An analytical approach for free vibration analysis of Euler-Bernoulli stepped beams with axial-bending coupling effect

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ABSTRACT
Free vibration of eccentrically stepped beams with one step change in cross-section is considered. It is assumed that the longitudinal symmetry axes of the beam segments are translationally shifted along the vertical direction with respect to each other. The effect of that arrangement of the segments on the coupling of axial and bending vibrations of the stepped beam is analyzed. The beam segments are modeled in the frame of the Euler-Bernoulli theory of elastic beams. Two numerical examples are presented.

KEYWORDS
Stepped beam, Step eccentricity, Axial-bending coupling, Euler-Bernoulli theory, Natural frequency, Free vibration

1. INTRODUCTION
Stepped beams often appear as an integral part of various devices and structures in mechanical and civil engineering. That is why the buckling and vibration analysis of stepped beams is extremely important for engineering practice. Despite the large number of papers published in connection with this problem (see e.g. [1-22]), this field of scientific research is still actual with intensive development and generation of new scientific problems. In the available literature, there are mainly references dealing with homogeneous stepped beams. Also, tapered beams can be modeled as stepped beams in a manner described in [23, 24]. In the recent years stepped beams made of functionally graded materials represent an actual research field due to special mechanical characteristics of this kind of materials [25-27]. Decoupled axial and bending vibrations are mainly considered for stepped beams. However, often these two types of vibrations can be coupled. The reason for this may be, for example, the shape of cross-section of beams [28, 29], rigid bodies eccentrically attached to the stepped beams [23, 30-33], angled-beam joints in the frame structures [31, 34] and the compliant mechanisms [35], varying material characteristics of beam segments through their thickness direction [36] as well as mutual eccentric positions of longitudinal symmetry axes of stepped beams segments [22].

The last-mentioned cause of coupling is considered in this paper. In this sense, the objective of our paper is to extend the approach described in [31, 33] to the case of Euler-Bernoulli eccentrically stepped beams with one step change in cross-section. In the frame of the Euler-Bernoulli theory of elastic beams, to the authors’ best knowledge of the literature, the appearance of axial-bending coupling effect in the case of this type of stepped beams was not considered.
2. FORMULATION OF GOVERNING EQUATIONS

An eccentrically stepped beam of rectangular cross-section with one step discontinuity is shown in Figure 1. The width and thicknesses of cross-sections of segments \((S_i)\) and \((S_j)\) are denoted by \(b_i\) \((i=1,2)\) and \(h_i\) \((i=1,2)\), respectively. In the undeformed configuration of the stepped beam, local stationary inertial coordinate frames \(\{x_i, y_i, z_i\}\) \((i=1,2)\) are placed in the manner shown in Figure 1. Also, the longitudinal symmetry axes of segments \((S_i)\) and \((S_j)\) are translationally shifted in the vertical direction by an amount \(e\). The quantities \(u_i(z_i,t)\) \((i=1,2)\) and \(w_i(z_i,t)\) \((i=1,2)\) represent the axial and transverse displacements respectively, of any point of the neutral axes of the beam segments. The material and geometric characteristics of segments \((S_i)\) and \((S_j)\) are: \(E_i\) is the modulus of elasticity, \(I_{xi(i)}\) is the cross-sectional area moment of inertia about axis \(x_i\), \(A_i\) is the cross-sectional area, \(\rho_i\) is the mass density, and \(L_i\) is the length of the \(i\)-th beam segment. The partial differential equations of bending and axial free vibrations of segments \((S_i)\) and \((S_j)\) are as follows [37, 38]:

\[
E_i I_{xi(i)} \frac{d^4w_i(z_i,t)}{dz_i^4} + \rho_i A_i \frac{d^2w_i(z_i,t)}{dt^2} = 0, \quad i = 1,2. \tag{1}
\]

\[
E_i A_i \frac{d^4u_i(z_i,t)}{dz_i^4} - \rho_i A_i \frac{d^2u_i(z_i,t)}{dt^2} = 0, \quad i = 1,2. \tag{2}
\]

Based on the method of separation of variables [37, 38], the displacements \(u_i(z_i,t)\) \((i=1,2)\) and \(w_i(z_i,t)\) \((i=1,2)\) can be written as:

\[
u_i(z_i,t) = W_i(z_i)T(t), \quad u_i(z_i,t) = U_i(z_i)T(t), \tag{3}
\]

where \(U_i(z_i)\) \((i=1,2)\) and \(W_i(z_i)\) \((i=1,2)\) are the mode shapes in free axial and bending vibrations, respectively, and \(T(t) = e^{\omega t}, \quad i = \sqrt{-1}\), and \(\omega\) is the natural angular frequency of vibration of the stepped beam.

![Figure 1: Stepped Euler-Bernoulli beam](image)

Introducing (3) into (1) and (2) yields:

\[
\frac{d^4W_i(z_i)}{dz_i^4} - k_i^4 W_i(z_i) = 0, \quad i = 1,2, \tag{4}
\]

\[
\frac{d^4U_i(z_i)}{dz_i^4} + \rho_i^2 U_i(z_i) = 0, \quad i = 1,2, \tag{5}
\]

\[
\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0, \tag{6}
\]

where:

\[
k_i^4 = \frac{\rho_i A_i}{E_i I_{xi(i)}} \omega^2, \quad \rho_i^2 = \frac{\rho_i}{E_i} \omega^2, \quad i = 1,2. \tag{7}
\]
Based on (7), the following relation can be established:

\[ \rho_1 = \sqrt{\frac{f_{d(i)}}{\lambda}}, \quad i = 1, 2. \]  

(8)

Combining (7) and (8) with the following expressions:

\[ k_1 = k, \quad \rho_1 = \sqrt{\frac{f_{d(i)}}{\lambda}}, \]  

(9)

yields:

\[ k_2 = \frac{E I_{s(i)} A_E}{E J_{s(i)} A_e} k, \quad \rho_2 = \sqrt{\frac{E I_{s(i)} A_E}{E J_{s(i)} A_e}} k, \]  

\[ \rho^2 = \frac{E I_{s(i)} A_E}{E J_{s(i)} A_e} k^2. \]

(10)

(11)

Introducing now the following dimensionless quantities:

\[ \bar{z} = \frac{z}{L}, \quad \bar{W}(\bar{z}) = \frac{W(Lz)}{L}, \quad \bar{U}_i(\bar{z}) = \frac{U(Lz)}{L}, \quad \frac{d^i}{dz^i} = \frac{1}{L} \frac{d^i}{dz^j}, \quad i = 1, 2, \ldots, \]  

(12)

the equations (4) and (5) as well as the relations (9) and (10) can be written in the following dimensionless forms:

\[ \frac{d^i \bar{W}(\bar{z})}{dz^i} - \bar{k}_i \bar{W}(\bar{z}) = 0, \quad i = 1, 2, \]

(13)

\[ \frac{d^i \bar{U}_i(\bar{z})}{dz^i} + \bar{p}_i \bar{U}_i(\bar{z}) = 0, \quad i = 1, 2, \]

(14)

\[ \bar{k}_1 = \bar{k}, \quad \bar{p}_1 = \bar{k}, \]

(15)

\[ \bar{k}_2 = \sqrt{\frac{E_{s(i)} A_E}{E_{s(i)} A_e}} k, \quad \bar{p}_2 = \sqrt{\frac{E_{s(i)} A_E}{E_{s(i)} A_e}} k^2, \]

(16)

where \( \bar{k} = k, L, \) \( \bar{p}_i = p_i L, \) \( \gamma = \rho_1 / \rho_1, \) \( \gamma = E_1 / E_1, \) \( \gamma_2 = E_2 / E_1, \) \( \gamma_1 = I_{s(i)} / I_{s(i)}, \) \( \bar{t} = \sqrt{\frac{E_{s(i)} A_E}{E_{s(i)} A_e}}, \) and:

\[ \bar{k} = k L, \quad \bar{p}_i = \bar{p}_i L, \]

(17)

are the dimensionless frequency coefficient and the dimensionless natural angular frequency, respectively. General solutions of the equations (13) and (14) are given as follows [37, 38]:

\[ \bar{W}_i(\bar{z}) = C_{\gamma(i)} \cos(\bar{k}_i \bar{z}_i) + C_{\delta(i)} \sin(\bar{k}_i \bar{z}_i), \]

\[ \bar{U}_i(\bar{z}) = C_{\gamma(i)} \cos(\bar{p}_i \bar{z}_i) + C_{\delta(i)} \sin(\bar{p}_i \bar{z}_i), \quad i = 1, 2, \]

(18)

(19)

where \( C_{\gamma(i)}, \ldots, C_{\delta(i)} \) are integration constants.

3. BOUNDARY CONDITIONS AND THE FREQUENCY EQUATION

3.1. Boundary conditions at the left end of the stepped beam

For the clamped left end of the considered stepped beam the following boundary conditions hold:

\[ \bar{U}_i(0) = 0, \quad \bar{W}_i(0) = 0, \quad \frac{d^i \bar{W}_i}{dz^i}(0) = 0, \]

(20)

whereas for the pinned left end one has:

\[ \bar{U}_i(0) = 0, \quad \bar{W}_i(0) = 0, \quad \frac{d^i \bar{W}_i}{dz^i}(0) = 0. \]

(21)
Introducing \( \mathbf{C}_1 = \begin{bmatrix} C_{y(1)} & \cdots & C_{y(n)} \end{bmatrix}^T \) as a vector of integration constants corresponding to segment \( (S_1) \) and putting (18) and (19) into (20) and (21) yields the following matrix relation:

\[
\mathbf{C}_1 = \mathbf{T}_0 \mathbf{C}_0
\]

(22)

where for the clamped end one has:

\[
\mathbf{C}_0 = \begin{bmatrix} C_{x(1)} & C_{x(2)} & C_{y(2)} \end{bmatrix}^T, \quad \mathbf{T}_0 = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \end{bmatrix}
\]

(23)

and for the pinned one:

\[
\mathbf{C}_0 = \begin{bmatrix} C_{u(1)} & C_{u(2)} & C_{v(2)} \end{bmatrix}^T, \quad \mathbf{T}_0 = \begin{bmatrix} 0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \end{bmatrix}
\]

(24)

3.2. Boundary conditions at the junction of the segments

In order to establish corresponding continuity conditions at the junction of the stepped beam segments, let us consider an infinitesimal part of the stepped beam at the step location as it is depicted in Figure 2.

Figure 2: Free-body diagram of an infinitesimal part of the stepped beam at the junction of segments

Here, \( F_{y(1)} \) and \( F_{y(2)} \) are the shear forces defined as [31, 33]:

\[
F_{y(1)} = -E_A \frac{dW_1}{dz_1} (L_1 - \delta), \quad F_{y(2)} = -E_A \frac{dW_2}{dz_2} (0 + \delta),
\]

(25)

\( F_{z(1)} \) and \( F_{z(2)} \) are the axial forces given as [31, 33]:

\[
F_{z(1)} = E_A \frac{dU_1}{dz_1} (L_1 - \delta), \quad F_{z(2)} = E_A \frac{dU_2}{dz_2} (0 + \delta),
\]

(26)

and, finally, \( M_{x(1)} \) and \( M_{x(2)} \) are the bending moments defined as [31, 33]:

\[
M_{x(1)} = -E_I \frac{dW_1}{dz_1} (L_1 - \delta), \quad M_{x(2)} = -E_I \frac{dW_2}{dz_2} (0 + \delta).
\]

(27)

Taking the length \( \delta \) approaches to zero yields the following continuity conditions at the level of forces and moments:
\[
F_{a0} = F_{b0} \Leftrightarrow \frac{d^3 W_i}{dz_i^3}(\gamma_{i1}) = \gamma_i \frac{d^3 W_i}{dz_i^3}(0), \\
F_{a1} = F_{b1} \Leftrightarrow \frac{d^2 U}{dz_1^2}(\gamma_{i1}) = \gamma_i \frac{d^2 U}{dz_1^2}(0), \\
M_{a0} = M_{b0} \Leftrightarrow \frac{d^3 W_i}{dz_i^3}(\gamma_{i1}) = \gamma_i \frac{d^3 W_i}{dz_i^3}(0) - \frac{\gamma_i}{r^3} \frac{d^2 U}{dz_i^2}(0),
\]

where \(\gamma_{i1} = L_i / L\) and \(\bar{e} = e / L\). Also, at the level of displacements one has the following continuity conditions:

\[
W_i(L) = W_i(0) \Leftrightarrow \bar{W}_i(\gamma_{i1}) = \bar{W}_i(0), \\
\frac{dW_i}{dz_i}(L) = \frac{dW_i}{dz_i}(0) \Leftrightarrow \bar{W}_i(\gamma_{i1}) = \bar{W}_i(0), \\
U_i(L) - \bar{e} \frac{dW_i}{dz_i}(L) = U_i(0) \Leftrightarrow \bar{U}_i(\gamma_{i1}) - \bar{e} \frac{dW_i}{dz_i}(\gamma_{i1}) = \bar{U}_i(0).
\]

The continuity conditions (28)-(33) generate the following matrix relation:

\[
T \mathbf{C}_1 = T \mathbf{C}_2 
\]

where \(\mathbf{C}_1 = [c_{a1}, \ldots, c_{a2}]^T\) is the vector of integration constants corresponding to segment \((S_i)\) and entries of the matrix \(T \in R^{6 \times 6}\) are:

\[
T_{1L} = \begin{bmatrix}
sin(\bar{k} \gamma_{L}) & -\cos(\bar{k} \gamma_{L}) & \sinh(\bar{k} \gamma_{L}) & \cosh(\bar{k} \gamma_{L}) & 0 & 0 \\
0 & 0 & 0 & -\sin(\bar{r} \bar{k}^2 \gamma_{L}) & \cos(\bar{r} \bar{k}^2 \gamma_{L}) \\
-\cos(\bar{k} \gamma_{L}) & -\sin(\bar{k} \gamma_{L}) & \cosh(\bar{k} \gamma_{L}) & \sinh(\bar{k} \gamma_{L}) & 0 & 0 \\
\cos(\bar{k} \gamma_{L}) & \sin(\bar{k} \gamma_{L}) & \sinh(\bar{k} \gamma_{L}) & \cosh(\bar{k} \gamma_{L}) & 0 & 0 \\
-\bar{k} \sin(\bar{k} \gamma_{L}) & -\bar{k} \cos(\bar{k} \gamma_{L}) & -\bar{k} \sinh(\bar{k} \gamma_{L}) & -\bar{k} \cosh(\bar{k} \gamma_{L}) & 0 & 0
\end{bmatrix},
\]

and of the matrix \(T_{1R} \in R^{6 \times 6}\):

\[
T_{1R} = \begin{bmatrix}
0 & -\gamma EY_1 \sqrt{\frac{\gamma Y_1 A}{L^3}} & 0 & \gamma EY_1 \sqrt{\frac{\gamma Y_1 A}{L^3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\gamma EY_1 \sqrt{\frac{\gamma Y_1 A}{L^3}} & 0 & \gamma EY_1 \sqrt{\frac{\gamma Y_1 A}{L^3}} & 0 & 0 & -\gamma EY \sqrt{\frac{L}{Y}} \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 4 \gamma EY_1 \sqrt{\frac{\gamma Y_1 A}{L^3}} & 0 & 4 \gamma EY_1 \sqrt{\frac{\gamma Y_1 A}{L^3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Solving (34) for \(\mathbf{C}_2\) yields:

\[
\mathbf{C}_2 = T \mathbf{C}_1
\]

where the matrix \(T \in R^{6 \times 6}\) is determined by:

\[
T = T_{1R}^{-1} T_{1L}
\]

and represents the transfer matrix between the integration constants of segments \((S_i)\) and \((S_j)\).

3.3. Boundary conditions at the right end of the stepped beam

In this section the following types of the right end of the stepped beam will be considered: free end, clamped end, and pined end. The boundary conditions for free right end read:
for clamped right end one has:
\[
\bar{U}_1 (\gamma_{_{12}}) = 0, \quad \bar{W}_1 (\gamma_{_{12}}) = 0, \quad \frac{d^2 \bar{W}_1}{dz^2} (\gamma_{_{12}}) = 0,
\]
and, finally, the corresponding boundary conditions for pinned right end are:
\[
\bar{U}_3 (\gamma_{_{12}}) = 0, \quad \bar{W}_3 (\gamma_{_{12}}) = 0, \quad \frac{d^2 \bar{W}_3}{dz^2} (\gamma_{_{12}}) = 0,
\]
where \( \gamma_{_{12}} = L_2 / L = 1 - \gamma_{_{11}} \). Introducing (18) and (19) into (39)-(41) yields the following matrix expression:
\[
\mathbf{T}_2 \mathbf{C}_2 = 0_{p_0}
\]
where \( 0_{p_0} \in \mathbb{R}^{p_0} \) is a zero matrix and the matrix \( \mathbf{T}_2 \in \mathbb{R}^{p_0} \) has the following entries:

- **free right end**
\[
\mathbf{T}_2 = \begin{bmatrix}
0 & -\cos \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & -\sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \cosh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) \\
\sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & -\cos \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \cosh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & -\sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) \\
0 & \cos \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \cosh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) \\
\sinh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & 0 & 0 & 0
\end{bmatrix}
\]

- **clamped right end**
\[
\mathbf{T}_2 = \begin{bmatrix}
0 & \cos \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \cosh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) \\
-\sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \cos \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \cosh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) \\
0 & \cos \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \cosh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) \\
\sinh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & 0 & 0 & 0
\end{bmatrix}
\]

- **pinned right end**
\[
\mathbf{T}_2 = \begin{bmatrix}
0 & \cos \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \cosh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) \\
-\cos \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \cosh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & 0 \\
0 & \cos \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \sin \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & \cosh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) \\
\sinh \left( \frac{\gamma_0 YA}{Y' E' I_1} \gamma_{_{1L2}} \right) & 0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & \cos\left(\sqrt{\gamma_E \gamma} \kappa L_2 \right) & \sin\left(\sqrt{\gamma_E \gamma} \kappa L_2 \right) \\
\sinh\left(\sqrt{\gamma_E \gamma} \kappa Y_{L2} \right) & 0 & 0 \\
\sinh\left(\sqrt{\gamma_E \gamma} \kappa Y_{L2} \right) & 0 & 0 \\
\end{bmatrix}
\]

(45)

3.4. Derivation of the frequency equation

Substituting (22) and (37) into (42) implies a homogeneous system of equations for unknown components of the vector \( \mathbf{C}_0 \). This equations system can be written in the matrix form as follows:

\[
\mathbf{T}\mathbf{C}_0 = 0_{3\times1},
\]

(46)

where \( \mathbf{T} \in \mathbb{R}^{m \times n} \) represents overall transfer matrix given as:

\[
\mathbf{T} = \mathbf{T}_2 \mathbf{T}_1 \mathbf{T}_0.
\]

(47)

Finally, the corresponding frequency equation for the problem analyzed reads:

\[
f(\tilde{k}) = \text{det} \mathbf{T} = 0.
\]

(48)

4. NUMERICAL EXAMPLES

In numerical calculations of this section, the stepped beam geometrical parameters given in [21] will be used as follows: \( h_1 = 19.05 \text{ mm}, \ h_2 = 5.49 \text{ mm}, \ b_1 = b_2 = 25.4 \text{ mm}, \ L_1 = 254 \text{ mm}, \ L_2 = 140 \text{ mm} \). Based on the theoretical considerations given in Sections 2 and 3, the effect of eccentricity on dimensionless natural angular frequencies of the stepped beam for various combinations of materials of the beam segments is shown in Table 1.

Table 1: Values of the lowest four dimensionless natural angular frequencies for various combinations of materials of beam segments

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>( \gamma_E )</th>
<th>( \gamma_\rho )</th>
<th>( \tilde{\varepsilon} )</th>
<th>( \tilde{\omega}_1 )</th>
<th>( \tilde{\omega}_2 )</th>
<th>( \tilde{\omega}_3 )</th>
<th>( \tilde{\omega}_4 )</th>
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<tbody>
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<td>C-F</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4.91792</td>
<td>11.5118</td>
<td>41.2372</td>
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<tr>
<td>(h_1, h_2)/2L</td>
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<td></td>
<td></td>
<td>4.91721</td>
<td>11.5160</td>
<td>41.1452</td>
<td>63.2388</td>
</tr>
<tr>
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<td>7800/2702</td>
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<td>3.56078</td>
<td>12.7772</td>
<td>38.0964</td>
<td>68.1344</td>
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<tr>
<td>(h_1, h_2)/2L</td>
<td></td>
<td></td>
<td></td>
<td>3.55984</td>
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<td>38.0964</td>
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</tr>
<tr>
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<td>7800/5700</td>
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<tr>
<td>(h_1, h_2)/2L</td>
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<td></td>
<td></td>
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The limit value \( \bar{\varepsilon} = \frac{(h_1 - h_2)}{(2L)} \) corresponds to the case of an eccentrically stepped beam with the flat bottom surface. The influence of the eccentricity \( \bar{\varepsilon} \) on the lowest four dimensionless frequency coefficients is examined in Figures 3, 4, 5, and 6. At that, the values of \( \bar{\varepsilon} \) is taken from the interval \( 0.001 \leq \bar{\varepsilon} \leq \frac{(h_1 - h_2)}{(2L)} \).

![Figure 3: The effect of the eccentricity \( \bar{\varepsilon} \) on the lowest four dimensionless frequency coefficients of the clamped-free stepped beam](image1)

![Figure 4: The effect of the eccentricity \( \bar{\varepsilon} \) on the lowest four dimensionless frequency coefficients of the clamped-clamped stepped beam](image2)

![Figure 5: The effect of the eccentricity \( \bar{\varepsilon} \) on the lowest four dimensionless frequency coefficients of the clamped-pinned stepped beam](image3)

![Figure 6: The effect of the eccentricity \( \bar{\varepsilon} \) on the lowest four dimensionless frequency coefficients of the pinned-pinned stepped beam](image4)
5. CONCLUSIONS

An analytical approach based on the transfer matrix method for free vibration analysis of stepped Euler-Bernoulli beams with coupled axial and bending vibrations has been presented. The mutual eccentric position of the beam segments longitudinal axes has been considered as the cause of coupling of axial and bending vibrations. The presented method can be also used in the cases of pure axial and pure bending vibrations of stepped beams. The numerical simulations show that the existence of eccentricity $\varepsilon$ causes small changes in the values of the first four natural frequencies. These changes are more pronounced in the case of the beam segments made of different materials. By changing the value of eccentricity $\varepsilon$, the crossing and veering phenomena have not been detected.

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