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# Critical Thinking and Equivalence of Propositional Functions as Tools for Solving Irrational Inequalities

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## Abstract

Critical thinking, a term referring to a broad range of cognitive skills and intellectual tendencies necessary for effectively identifying, analyzing, and evaluating arguments, is an essential tool for solving any problem. It is especially crucial in solving mathematical problems. Specifically, in this paper, using critical thinking as well as fundamental knowledge of mathematical logic, we highlight certain shortcomings, in terms of precision, in some approaches to solving irrational inequalities.

**Keywords:** Critical thinking, Propositional function, Domain of definition, Irrational inequalities.

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## 1. Introduction

The ability to think critically and solve various problems has become a central focus for researchers, educators, entrepreneurs, and even the media. The ability to analyze information in an objective and thoughtful manner is particularly valued. These abilities are inherent characteristics of critical thinking. Developing critical thinking allows for a better assessment of accuracy and evaluation of the relevance of information. Critical thinking involves questioning assumptions and evaluating arguments from all perspectives, asking questions, and seeking evidence before making a final conclusion. The skill of critical thinking is crucial for identifying logical fallacies and misconceptions in arguments, and in a world full of information, critical thinking is the tool that separates facts from misinformation. Critical thinkers do not take information at face value; they carefully analyze it, thereby gaining the ability to recognize biases and manipulations, as described by Bassham et al. (2011).

The tendencies of modern education are such that critical thinking as a skill should be developed as much as possible in every student. It is naturally assumed that teachers should adopt an approach that supports and nurtures critical thinking in their work and professional development. For mathematics teachers, if they emphasize reasoning, logic, and validity, they provide their students with an approach to mathematics as an effective way to practice critical thinking. It can rightly be said that all students have the abilities and predispositions to enhance and expand their critical thinking skills when learning or discovering mathematical content. Each student can develop critical thinking when faced with a mathematical problem, identifying all possible solutions and evaluating the results. This enables every student to become a confident critical thinker.

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There is a wealth of research on the influence of critical thinking in mathematics education. Susandi (2021), in his paper, describes the critical thinking skills of eighth-grade students when solving systems of linear equations with two variables and identifies components of critical thinking in the observed students. This can be seen as a study with a qualitative descriptive approach. Cahyono et al. (2019), in their work, describe the understanding of the critical thinking process from the perspective of cognitive style in solving algebraic problems of mathematics students. Su et al. (2016) point out the necessity of applying critical thinking and give examples of how critical thinking, creativity, and flexibility help students better understand mathematical concepts. Elliott et al. (2001) briefly describe a newly designed interdisciplinary course titled "Algebra for the Sciences" taught at Southeastern Oklahoma State University and then studied the effects the course had on students' critical thinking skills, problem-solving abilities, and attitudes towards mathematics. Fadhilah et al. (2021), in their paper, describe the critical thinking processes of upper-grade students when solving problems involving systems of linear equations. Prayitno (2018) describes the critical thinking processes of students in solving mathematical tasks based on Polya's framework. Chandra (2021), in his work, describes some of the mistakes made when solving irrational inequalities in the 10th-grade mathematics textbook, while similar issues are addressed by Bagni (1996) in his work discussing the solving of irrational inequalities. In addition to these studies on common errors in solving irrational inequalities, there are also many qualitative studies aimed at analyzing students' responses when solving inequalities. Pratiwi and Rosjanuardi (2020) analyzed mistakes in solving inequalities and examined whether these mistakes depend on whether or not students use a number line during the solving process. The mentioned research, particularly those describing the application of critical thinking in solving algebraic problems, motivated us to consider the approaches to solving irrational inequalities in high school.

To better present the ideas and results of our research, we organized the paper into five sections, where Section 1 outlines introductory considerations. In Section 2, concepts such as the domain of definition of propositions and formulas necessary for further work are discussed, along with the concept of a propositional function. Section 3 explores the concept of equivalent propositional functions through examples of equation solving. Formulas for solving irrational inequalities presented in many textbooks are discussed in Section 4, where some limitations of these formulas are highlighted. The summary of results and concluding remarks are provided in Section 5.

## 2. Propositional Functions

From a grammatical standpoint in any language, the sentence "The number 2 is healthier to drink than bread" does not contain any grammatical errors. However, if we consider the sentence semantically, i.e., if we analyze its meaning, we cannot state that it is true, nor can we state that it is false. In one word, it is nonsensical, like many similar sentences. In contrast to such nonsensical sentences, propositions that assert something and such assertions that we accept as either true or false are called propositions, a concept well-known in the field of mathematical logic. Often, instead of using the term "true proposition", we use "correct proposition", and instead of "false proposition", we use "incorrect proposition". Another well-known concept in mathematical logic is that of a term and a formula. The formal definitions of these concepts, along with many of their properties, can be found in the works of Manin (2010) and Марјановић (1996).

**Example 2.1.** The expression $\frac{1}{(x+1)(x-1)}$ represents a term involving one variable. Besides this term, the following are also terms obtained by taking $-1$, $0$, and $1$ for the variable $x$,

$$\frac{1}{(-1+1)(-1-1)},\ \frac{1}{(0+1)(0-1)},\ \frac{1}{(1+1)(1-1)}.$$

Notice that only the second term represents a number, namely $-1$, while the other two do not. Thus, we can speak of the term $\frac{1}{(-1+1)(-1-1)}$, but not of a number that represents it.

The discussion in the previous example leads us to consider the domain of the definition of a term and how to determine it. If the set of real numbers is from which the variable takes its values, it is easy to determine the set representing the domain of the definition of the given term in the previous example. This problem can be generalized by introducing the concept of the domain of definition of any term with a finite number of variables.

**Definition 2.1.** Let term $A$ contain variables $x_1, \dots, x_n$ whose universal sets are $U_1, \dots, U_n$. The set $D$, consisting of all values $(x_1, \dots, x_n) \in U_1 \times \dots \times U_n$, for which $A$ represents some object, is called the domain of definition of the term $A$.

**Example 2.2.** According to Definition 2.2, for the term $\sqrt{x-1} + \ln y$, the domain of definition is the set of ordered pairs

$$D = [1, +\infty) \times (0, +\infty) \subset R \times R.$$

Considering the domain of definitions of terms, a natural question arises: Should we also take into account the domains of definitions of formulas, which by their structure are formed from terms? To address this, let's consider the following example:

**Example 2.3.** Consider the formula $\sqrt{x} > 1 - x$. If we substitute $-1$, 0, and 1 for the variable $x$, respectively, we get the following formulas:

$$\sqrt{-1} > 1 - (-1),\ \sqrt{0} > 1 - 0,\ \sqrt{1} > 1 - 1.$$

Among these, the first one is not a proposition (i.e., the formula makes no sense since it does not define a relational comparison between objects), the second is a false proposition, and the third is a true proposition. Now, we can easily draw a conclusion: For $x \geq 0$, the formula becomes a proposition, i.e., its domain of definition is the set $D = [0, +\infty)$ and whenever we take some $x_0 \in [0, +\infty)$, we get that $\sqrt{x_0} > 1 - x_0$ is a proposition.

We can generalize the discussion from the previous example to arbitrary formulas.

**Definition 2.2.** Let $A\rho B$ be a formula with variables $x_1, \dots, x_n$ whose universal sets are $U_1, \dots, U_n$. The set $D$ consisting of all values $(x_1, \dots, x_n) \in U_1 \times \dots \times U_n$, for which $A\rho B$ is a proposition, is called the domain of definition of the formula $A\rho B$.

**Example 2.4.** Consider the formula $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$. Clearly, the domain of definition of the formula is the set $D = [0, +\infty) \times [0, +\infty)$, obtained as the intersection of the sets $\{(x, y) \mid x + y \geq 0\}$ and $\{(x, y) \mid x \geq 0,\ y \geq 0\}$ which represent the domains of definition of the terms on the left and right sides of the inequality.

Let us return to the formula from Example 2.4. As we have seen, the formula $\sqrt{x} > 1 - x$ has a domain of definition $D = [0, +\infty)$. Without considering the truth of the proposition, for every $x_0 \in D$, we have that $\sqrt{x_0} > 1 - x_0$ is a proposition. Let $S$ be the set of all such propositions, i.e., the set

$$S = \left\{\sqrt{x_0} > 1 - x_0 \mid x_0 \in [0, +\infty)\right\}.$$

Notice that the function $x_0 \mapsto \sqrt{x_0} > 1 - x_0$ assigns each number $x_0$ from $D$, to a proposition in the set of propositions $S$, and we observe that within the set $D$ there is a subset of values for which the proposition "$\sqrt{x_0} > 1 - x_0$" is true. In Example 2.4, this subset is $\left(\frac{3-\sqrt{5}}{2}, +\infty\right) \subset D$ and we call it the truth set of the given propositional formula or propositional function. In general, we have the following definition:

**Definition 2.3.** Let $D$ be a given non-empty set, and $S$ be a set of propositions. Then the function $\varphi: D \to S$, is called a propositional function, and the set of all values $x \in D$ for which $\varphi(x)$ is a true proposition is called the truth set of the propositional function $\varphi$.

In the previous discussion, let us recall the way of solving problems where propositional functions are given and where it is necessary to determine their truth sets. It is not difficult to conclude that these are all problems where we need to determine the solution sets of the given equation or inequality. In all such problems, when we mention formulas representing equations or inequalities and say that propositional functions are thus given, we are actually referring to functions defined by those formulas.

## 3. Equivalence of Propositional Functions

A significant portion of the mathematics curriculum in elementary and secondary education consists of elementary functions, as well as solving equations and inequalities. The method of solving equations and inequalities that we are accustomed to involves starting from the given equation or inequality and applying established rules to transition to simpler equations or inequalities. However, it is strictly required that each resulting equation or inequality be equivalent to all the previous ones in the sequence, meaning they all must have the same solution set. In the context of the terminology introduced in Section 2, solving an equation or inequality refers to transitioning from one formula to progressively simpler formulas while ensuring that they always define propositional functions whose truth sets are identical. The way we solve inequalities is illustrated in the following example.

**Example 3.1.** Let $3x - 2 < 7$ be the given inequality, and we need to determine its solution set. In the process of solving the inequality

$$3x - 2 < 7,$$

by adding the same number to both sides of the inequality, we obtain

$$3x < 9,$$

and by multiplying both sides by the same value, we find that

$$x < 3,$$

which can be written as

$$x \in (-\infty, 3).$$

In the previous example, we observe that all the formulas have the same truth set, even though all the formulas obtained in the sequence of solving represent different propositional functions. We will say that such propositional functions are equivalent. We can generalize this conclusion and define the concept of equivalence of propositional functions.

**Definition 3.1.** For propositional functions $p(x)$ and $q(x)$ that have the same domain and the same truth set, we say that they are equivalent propositional functions and write $p(x) \Leftrightarrow q(x)$.

When solving equations and inequalities, the equivalence of propositional functions is often implied in the sequence representing the solution process, and the equivalence sign is omitted, with the equivalent formulas written one below the other. It is assumed that the propositional functions have the same domain. On the other hand, let us consider the following example.

**Example 3.2.** We solve the equation $\frac{x^2-4}{x^2-1} = 0$ as follows:

$$\frac{x^2 - 4}{x^2 - 1} = 0$$

$$x^2 - 4 = 0$$
$$x \in \{-2,2\}.$$

Although the propositional functions $\frac{x^2-4}{x^2-1} = 0$ and $x^2 - 4 = 0$ are not equivalent, since they do not have the same domain, the domain of the first is $(-\infty, -2) \cup (-2,2) \cup (2, +\infty)$, while the domain of the second equation is $(-\infty, +\infty)$. However, both the equation $\frac{x^2-4}{x^2-1} = 0$ and the equation $x^2 - 4 = 0$ have the same truth set $R = \{-2,2\}$. Although they are not equivalent, we can consider them equivalent over the common domain, which is the set $(-\infty, -2) \cup (-2,2) \cup (2, +\infty)$. In this context, when solving equations and inequalities, we assume the equivalence of formulas over the common domain, even though this is often not explicitly mentioned during the solving process.

## 4. Solving Irrational Inequalities

In this section, we discuss a class of inequalities and the method of solving them, as presented to students in many textbooks, which does not align with the standards of critical thinking. Specifically, this concerns solving irrational inequalities of the form $\sqrt{P(x)} < Q(x)$ and $\sqrt{P(x)} > Q(x)$ and the so-called formulas for representing the solutions of these inequalities. Using logical connectives but without determining the domain of the given formulas, many textbook authors present the solving of irrational inequalities in the form of the following formulas:

$$\sqrt{P(x)} < Q(x) \Leftrightarrow P(x) \geq 0 \wedge Q(x) > 0 \wedge P(x) < Q^2(x)$$

and

$$\sqrt{P(x)} > Q(x) \Leftrightarrow [P(x) \geq 0 \wedge Q(x) < 0] \vee [P(x) > Q^2(x) \wedge Q(x) \geq 0].$$

However, let us consider the following example.

**Example 4.1.** Using the given formulas when solving the inequality $\sqrt{3x - x^2} < 4 - x$ we obtain the following equivalence:

$$\sqrt{3x - x^2} < 4 - x \Leftrightarrow 3x - x^2 \geq 0 \wedge 4 - x > 0 \wedge 3x - x^2 < (4 - x)^2,$$

even though the formulas are not equivalent because they do not have the same domain. On the other hand, if they are considered equivalent over the common domain, the condition $3x - x^2 \geq 0$ on the right-hand side of the equivalence becomes redundant.

Based on the previous example, and similarly in cases where we solve inequalities of the form $\sqrt{P(x)} > Q(x)$ we conclude that it is necessary first to determine the domain of the given formulas and then, within that set, consider the corresponding equivalent formulas. Below, we provide examples of how to solve the inequalities discussed in this section.

**Example 4.1.** Let us determine the truth set of the formula from Example 4.1, i.e., the formula $\sqrt{3x - x^2} < 4 - x$. To determine the domain of the given formula, it is necessary to find all the values of the variable for which $3x - x^2 \geq 0$. Thus, we easily find that the domain of the given formula is the set $D = [0, 3]$. The resulting set represents two sets: the set of values for which the given formula is true and the set of values for which the formula is false. To determine the subset of values for which the formula is true, it is necessary that $4 - x > 0$ and $3x - x^2 < (4 - x)^2$. We easily conclude that the obtained formulas are valid across the entire domain. Therefore, the truth set, i.e., the solution set of the observed inequality, is $D = [0, 3]$. An important observation is that there are no values for which the formula $\sqrt{3x - x^2} < 4 - x$ is false.

**Example 4.2.** When solving the inequality

$$\sqrt{x^2 - x - 12} > 7 + x,$$

we first need to determine the domain of the formula that represents this inequality. Indeed, we easily arrive at the set $D = (-\infty, -3] \cup [4, +\infty)$. On the obtained set $D$ we determine those values for which the proposition is true as opposed to those values for which the proposition is false. To do this, let us consider two cases. First, if $7 + x < 0$, we conclude that all values in the interval $(-\infty, -7)$ represent part of the truth set because they fall within the domain. In the second case, $7 + x \geq 0$, we need to isolate those values for which the inequality $x^2 - x - 12 > (7 + x)^2$, holds. Therefore, in the second case, we find that part of the truth set is the interval $[-7, -\frac{61}{15})$. Finally, the truth set of the observed formula is the interval $\left(-\infty, -\frac{61}{15}\right)$, which also represents the solution set of the observed inequality. It is worth noting in this example that, in addition to the truth set representing the solution set of the given inequality, the values within the domain that are not part of the truth set are those for which the formula is false, while for values outside the domain, the inequality has no meaning.

## 5. Summary of Results and Conclusion

This paper has examined concepts of mathematical logic, such as propositions, formulas, propositional functions, and the equivalence of propositional functions. A significant segment of our work focused on the domains of propositions and formulas, as well as the examination of the truth sets of formulas. By applying critical thinking and its standards, the paper highlights inaccuracies that arise when solving irrational inequalities using formulas presented in certain literature. Additionally, the study discusses sets of values for which the formula is true, sets of values for which it is false, and sets of values for which the formula is meaningless.

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