A_1A} + \overrightarrow{AM_j}, \quad j = 1, \ldots, n \) (see Fig. 1). Now, if we put those vectors in the Eq.(2), and taking into account the Eq.(1), we get

\[
m \overrightarrow{A_1A} + m \overrightarrow{AC} + m \overrightarrow{CC_1} = \sum_{j=1}^{n} m_j \overrightarrow{AM_j} + \sum_{j=1}^{n} m_j \overrightarrow{AM_j}
\]

which implies \( \overrightarrow{CC_1} = 0 \). Thus, the center \( C_1 \) coincides with the original one.

**Remark 2.** According to Theorem 1, it follows that the position of the center depends only on the size of the masses and the distribution of these masses in the area. Therefore, the center of masses is an important natural point of any material system.

**Theorem 3.** The sum of the position vectors of points \( M_j \), in relation to the center \( C \) and weighted by the masses \( m_j \), is equal to zero, i.e.

\[
\sum_{j=1}^{n} m_j \overrightarrow{CM_j} = 0.
\]

**Proof.** Let us consider again the Eq.(1). In order to determine the point \( C \), into this equation let we put \( \overrightarrow{AM_j} = \overrightarrow{AC} + \overrightarrow{CM_j}, \quad j = 1, \ldots, n \). Thus, we have:

\[
\overrightarrow{AC} = \sum_{j=1}^{n} m_j \overrightarrow{CM_j},
\]

or, equivalently:

\[
\sum_{j=1}^{n} m_j \overrightarrow{CM_j} = 0,
\]

which was supposed to be proven.

**Remark 4.** From the basic vector Eq.(1) for the center \( C \) it follows that:

\[
\overrightarrow{AC} = \sum_{j=1}^{n} \lambda_j \overrightarrow{AM_j}
\]

where \( \lambda_j = m_j/m \), \( j = 1, \ldots, n \) and \( \sum_{j=1}^{n} \lambda_j = 1 \). Thus, vector \( \overrightarrow{AC} \) is the convex combination of vectors \( \overrightarrow{AM_1}, \ldots, \overrightarrow{AM_n} \). In the Cartesian coordinates, Eq.(4) can be written as:

\[
\overrightarrow{ic} = \sum_{j=1}^{n} \lambda_j \overrightarrow{ij},
\]

where \( \overrightarrow{ic} = \overrightarrow{AC} = (x_c, y_c, z_c) \) and \( \overrightarrow{ij} = \overrightarrow{AM_j} = (x_j, y_j, z_j) \). In this way, to the vector Eqs.(4)-(5) correspond the following three scalar equations

\[
x_c = \sum_{j=1}^{n} \lambda_j x_j, \quad y_c = \sum_{j=1}^{n} \lambda_j y_j, \quad z_c = \sum_{j=1}^{n} \lambda_j z_j.
\]
These are the basic scalar equations for determining the position of the center of masses.

**Remark 5.** If the masses of a material system are distributed in a certain area continuously, the sums extended to all material points of the system pass into defined integrals extended to the areas of continuous matter. Then we have the vector equations:

\[ \mathbf{AC} \cdot \iint_V \, d\mathbf{m} = \iint_V \mathbf{r} \, d\mathbf{m} \]

or

\[ \mathbf{AC} \cdot \iint_V \sigma \, dV = \iint_V \sigma \mathbf{r} \, d\mathbf{V} \]

The scalar form of \( O_x \) is:

\[ x_c = \frac{\iint_V \sigma \mathbf{x} \cdot d\mathbf{V}}{\iint_V \sigma \, d\mathbf{V}} \]

The analog integrals apply in the case of masses continuously distributed over the surface and along the lines.

**Remark 6.** If the matter of body is homogeneous, the multiplier of density \( \sigma \) entering the numerator and denominator can be shortened, and thus for the homogeneous bodies we have patterns in which the mass is not included. These patterns can be considered as forms for the determination of the center of inertia of a volume i.e. of a surface or line. Thus, we have:

\[ x_c = \frac{\iint_V \mathbf{x} \cdot d\mathbf{V}}{\iint_V \mathbf{r} \, d\mathbf{V}} \]

\[ x'_c = \frac{\iint_A \mathbf{r} \cdot d\mathbf{A}}{\iint_A \mathbf{r} \, d\mathbf{A}} \]

**Remark 7.** If the arrangement is symmetrical with respect to a plane, or a line or point, the center of masses must lie in that plane, on the line or at that point.

**PAPPOS-GULDIN’S THEOREMS**

Regarding the concept of the center of masses we list two so-called Pappos-Guldin’s theorems.

**Theorem 8.** The area which is obtained by rotating the arc of a curve in the plane about an axis in the plane, which does not cut the curve, is equal to the product of the arc length and circumference of a circle described by the mass center of this arc.

**Proof.**

Let a part of the flat curve, from point \( A \) to point \( B \), rotate about an axis \( p \) on the curve plane and does not cut the curve, but only points \( A \) and \( B \) can belong to the axis. Let \( \Delta l \) be the distance between two very close points of the curve (see Fig. 2). Area formed by \( \Delta l \) in reversing (i.e. the side surface of the cylinder, coupe, truncated cone or a circular ring) is equal to \( 2\pi r \Delta l \), where \( r \) is the distance of the middle of the line \( \Delta l \) from the axis of rotation. Since in the border case the area is expressed by \( 2\pi rd \), where \( dl \) is the differential of the curve arc and \( r \) is the distance of an arbitrary point of the arc from the rotation axis, accordingly the surface \( S \), described by the arc \( AB \) is equal to:

\[ S = 2\pi \int L \, dl, \quad \text{where } L \text{ is the arc of } AB. \]

If we use \( r \) to mark the distance of the center of mass \( C \) of the arc \( L \) from the straight line \( p \), then we have the equation:

\[ Lr = \int L \, dl. \]

If this value of integral is put into the previous equation it gives:

\[ S = 2\pi rL. \]

**Theorem 9.** The volume which is obtained by rotating the flat surface around the axis in the plane, which does not cut the surface, equals to the product of the surface value and the circumference of a circle described by the center of masses of the surface.

**Proof.** Suppose that the flat surface \( P \) rotates around the axis \( p \), which does not cut the contour of the surface (see Fig. 3). Volume \( V \), which is obtained by reversing the surface, can be calculated using the double integral:

\[ V = 2\pi \iint r \, dr \, dh \]

where \( r \) is the distance of the point of elementary rectangle of dimensions \( dr \) and \( dh \) from the axis of rotation. If we write down the equation for determining the center of mass \( C \) of the area \( P \) with respect to line \( p \), the following is obtained

\[ Pr = \iint r \, dr \, dh \]
where \( r_e \) is the distance of the center of mass of the surface \( P \) from the line \( p \). When this integral value is put into the previous equation, we get:

\[
V = 2\pi r_e P
\]

which is supposed to be obtained.

**AXIAL SQUARE MOMENT OF INERTIA**

Let there be given the line \( p \) and the material point of the mass \( m \) at the distance \( d \) from the line. The product \( md^2 \) is called the axial square moment of the given material point relative to the given line. If there are more material points with masses \( m_1, m_2, \ldots, m_n \) at distances \( d_1, d_2, \ldots, d_n \) of the given line \( p \), for the moment of inertia of the system we have the equation and the mark:

\[
I_p = \sum_{i=1}^{n} m_i d_i^2
\]

If the coordinates of mass \( m_i \) are marked by \( x_i, y_i, z_i \) in relation to the Cartesian coordinate system, then three moments of inertia can be introduced:

\[
I_x = \sum m_i \left( y_i^2 + z_i^2 \right), \quad I_y = \sum m_i \left( x_i^2 + z_i^2 \right), \quad I_z = \sum m_i \left( x_i^2 + y_i^2 \right).
\]

For continuously distributed masses the sums are replaced by integrals. For the moment of inertia about the \( x \)-axis we have:

\[
I_x = \iiint_V \rho \left( y^2 + z^2 \right) \, dxdydz.
\]

It is similar for the other axis.

The term axial moment of inertia can be applied both to masses spread across the surface and the curve, both in the case when the curve and axis lie in the same plane, and in the case when the curve, which can also be spatial, occupies an arbitrary position in relation to the axis of moment of inertia.

To better explain the afore mentioned mechanical phenomena, the concept of kinetic energy i.e. energy of movement can further be introduced. This highlights the importance of multiple integrals in mechanics. For a material point of mass \( m \), moving at speed of intensity \( v \), the kinetic energy is calculated by the form \( \frac{1}{2}mv^2 \).

When a solid body is moving translational its points have the same speed. Then taken for the representative of the body is the center of masses point \( C \), and as the speed of all points \( v_p \) velocity is determined with intensity \( v_p \). If \( T \) marks the kinetic energy of a solid in the case of translational movement, then it follows:

\[
T = \frac{1}{2} \sum_{i=1}^{n} m_i v_p^2 = \frac{1}{2} m v_c^2
\]

where \( m \) is the overall mass of the body. Accordingly, in the translational motion, the kinetic energy of the body is expressed in the same way as the kinetic energy of a single material point.

Now we look at a rotary motion of a solid body, i.e. rotation about a fixed axis. The velocity intensity \( v_i \) of some point \( M_i \) of the body which is at the distance \( d_i \) from the axis can be calculated using the formula:

\[
v_i \lim_{\Delta t \to 0} \frac{\Delta S_i}{\Delta t} = \lim_{\Delta \omega \to 0} \frac{d_i \Delta \alpha}{\Delta t} = d_i \lim_{\Delta \alpha \to 0} \frac{\Delta \alpha}{\Delta t} = d_i \omega
\]

where \( \Delta S_i \) is the element of the roundabout of point \( M_i \), \( \Delta \alpha \) infinitely small rotation angle and:

\[
\omega = \lim_{\Delta \alpha \to 0} \frac{\Delta \alpha}{\Delta t}
\]

the intensity of angular velocity of the body. Thus, for the kinetic energy of each point of a solid body it can be put as follows:

\[
\frac{1}{2} m_i d_i^2 \omega^2 = \frac{1}{2} I_m \omega^2 = \frac{1}{2} I \omega^2 (m_i d_i^2),
\]

and for the whole body we will get:

\[
T = \frac{1}{2} I \omega^2 \quad \text{where} \quad I = \lim_{n \to \infty} \sum_{i=1}^{n} m_i d_i^2,
\]

i.e. moment of inertia of the body about the axis of rotation.

If we compare the expressions:

\[
T = \frac{1}{2} I \omega^2 \quad \text{and} \quad T = \frac{1}{2} I \omega^2
\]

we see that in the first expression the kinetic energy depends on the square of the linear velocity, and in the other on the square of the angular velocity. In the first expression we have a coefficient \( m \), and in the second \( I \) and both are called inertial coefficients-the first for the translational movement and the second for rotational movement.

**REFERENCES**


