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NEW PROOF OF AHLFORS LEMMA ABOUT GREEN STOKES FORMULA FOR DISTRIBUTIONS

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ABSTRACT

The paper presents a new proof of Ahlfors lemma about Green Stokes formula for distributions. The proof is performed directly using test functions instead of using convolutions.

Keywords: Convolutions, Distributions, Green Stokes formula, Test functions.

INTRODUCTION

The paper presents a new proof of Ahlfors lemma about Green Stokes formula for distributions (Ahlfors, 2006). The new proof is performed directly using test functions instead of using convolutions.

If there is interest in solving the equation

$$f_{\bar{z}} = \mu f_z \tag{1}$$

where $\|\mu\|_\infty \leqslant k < 1$, Arsenović et al. (2012); Arsenović & Mateljević (2021) let's treat at the beginning the case where μ has compact support so that f will be analytic at ∞. The fixed exponent $p > 2$ will be used, such that $kC_p < 1$.

Theorem 1. If μ has compact support there exist a unique solution f of (1) such that $f(0) = 0$ and $f_z - 1 \in L_p$.

According to Theorem 1 following conclusion can be made:

Lemma 1. $\|g_z - f_z\|_p \to 0$ and $g \to f$ uniformly on compact sets.

In order to show that f has derivatives if μ has, a slight generalization of Weyl's lemma is made:

Lemma 2. If p and q are continuous and have locally integrable distributional derivatives that satisfy $p_{\bar{z}} = q_z$, then there exists a function $f \in C^1$ with $f_z = p$ and $f_{\bar{z}} = q$. (Ahlfors, 2006)

Such results are useful in deriving solutions of Beltrami's equation (1). One method, which works in higher dimensions as well is to use convolutions, see (Arsenović at al., 2021). Here we work directly with test functions. A very concise presentation is given in (Ahlfors, 2006).

The new proof of Lemma 2 will be performed directly using test functions.

In the first part of the paper, Lemma 3, Lemma 4 and Lemma 5 with their proofs are presented. Cantor's theorem on uniform continuity (Carleson & Jones, 1992), Fubini's theorem (Arsenović at al., 2012), (Mateljević, 2013a, 2012, 2013b) and the Mean value theorem (Duren, 2004) were used in the proofs.

Below are given Theorem 2 and Theorem 3 with proofs, which represent a new proof of Ahlfors lemma about Green Stokes formula for distributions. A new proof is provided directly using test functions instead of using convolutions.

THEORETICAL PART

Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as

$$F(t) = \begin{cases} e^{-1/t}, & t > 0. \\ 0, & t \leqslant 0. \end{cases} \tag{2}$$

Lemma 3. The function F is infinitely differentiable. Let C be a set of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that there exists a real rational function r such that we have

$$f(t) = \begin{cases} r(x)e^{-1/t}, & t > 0. \\ 0, & t \leqslant 0. \end{cases} \tag{3}$$

Let f be an arbitrary function from the C class. Then it holds that

$$\lim_{t\to 0+} f(t) = \lim_{t\to 0+} \frac{r(t)}{e^{1/t}} = \lim_{x\to\infty} \frac{r(1/x)}{e^x} = 0 = f(0), \tag{4}$$

so the function f is continuous. It also holds that

$$\lim_{t\to 0+} \frac{f(t) - f(0)}{t} = \lim_{t\to 0+} \frac{f(t)}{t} = \lim_{t\to 0+} \frac{r(t)}{t\, e^{1/t}} = \lim_{x\to\infty} \frac{x\, r(1/x)}{e^x} = 0, \tag{5}$$

so the function f is differentiable and we have

$$f'(t) = \begin{cases} (r(t)' + r(t)/t^2)e^{-1/t}, & t \geq 0. \\ 0, & t \leqslant 0. \end{cases} \tag{6}$$

Hence it follows that $f' \in C$.

It follows in particular that the functions from the C class are differentiable and that the class C is closed for derivatives. It clearly implies that all functions from the class C are infinitely differentiable. In particular, it holds that F is infinitely differentiable.

Let $\varepsilon > 0$. Since the function $x \mapsto F(\varepsilon^2 - x^2)$ is indefinitely differentiable and positive on the interval $(-\varepsilon, \varepsilon)$, and equals zero outside it, its integral in $\mathbb{R}$ is finite and positive, and the C_ε constant can be defined as

$$C_\varepsilon = \Big(\int_{\mathbb{R}} F(\varepsilon^2 - x^2)\, dx\Big)^{-1}. \tag{7}$$

Let us define the function $\omega_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ with $\omega_\varepsilon(x) = C_\varepsilon F(\varepsilon^2 - x^2)$. It is infinitely differentiable, non-negative and it holds that $\int_{\mathbb{R}} \omega_\varepsilon(x)\, dx = 1$.

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Let us define the function $\mu_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ in the following way: $\mu_\varepsilon(x) = \int_0^x \omega_{\varepsilon/2}(t-\varepsilon/2)\,dt$. It is also infinitely differentiable, non-decreasing and it holds that $\mu_\varepsilon(x) = 0 \Leftrightarrow x \leqslant 0$ and $\mu_\varepsilon(x) = 1 \Leftrightarrow x \geqslant \varepsilon$.

For the given $a, b \in \mathbb{R}$ such that $a < b$ and $b - a > 2\varepsilon$ let us define the function $\eta_{a,b,\varepsilon} : \mathbb{R} \longrightarrow \mathbb{R}$ in the following way:

$$\eta_{a,b,\varepsilon}(x) = \{\begin{cases} \mu_\varepsilon(x-a), & x < (a+b)/2, \\ \mu_\varepsilon(b-x), & x \geqslant (a+b)/2. \end{cases} \tag{8}$$

This is an infinitely differentiable function equal to zero outside the interval (a, b), equal to one in the segment $[a+\varepsilon, b-\varepsilon]$, increasing in the segment $[a, a+\varepsilon]$ and decreasing in the segment $[b-\varepsilon, b]$.

Lemma 4. Let $a, b \in \mathbb{R}$ such that $a < b$ and $f, g : (a, b) \longrightarrow \mathbb{R}$ functions such that f is continuous and bounded and g s integrable without changing the sign. Then there exists $c \in (a, b)$ such that the following holds:

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx. \tag{9}$$

Proof. Since g does not change the sign, $\int_a^b g(x)\,dx = 0$ is possible only when $g \equiv 0$ almost everywhere and in that case the claim holds true for each $c \in (a, b)$.

Let us assume, therefore, that $\int_a^b g(x)\,dx \neq 0$. In addition, if there is a constant v such that $\mu(\{x \in (a, b) | g(x) > 0, f(x) \neq v\}) = 0$ holds, then the claim holds for each c $\in$(a,b) such that g(c)>0 and f(c)=v holds.

Without loss of generality, we can assume that there is no such constant v.

For

$$m = \inf_{x\in(a,b)} f(x), \quad M = \sup_{x\in(a,b)} f(x) \tag{10}$$

we have

$$\begin{aligned} m\int_a^b g(x)\,dx &= \int_a^b m\,g(x)\,dx < \int_a^b f(x)g(x)\,dx \\ &< \int_a^b M\,g(x)\,dx = M\int_a^b g(x)\,dx, \end{aligned} \tag{11}$$

that is

$$m < \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} < M. \tag{12}$$

Due to the continuity of the f function, there exists a $c \in (a, b)$ for which

$$f(c) = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}, \tag{13}$$

holds and thus completes the proof of the lemma.

Lemma 5. Let $a, b, c, d \in \mathbb{R}$ be such that $a < b$, $c < d$ and $f : D \longrightarrow \mathbb{R}$ continuous function where $D = [a, b] \times [c, d]$. Then we have

$$\begin{aligned} &\lim_{\varepsilon\to 0+} \iint_D f(x,y)\eta_{a,b,\varepsilon}(x)\eta'_{c,d,\varepsilon}(y)\,dx\,dy \\ &\quad = \int_a^b f(x,c)\,dx - \int_a^b f(x,d)\,dx, \end{aligned} \tag{14}$$

$$\begin{aligned} &\lim_{\varepsilon\to 0+} \iint_D f(x,y)\eta'_{a,b,\varepsilon}(x)\eta_{c,d,\varepsilon}(y)\,dx\,dy \\ &\quad = \int_c^d f(a,y)\,dy - \int_c^d f(b,y)\,dy. \end{aligned} \tag{15}$$

Proof. For the given $\varepsilon > 0$, define the r_ε function as

$$r_\varepsilon(x) = \int_c^d f(x,y)\eta'_{c,d,\varepsilon}(y)\,dy - (f(x,c) - f(x,d)). \tag{16}$$

From Cantor's theorem on uniform continuity, it follows that the function f in uniformly continues and therefore the r_ε function is continuous. According to the mean value theorem, it holds that

$$\begin{aligned} \int_c^d f(x,y)\eta'_{c,d,\varepsilon}(y)\,dy &= \int_c^{c+\varepsilon} f(x,y)\omega_{\varepsilon/2}(y-c+\varepsilon/2)\,dy \\ &- int_{d-\varepsilon}^d f(x,y)\omega_{\varepsilon/2}(d-y-\varepsilon/2)\,dy \\ &= f(x,c')\int_c^{c+\varepsilon} \omega_{\varepsilon/2}(y-c+\varepsilon/2)\,dy \\ &- f(x,d')\int_{d-\varepsilon}^d \omega_{\varepsilon/2}(d-y-\varepsilon/2)\,dy \\ &= f(x,c') - f(x,d') \end{aligned} \tag{17}$$

for some $c' \in (c, c+\varepsilon)$ and $d' \in (d-\varepsilon, d)$. From there as well as from Cantor's theorem on uniform continuity, it follows that r_ε uniformly tends to zero when ε tends to zero. According to Fubini's theorem, the following holds

$$\begin{aligned} &\iint_D f(x,y)\eta_{a,b,\varepsilon}(x)\eta'_{c,d,\varepsilon}(y)\,dx\,dy \\ &= \int_a^b\int_c^d f(x,y)\eta_{a,b,\varepsilon}(x)\eta'_{c,d,\varepsilon}(y)\,dy\,dx \\ &= \int_a^b \eta_{a,b,\varepsilon}(x)\int_c^d f(x,y)\eta'_{c,d,\varepsilon}(y)\,dy\,dx \\ &= \int_a^b \eta_{a,b,\varepsilon}(x)(r_\varepsilon(x) + f(x,c) - f(x,d))\,dx \\ &= \int_a^{a+\varepsilon} (\eta_{a,b,\varepsilon}(x) - 1)(r_\varepsilon(x) + f(x,c) - f(x,d))\,dx \\ &+ \int_{b-\varepsilon}^b (\eta_{a,b,\varepsilon}(x) - 1)(r_\varepsilon(x) + f(x,c) - f(x,d))\,dx \\ &+ \int_a^b r_\varepsilon(x)\,dx + \int_a^b f(x,c)\,dx - \int_a^b f(x,d). \end{aligned} \tag{18}$$

When ε tends to zero, the first two addends tend to zero because the subintegral function is bounded as continuous on a compact set, while the third addend tends to zero on the basis of the uniform convergence of the r_ε function to zero. From there, the first part of the claim follows while the second part can be proven analogously.

Let us further identify the complex plane with $\mathbb{R}^2$. Let Ω denote a simply connected region in the complex plane. Let $f : \Omega \longrightarrow \mathbb{C}$ which has partial derivatives f_x and f_y. The partial derivatives with respect to z and $\bar{z}$ are defined as

$$f_z = \frac{f_x - i f_y}{2}, \quad f_{\bar{z}} = \frac{f_x + i f_y}{2}. \tag{19}$$

Let p and q are now continuous functions which map Ω into $\mathbb{C}$ let γ denote a rectifiable curve in Ω. By the integral

$$\int_\gamma p\, dz + q\, d\bar{z} \tag{20}$$

we mean the integral

$$\int_\gamma (p+q)\, dx + i(p-q)\, dy. \tag{21}$$

NUMERICAL RESULTS

Theorem 2. The following conditions are equivalent:

A *Integral* (20) *is equal to zero with respect to each rectifiable loop γ in Ω.*

B *Integral* (20) *is equal to zero with respect to each rectangle loop γ in Ω.*

C *There exists a function $F : \Omega \longrightarrow \mathbb{C}$ which has partial derivatives and it is such that $F_z = p$ and $F_{\bar{z}} = q$.*

Proof. Since each rectangle loop is rectifiable, B follows from A. Let us derive A from C. Let us derive 1 from 3.

$$\begin{aligned} \oint_\gamma p\, dz + q\, d\bar{z} &= \oint_\gamma (p+q)\, dx + i(p-q)\, dy \\ &= \oint_\gamma (F_z + F_{\bar{z}})\, dx + i(F_z - F_{\bar{z}})\, dy \\ &= \oint_\gamma F_x\, dx + F_y\, dy = 0. \end{aligned} \tag{22}$$

Suppose that B holds and let us derive C. By a special path determined by the vertices

$$(x_0, y_0), \ldots, (x_n, y_n) \tag{23}$$

in the denotation $s[x_0, y_0; \ldots; x_n, y_n]$ we mean a zig-zag line determined by the vertices (23) so that the whole line is in Ω, that it is parameterized by a parameter from the segment $[0,1]$, that it is linear part by part, and that, for each $k < n$ it holds $x_k = x_{k+1}$ or $y_k = y_{k+1}$. By the value along the rectifiable path γ in Ω in the denotation $V(\gamma)$ we mean value (20).

Let γ and δ be special paths such that $\gamma(0) = \delta(0)$ and $\gamma(1) = \delta(1)$. Let us prove that the following holds:

$$\int_\gamma p\, dz + q\, d\bar{z} = \int_\delta p\, dz + q\, d\bar{z}. \tag{24}$$

Let us prove it first in the case when the paths γ and δ are in the $I \times J$ set for some non-empty open intervals I and J of a real line such that $I \times J \subseteq \Omega$. Based on assumption B, for any $x_0, x_1, x_2 \in I$ and $y_0, y_1, y_2 \in J$

$$V(s[x_0, y_0; x_1, y_0; x_2, y_0]) = V(s[x_0, y_0; x_2, y_0]), \tag{25}$$

$$V(s[x_0, y_0; x_1, y_0; x_1, y_1]) = V(s[x_0, y_0; x_0, y_1; x_1, y_1]), \tag{26}$$

hold as well as

$$\begin{aligned} &V(s[x_0, y_0; x_1, y_0; x_1, y_1; x_2, y_1]) \\ &= V(s[x_0, y_0; x_1, y_0; x_2, y_0; x_2, y_1]) \\ &\quad = V(s[x_0, y_0; x_2, y_0; x_2, y_1]). \end{aligned} \tag{27}$$

Let $x_0, \ldots, x_n \in I$ and $y_0, \ldots, y_n \in J$ for $n \geqslant 2$. If there exists $i \in \{1, \ldots, n-1\}$ holds that $x_{i-1} = x_i = x_{i+1}$ or $y_{i-1} = y_i = y_{i+1}$, then there exist $x'_0, \ldots, x'_{n-1} \in I$ and $y'_0, \ldots, y'_{n-1} \in J$ such that we obtain

$$\begin{aligned} V(s[x_0, y_0; \ldots; x_n, y_n]) &= V(s[x'_0, y'_0; \ldots; x'_{n-1}, y'_{n-1}]), \\ (x_0, y_0) &= (x'_0, y'_0), \\ (x_n, y_n) &= (x'_{n-1}, y'_{n-1}). \end{aligned} \tag{28}$$

The same also holds in the case when such i does not exist but when $n \geqslant 3$. Hence, for any $x_0, \ldots, x_n \in I$ and $y_0, \ldots, y_n \in J$ it holds that

$$\begin{aligned} V(s[x_0, y_0; \ldots; x_n, y_n]) &= V(s[x_0, y_0; x_n, y_0; x_n, y_n]) \\ &= V(s[x_0, y_0; x_1, y_n; x_n, y_n]). \end{aligned} \tag{29}$$

This means that $V(\mu)$ for the special path μ in $I \times J$ depends only on the initial and the final point of the path μ, which thus proves the claim in this case. Let us further deal with a general case.

Since Ω is a simply connected region, we can choose continuous mapping $H : [0,1]^2 \longrightarrow \Omega$ such that for every $t \in [0,1]$ it holds that $H(0,t) = \gamma(t)$ and $H(1,t) = \delta(t)$, and also that for every $x \in [0,1]$ it holds that $H(x,0) = \gamma(0) = \delta(0)$ and $H(x,1) = \gamma(1) = \delta(1)$.

The set $H[[0,1]^2]$ is a compact subset of the open set Ω, so there exists some $\varepsilon > 0$ such that for every $x, t \in [0,1]$ it holds that $B(H(x,t), \varepsilon) \subseteq \Omega$.

According to Cantor's theorem on uniform continuity, there exists $n \in \mathbb{N}$ such that it holds

$$(\forall x_1, t_1, x_2, t_2 \in [0,1])(|x_2 - x_1| + |t_2 - t_1| < 1/n \Rightarrow$$

$$d(H(x_1, t_1), H(x_2, t_2)) < \varepsilon).$$

For $i, j \in \{0, \ldots, n\}$ let us define X_i^j and Y_i^j as

$$(X_i^j, Y_i^j) = H(i/n, j/n). \tag{30}$$

Let $i_1, i_2, j_1, j_2 \in \{0, \ldots, n\}$ be such that it holds $|i_1 - i_2|, |j_1 - j_2| \leqslant 1$. Then

$$d((X_{i_1}^{j_1}, Y_{i_1}^{j_1}), (X_{i_2}^{j_2}, Y_{i_2}^{j_2})) \leqslant \varepsilon/5. \tag{31}$$

Let us also, for every $i, j \in \{0, \dots, n\}$ define a path

$$\pi_i^j = s[x_0, y_0; \dots; x_{2j-1}, y_{2j-1}], \tag{32}$$

where

$$(x_0, y_0) = (X_i^0, Y_i^0) = H(i/n, 0) = \gamma(0) = \delta(0), \tag{33}$$

$$(x_{2j-1}, y_{2j-1}) = (X_i^j, Y_i^j) = H(i/n, 1) = \gamma(1) = \delta(1),$$

$$x_{2k-1} = X_i^{k-1}, \quad x_{2k} = X_i^k, \quad y_{2k-1} = y_{2k} = Y_i^k, \quad \text{for } 1 \leqslant k < n.$$

For $i, k < n$ let us define the special path μ_i^k as

$$\mu_i^k = s[X_i^k, Y_i^k; X_i^k, Y_i^{k+1}; X_i^{k+1}, Y_i^{k+1}; X_{i+1}^{k+1}, Y_i^{k+1}; X_{i+1}^{k+1}, Y_{i+1}^{k+1}; \tag{34}$$
$$X_{i+1}^k, Y_{i+1}^{k+1}; X_{i+1}^k, Y_{i+1}^k; X_{i+1}^k, Y_i^k; X_i^k, Y_i^k].$$

All points of the path μ_i^k are at a mutual distance smaller than $\varepsilon/4$, so the path μ_i^k lies in $B((X_i^k, Y_i^k), \varepsilon/4)$, so there are intervals I and J of real line, which are of the width $\varepsilon/2$ and such that it holds $B((X_i^k, Y_i^k), \varepsilon/4) \subseteq I \times J \subseteq B((X_i^k, Y_i^k), \varepsilon)$. It thus follows that the path μ is in $I \times J$ and that $I \times J \subseteq \Omega$, resulting in $V(\mu_i^k) = 0$. On the basis of this, it is proven by induction that for all $i, k < n$ the following holds:

$$V(\pi_{i+1}^k) - V(\pi_i^k) = V(s[X_i^k, Y_i^k; X_{i+1}^k, Y_i^k; X_{i+1}^k, Y_i^{k+1}]). \tag{35}$$

In particular, for $k = n$ it holds that $V(\pi_i^n) = V(\pi_{i+1}^n)$, from where it follows that $V(\pi_0^n) = V(\pi_n^n)$. For any $k < n$ the restrictions of the paths π_0^n and γ on the segment $[k/n, (k+1)/n]$ have a common beginning (X_0^k, Y_0^k) and a common end (X_0^{k+1}, Y_0^{k+1}) and they are within the disc $B((X_0^k, Y_0^k), \varepsilon/4)$, so it is similarly proven as before that the integrals of $p\,dz + q\,d\bar{z}$ are equal by themselves, which therefore proves that both $V(\gamma) = V(\pi_0^n)$ i $V(\delta) = V(\pi_n^n)$.

Let us chose arbitrary(a, b) $\in \Omega$. For each $(u, v) \in \Omega$, let us define $F(u, v)$ as follows. Let γ is arbitrary special path such that $\gamma(0) = (a, b)$ and $\gamma(1) = (u, v)$ hold. Let

$$F(u, v) = \int_\gamma (p + q)\,dx + i\,(p - q)\,dy.$$

This definition is correct because of independence of the value on the choice of γ .Then $F_x = p + q$ and $F_y = i\ (p - q)$ hold and therefore $F_z = p$ and $F_{\bar{z}} = q$ hold.

Theorem 3. The following conditions are equivalent:

A *For any infinitely differentiable function $\varphi : \Omega \longrightarrow \mathbb{C}$ with compact support, it holds that*

$$\iint_\Omega p(x, y)\varphi_{\bar{z}}\,dx\,dy = \iint_\Omega q(x, y)\varphi_z\,dx\,dy.$$

B *For any closed rectangular domain $D \subseteq \Omega$ and any infinitely differentiable function $\varphi : \Omega \longrightarrow \mathbb{C}$ with support D it holds that*

$$\iint_D p(x, y)\varphi_{\bar{z}}\,dx\,dy = \iint_D q(x, y)\varphi_z\,dx\,dy.$$

C *There exists a function $F : \Omega \longrightarrow \mathbb{C}$ with continuous partial derivatives such that $F_z = p$ and $F_{\bar{z}} = q$ hold.*

Proof. It is clear that B follows from A. Suppose B and let us prove C. Let $D = [a, b] \times [c, d]$ for some $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$. For $\varepsilon > 0$ such that $\varepsilon < (b-a)/2,\ (d-c)/2$ let us define h_ε, k_ε and φ_ε as

$$h_\varepsilon = \eta_{a,b,\varepsilon}, \quad k_\varepsilon = \eta_{c,d,\varepsilon}, \tag{36}$$

$$\varphi_\varepsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \varphi_\varepsilon(x, y) = h_\varepsilon(x)k_\varepsilon(y). \tag{37}$$

With the truncated forms of $p = p(x, y)$, $\varphi_\varepsilon = \varphi_\varepsilon(x, y)$, $h = h(x)$ and $k = k(y)$, it follows from B that

$$\iint_D p\,(\partial_x\varphi_\varepsilon + i\partial_y\varphi_\varepsilon)\,dx\,dy = \iint_D q\,(\partial_x\varphi_\varepsilon - i\partial_y\varphi_\varepsilon)\,dx\,dy, \tag{38}$$

$$\iint_D (p - q)\,\partial_x\varphi_\varepsilon\,dx\,dy + i\iint_D (p + q)\,\partial_y\varphi_\varepsilon\,dx\,dy = 0, \tag{39}$$
$$\iint_D (p + q)\,\partial_y\varphi_\varepsilon\,dx\,dy - i\iint_D (p - q)\,\partial_x\varphi_\varepsilon\,dx\,dy = 0.$$

On the basis of Lemma 5, allowing ε to tend to zero, we conclude that the following holds

$$\int_a^b (p + q)(x, c)\,dx - \int_a^b (p + q)(x, d)\,dx + i$$
$$+ \int_c^d (p - q)(a, y)\,dy - i\int_c^d (p - q)(b, y)\,dy = 0, \tag{40}$$

namely

$$\oint_{\partial D} (p + q)\,dx + i(p - q)\,dy = 0, \tag{41}$$

so on the basis of Theorem 2 it holds B. Let us suppose C and let us derive A. It should be actually proven that the following holds

$$\iint_\Omega F_z\varphi_{\bar{z}}\,dx\,dy = \iint_\Omega F_{\bar{z}}\varphi_z\,dx\,dy, \tag{42}$$

i.e.

$$\iint_\Omega (F_x - iF_y)(\varphi_x + i\varphi_y)\,dx\,dy \tag{43}$$
$$= \iint_\Omega (F_x + iF_y)(\varphi_x - i\varphi_y)\,dx\,dy,$$

which is equivalent to

$$\iint_\Omega F_x\varphi_y\,dx\,dy = \iint_\Omega F_y\varphi_x\,dx\,dy. \tag{44}$$

Let $K \subseteq \Omega$ be a compact support of the function φ. Each point of the set K can be paired with an open rectangle whose sides are parallel to the coordinate axes and whose closure lies in Ω. Such a cover has a definite subcover; therefore, without loss of generality, we can assume that K is a non-empty finite union of closed rectangles with non-empty interior and sides parallel to the coordinate axes.

Suppose that $D = [a, b] \times [c, d]$ is the smallest closed rectangle with its sides parallel to the coordinate axes, such that $K \subseteq D$. Let us pair each $x \in [a, b]$ with the set

$$D_x = \{y \in [c, d]\ :\ (x, y) \in K\}. \tag{45}$$

For each x, the set D_x is a finite union of disjoint closed intervals. On the endpoints of each of these intervals, the function φ is annuled together with its partial derivatives. Hence, if $[u, v]$ is some of these intervals, then it holds that

$$\begin{aligned}\int_u^v F_y(x,y)\varphi_x(x,y)\,dy &= F(x,v)\varphi_x(x,v) - F(x,u)\varphi_x(x,u) \\ &- \int_u^v F(x,y)\varphi_{xy}(x,y)\,dy \\ &= -\int_u^v F(x,y)\varphi_{xy}(x,y)\,dy.\end{aligned} \tag{46}$$

By adding the integrals as a function of all disjoint closed intervals which give the set D_x in the union, we conclude that the following holds

$$\int_{D_x} F_y(x,y)\varphi_x(x,y)\,dy = -\int_{D_x} F(x,y)\varphi_{xy}(x,y)\,dx\,dy. \tag{47}$$

According to Fubini's theorem, we have

$$\begin{aligned}\iint_K F_y(x,y)\varphi_x(x,y)\,dx\,dy &= \int_a^b \int_{D_x} F_y(x,y)\varphi_x(x,y)\,dx\,dy \\ &= -\int_a^b \int_{D_x} F(x,y)\varphi_{xy}(x,y)\,dx\,dy \\ &= -\iint_K F(x,y)\varphi_{xy}(x,y)\,dx\,dy.\end{aligned} \tag{48}$$

Since K is the support of the function φ, it holds that

$$\iint_\Omega F_y\varphi_x\,dx\,dy = \iint_K F_y\varphi_x\,dx\,dy = \tag{49}$$

$$-\iint_K F(x,y)\varphi_{xy}(x,y)\,dx\,dy = -\iint_\Omega F(x,y)\varphi_{xy}(x,y)\,dx\,dy.$$

The formula

$$\iint_\Omega F_x\varphi_y\,dx\,dy = -\iint_\Omega F(x,y)\varphi_{xy}(x,y)\,dx\,dy, \tag{50}$$

is similarly proven thus finally leading to (44).

APPENDIX

Recall that this paper is related to a paper by authors Arsenović, M. and Mateljević, M. titled "On Ahlfors-Beurling Operator" published in Journal of Mathematical Sciences, 2021 (referenced below Arsenović, M., Mateljević, M. On Ahlfors–Beurling Operator. J Math Sci 259, 1–9 (2021). https://doi.org/10.1007/s10958-021-05596-9)

In the above mentioned paper authors Miodrag Mateljević and Miloš Arsenović investigate regularity properties of solutions of Beltrami equation expressed in terms of moduli of continuity. Authors Miodrag Mateljević and Miloš Arsenović prove that a class of Calderon-Zygmund operators, including Ahlfors-Beurling operator, preserves certain type of modulus of continuity of compactly supported functions. They have also proved a purely topological result which easily gives injectivity of normal solutions of Beltrami equation.

Theorem 1 and Theorem 2 from the above mentioned paper are given below. In order to make it easier for readers to understand the problems that the authors Miodrag Mateljević and Miloš Arsenović deal with in the mentioned paper, a complete proof of Theorem 1 is given.

For the convenience of the readers, some definitions, terms and considerations will be mentioned (in the same way as presented in the mentioned paper).

Authors choose a majorant ω i.e. a continuous increasing and concave function, $\omega(t)$, $t \geqslant 0$ such that $\omega(0) = 0$ and $\omega(\lambda t) \leqslant C_\lambda \omega(t)$, $\lambda > 1$.

Authors impose the following two conditions on majorant ω:

$$\int_0^1 \frac{\omega(t)}{t}\,dt \leqslant A_1\omega(\delta), \quad 0 < \delta < 1. \tag{1}$$

$$\int_0^1 \frac{\omega(t)}{t}\,dt \leqslant A_2\frac{\omega(\delta)}{\delta}, \quad 0 < \delta < 1. \tag{2}$$

The operator T is defined by the following formula:

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} K_\varepsilon(y) f(x-y)\,dy,$$

where K_ε denotes a truncated kernel

$$K_\varepsilon(x) = \begin{cases} K(x), & |x| \geqslant \varepsilon, \\ 0, & |x| < \varepsilon. \end{cases}$$

In the same paper it is proved:

$$Tf(x) = \int K(y)[f(x-y) - f(x)]\,dy, \quad x \in \mathbb{R}^n \tag{3}$$

Theorem 1. Assume a majorant ω satisfies conditions (1) and (2). Then for every $R > 0$ there is a constant $C = C(R, n, \Omega, \omega)$ such that

$$\|Tf\|_\omega \leqslant C\|f\|_\omega, \quad f \in \wedge^R_\omega. \tag{4}$$

We note the following well known estimate for the kernel K:

$$|K(x+h) - K(x)| \leqslant C(\Omega)\frac{|h|}{|x|^{n+1}}, \quad x \neq 0, h \leqslant |x|/3. \tag{5}$$

Let us choose $f \in \Lambda^R_\omega$ and $x, x+h \in \mathbb{R}^n$, where $|h| \leqslant 1$. From (3) we obtain

$$Tf(x) = \left(\int_{|y|\leqslant 3|h|} + \int_{3|h|\leqslant y\leqslant |x|+R}\right) K(y)[f(x-y) - f(x)]\,dy = I_1 + I_2. \tag{6}$$

Analogously to the argument leading to (11) we estimate I_1:

$$\|I_1\| \leqslant \|\Omega\|_\infty \|f\|_\omega |S^{n-1}| \int_0^{3|h|} \frac{\omega(t)}{t}\,dt \leqslant C(n, \Omega, \omega)\|f\|_\omega \omega(|h|). \tag{7}$$

where we used condition (1). Replacing x with $x+h$ in (6) we obtain

$$Tf(x) = \left(\int_{|y|\leqslant 3|h|} + \int_{3|h|\leqslant y\leqslant |x|+R}\right) K(y)[f(x+h-y) - f(x+h)]\,dy = J_1 + J_2, \tag{8}$$

where $|J_1| \leqslant C(n,\Omega,\omega)\|f\|_\omega \omega(|h|)$ and

$$\begin{aligned} J_2 &= \int_{3|h|\leqslant|y|\leqslant|x+h|+R} K(y)[f(x+h-y)-f(x+h)]\,dy \\ &= \int_{3|h|\leqslant|y|\leqslant|x+h|+R} K(y)[f(x+h-y)-f(x)]\,dy \qquad (9) \\ &= \int_{3|h|\leqslant|z+h|\leqslant|x+h|+R} K(z+h)[f(x-z)-f(x)]\,dz \\ &= \int_{3|h|\leqslant|z|\leqslant|x|+R} K(z+h)[f(x-z)-f(x)]\,dz + E = \tilde{J}_2 + E. \end{aligned}$$

Note that cancellation property enabled replacement of $f(x+h)$ with $f(x)$. Now we estimate the error term E, which results from a change of domain of integration from one spherical ring $A(-h;3|h|;|x+h|+R)$ to another one $A(0;3|h|;|x|+R)$. Regarding the change of inner limits, the size of $K(z+h)$ is estimated by $C(n)\|\Omega\|_\infty |h|^{-n}$, the measure of the symmetric difference of $B(0;3|h|)$ and $B(-h;3|h|)$ is estimated by $C(n)|h|^n$ and the size of $f(x-z)-f(x)$ is estimated by $C(\omega)\|f\|_\omega\omega(|h|)$. Hence the error due to the change of inner limits is estimated by $C(n,\Omega,\omega)\|\Omega\|_\infty\omega(|h|)$.

Regarding the change of outer limits, the measure of the symmetric difference of domains of integration is estimated by $C(n)|h|(|x|+R)^{n-1}$, the size of $K(z+h)$ by $\|\Omega\|_\infty(|x|+R)^{-n}$ and the size of $f(x-z)-f(x)$ by $2\|f\|_\infty$, hence the contribution to the error term is bounded by $C\|\Omega\|_\infty\|f\|_\infty|h|$, where C is a constant depending only on n. Since $\|f\|_\infty \leqslant C(\omega,R)\|f\|_\omega$ and $\delta \leqslant C_\omega(\delta)$ for $0 \leqslant \delta \leqslant 1$ we obtain

$$|E| \leqslant C(R,n,\Omega,\omega)\|f\|_\omega\omega(|h|). \qquad (10)$$

Combining (6), (7), (8), (9) and (10) we obtain

$$\begin{aligned} |Tf(x+h)-Tf(x)| &= |J_1 + \tilde{J}_2 + E - I_1 - I_2| \\ &\leqslant |\tilde{J}_2 - I_2| + |E| + |I_1| + |J_1| = E_1 + E_2 \end{aligned}$$

where

$$E2 = |E| + |I_1| + |J_1| \leqslant C(R,n,\Omega,\omega)\|f\|_\omega\omega(|h|)$$

and

$$E_1 = \left|\int_{3|h|\leqslant|z|\leqslant|x|+R} [K(z+h)-K(z)][f(x-z)-f(x)]\,dz\right|.$$

Since f is supported in $B(0;R)$ we can assume $\omega(t)$s is constant for $t \geqslant 2R$, in particular $\omega(t) \leqslant \omega(2R)$. We use (5) and (2) to estimate E_1:

$$\begin{aligned} E_1 &\leqslant C(\Omega)\int_{3|h|\leqslant|z|\leqslant|x|+R} \frac{|h|}{|z|^{n+1}}|f(x-z)-f(x)|\,dz \\ &\leqslant C(\Omega)|h||S^{n-1}|\|f\|_\omega \int_{3|h|}^{|x|+R} \frac{\omega(t)}{t^2}\,dt \\ &= |h|C(n,\Omega)\|f\|_\omega\left(\int_{3|h|}^1 + \int_1^\infty\right)\frac{\omega(t)}{t^2}\,dt \\ &\leqslant C_n\|f\|_\omega(A_2\omega(|h|) + \omega(2R)|h|) \\ &\quad C(R,n,\Omega,\omega)\|f\|_\omega\omega(|h|). \end{aligned}$$

Note that this was the only estimate in the proof that relied on (2). This gives desired estimate for $|h| \leqslant 1$, the estimate for $|h| > 1$ follows easily from the vanishing of Tf at infinity. In fact, since the support of f is compact we have the following asymptotic: $Tf(x) = O(|x|^{-n}))$ as $x \to +\infty$; we leave details to the reader.

Theorem 2. Let f be a quasiconformal mapping between bounded planar domains D and G with $C^{1,\alpha}$ boundaries, where $0 < \alpha < 1$. Then the following three conditions are equivalent.

(A) $f \in C^{1,\alpha}(\overline{D})$ and $f^{-1} \in C^{1,\alpha}(\overline{G})$.

(B) The complex dilatation μ_f is α Hölder continuous on D.

(C) The complex dilatation $\mu_{f^{-1}}$ is α Hölder continuous on G.

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