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A visualization of the limit: Experimental Mathematics

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Abstract

The limit value of a function is a fundamental mathematical notion that describes how a function behaves close to a given point $\lim_{x \to x_0} f(x)$. We first met the limits of a function in high school mathematics classes, and we later applied it in various areas, most notably physics. Understanding the concept of a function's limit value is far from straightforward. There are no universal rules, although visualization can be an effective technique for overcoming difficulties in this procedure. This work presents an idea for visualization through a physical experiment. An example of measuring the acceleration of the system using Atwood's machine was presented. The system consists of a pulley of radius R and mass M, and two objects with masses m_1 and m_2 that are connected by an inextensible massless string. The acceleration of the system is measured for the variable mass m_2 and behavior of the function $a(m_2)$ at infinity $(m_2 \to +\infty)$ is examined. The graphical presentation of the results illustrates that the limit value of the function gives the horizontal asymptote $a = g$. The proposed concept would enable the introduction of more effective visual learning strategies not only in the teaching of mathematics but also in other natural sciences.

Keywords: function, limit value, acceleration, experiment, Atwood's machine.

MSC2020: 97M50

1. Introduction

The subjects of mathematics and physics are particularly suitable for interdisciplinary cooperation because mathematics and physics as subject sciences exhibit numerous synergies. Recently, physics teaching has been subject to ongoing discussions concerning aims and learning goals, relation to modern society, and the role of mathematics (Hansson et al.,2023; Kraus & Krause, 2020; Dominguez et al., 2024). A recent example is the study by Hansson et al., 2023 reports on physics teaching in upper-secondary schools with a focus on communication and relations made between mathematics, theoretical models in physics, and reality. Mathematics and physics have large overlapping content and are historically interwoven; the combined teaching of both subjects poses numerous challenges. For example, if the mathematics teacher chooses physical context to motivate and apply mathematical terms and theorems, he must be familiar with this physical content. The studies (Kraus & Krause, 2020; Dominguez et al., 2024) focus on an integrated physics and mathematics course in STEM education based on modeling.

George Atwood (George Atwood 1746-1807), a well-known English physicist and mathematician, invented and detailed a mechanism in 1784 to test the fundamental rules of

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mechanical motion (Atwood, 1784; Greenslade, 1985). Even today, this device allows for simple experimental verification and confirmation of fundamental mechanical rules. In the extensive literature on this subject, we highlight a few papers that contain various forms of Atwood's machine (Connolly, 1987; Matehkolaee, 2011; Lopresto, 1999; Beeken, 2011; Wang, 1973; Kovačević, 2012; De Sousa, 2012; West & Weliver, 1999; Martell & Martell, 2013). The purpose of this work is to determine the acceleration using the Atwood machine, with a particular emphasis on visualizing the limit of the function, i.e., $\lim_{x \to x_0} f(x)$. For all of this, we employed an Arduino board equipped with pairs of sensors in the form of emitting and receiving diodes for precise time measurement, and we digitalized the traditional Atwood machine (Kovačević et al., 2024). The time interval in which the inputs or pins on the Arduino platform may monitor voltage is a hundred microseconds. The wheel's rim has a groove (Fig. 1) that allows a thin thread to be passed over it, from which weights $m_1$and m_2 ($m_2 > m_1$) hang. As a result, the weight m_2 will be lowered while the weight m_1 will be raised. The entire system's motion will be uniformly accelerated with zero initial velocity. The microcontroller records the exact moment that mass m_2 intersect the infrared diode's beam because it detects a change in voltage, which results from a decrease in the photodiode's resistance. To accomplish the movement without initial velocity, the weight needs to start the movement from the edge of the beam, which is impractical to establish, and the weight is starting to fall at some distance from the beam. Therefore, the weight has an initial velocity when it intersects the first beam when the timer is set. Initial velocity is considered while analyzing the measurement data.

The mass of the weights, the radius and the mass of the pulley, respectively, and the friction force in the wheel axle will all affect acceleration. First, we developed a formula for the system's acceleration, neglecting friction, the wheel's mass, and the thread's mass. If the masses m_1 and m_2 move according to Newton's Second Law (Fig. 1b):

$$a = \frac{m_2 - m_1}{m_1 + m_2} g. \tag{1}$$

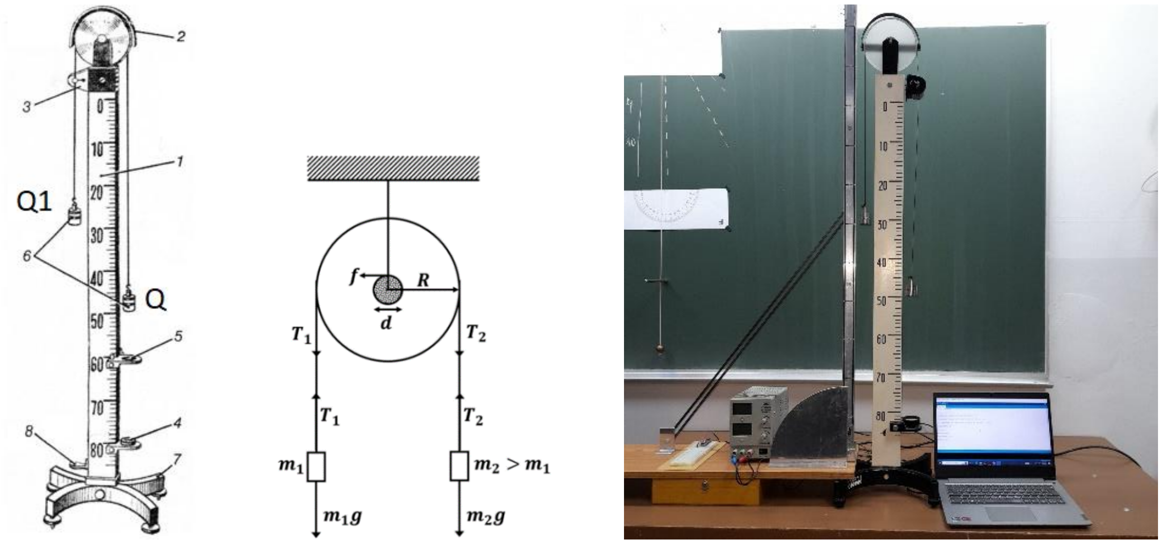
Considering the wheel's moment of inertia $I = MR^2/2$ with the relation between angular and linear acceleration $a = \alpha R$, where M represents the wheel's mass, expression (1) becomes

$$a = \frac{m_2 - m_1}{m_1 + m_2 + \frac{M}{2}} g. \tag{2}$$

In a practical experiment using Atwood's machine, friction force in the wheel axle is unavoidably present (Fig. 1b). When this force acting on the wheel axle is considered, the formula for the system's acceleration becomes slightly more complicated

$$a = \frac{m_2 - m_1}{\frac{M}{2} + m_1 + m_2} g - \left(\frac{r}{R}\right) \frac{f}{\frac{M}{2} + m_1 + m_2} \tag{3}$$

where r and f are the wheel axle radius and friction force, respectively. To simplify the approach, subsequent analysis is based on equation (1) and the examination of the accelerated movement of the system for different masses of weights m_2 (the mass of weight m_1 remains constant in all measurements) when the mass of the pulley and the friction force in the wheel axle are neglected.



(a) (b) (c)

Fig. 1. (a) The classic Atwood machine consists of the following parts: 1 - a wooden or metal pole, 2 - a wheel, 3 - an electromagnet for starting, 4 - a reception table, 5 - a moving permeable ring, 6 - moving weights, 7 - a base, and 8 - a stop. Schematic of the forces in Atwood's machine (b); the digital Atwood machine in (c).

2. The measurements made before doing a function analysis

Rational functions of type (1) allow for more extensive examination using mathematical apparatus, which includes the concept of a limit value in both mathematics and physics curricula. This study, known as boundary problems, allows for the generalization of conclusions and the possibility of drawing meaningful conclusions in circumstances when direct laboratory measurements are not available. All of this needs the learner to understand the mathematical fundamentals of the limit value of a rational function, i.e., an analysis of the function's behavior near a certain point or at the endpoints of a defined interval. The student's activity implies measuring and studying the movement of the two-mass system, as shown in Figure 1. The limiting situation is considered when the mass m_2 increases in discrete quantities and approaches an endlessly large number. Since turning a physical formula into an algebraic function is uncommon in physics lectures, the teacher must provide additional explanations for this activity.

Consider a system of two masses where the mass $m_1 = 155$ g remains constant during the experiment. The mass m_2 is increased by adding weights Δm whose masses are 4 g, 6 g, and 8 g... and so on, increasing this mass to a very big value compared to mass m_1 - in our experiment up to 1 kg (Fig. 2 and Fig. 3). The wheel has a mass of $M = 108.8$ g, a mean radius of $r = 1.5$ cm for the wheel shaft, and a radius of $R = 7$ cm for the wheel itself. These data are of interest when analyzing the motion, taking into consideration the moment of inertia of the pulley and the friction force in the wheel axle (expressions 2 and 3). Neglecting the friction in the wheel axle, students are presented with two tasks:

(a) Students should compute the system's acceleration for various mass m_2 values using measured data. By extrapolating the measured data, one may anticipate how big the system's acceleration will be when the weight mass m_2 tends to an endlessly large number,

(b) using Newton's Second law, students should derive the formula for the compression of the system, convert this formula into a function, and analyze the behavior of the function as $m_2$tends to an infinitely large value. Based on this analysis, it is necessary to confirm or refute the prediction that they reached by solving the task under (a).

The measurements imply the precise measurement of the time for which the weight of mass m_2 crosses certain equidistant paths. Based on the measured values, students draw the dependence of the traveled distance on time during accelerated movement and obtain a characteristic quadratic dependence. A series of graphs is obtained, only some of which are shown in Fig. 2. The acceleration values of the system are read from the graphs, and the dependence of the acceleration on the mass m_2 is drawn. These results are shown in Fig. 3. The obtained function is extrapolated for values of m_2 that significantly exceed the maximum value of mass m_2 that we used in our experiment.

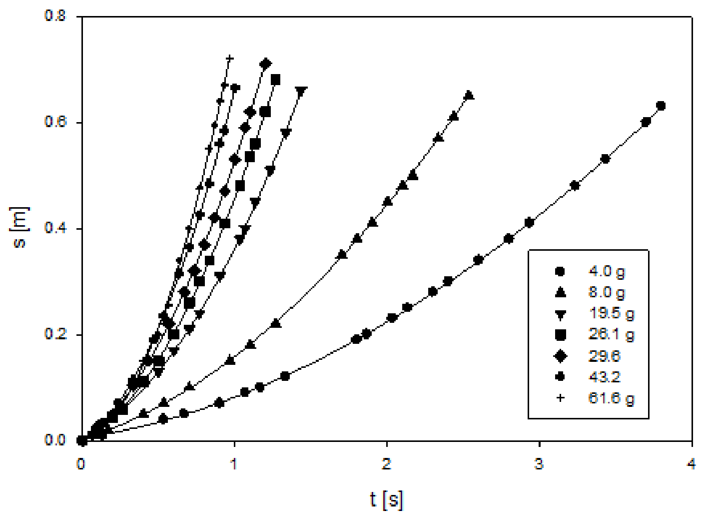


Fig. 2. The distance traveled by mass m_2 as a function of time t.

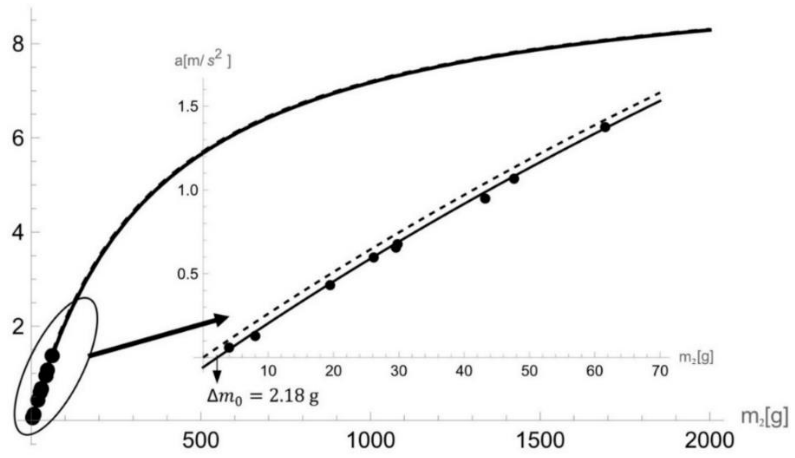


Fig. 3. System acceleration as a function of mass m_2.

From Fig. 3, students conclude the behavior of acceleration when the mass m_2 increases. The dash line in Fig.3 represents the curve defined by equation 1. As there is friction in real systems, it is necessary to use equation 3. However, the force of friction is an unknown factor. The curve in Fig. 3, represented by the solid line, is the function given by Equation 3 in which the friction force as a parameter is varied so that Equation 3 best fits the experimental data. Circles show the experimental values on the graph. It can be observed that when the value of the mass m_2 tends to an infinitely large value, the acceleration of the system reaches a certain saturation, i.e., tends to the maximum value, which in the considered problem represents the limiting value of the acceleration.

3. System acceleration limit value

Applying Newton's Second Law, students derive the formula for acceleration (expression 1). First, it is necessary to specify the independent and dependent variables in this expression. In the case under analysis, the mass m_2 constitutes the independent variable, while the function is the acceleration of the system $a(m_2)$. To emphasize the functional nature of acceleration and its dependence on the mass m_2, we write the expression (1) in the form $a(m_2)$. The value of mass m_1 is constant. Equation (1) takes the form of a rational function

$$a(m_2) = \frac{m_2 - m_1}{m_1 + m_2} g \tag{4}$$

We can simply show the function (4) graphically and, from the graph, estimate the limiting value of the acceleration when the mass m_2 tends to an infinitely large value (Fig. 4). It is obvious from the graph that when the mass $m_2 \to \infty$, the acceleration approaches the value of 9.8 m/s^2. This confirms that the equation of the horizontal asymptote of the function (4) is exactly $a = g$. Using a simple algebraic procedure (Lovatt, 2019; Brown, 2013), we find

$$a = \lim_{m_2 \to \infty} \frac{m_2 - m_1}{m_1 + m_2} g \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{m_2 \to \infty} \frac{m_2\left(1 - \frac{m_1}{m_2}\right)}{m_2\left(\frac{m_1}{m_2} + 1\right)} g = \lim_{m_2 \to \infty} \frac{1 - \frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1} g = g \tag{5}$$

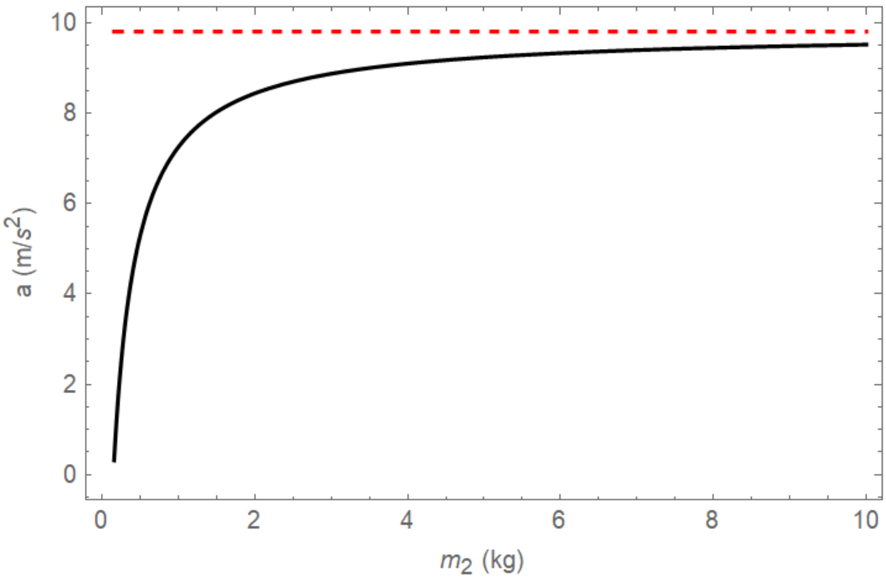


Fig. 4. Graphic representation of function (4).

The same result is obtained by applying the first derivative of the numerator and denominator function (4):

$$a = \lim_{m_2 \to \infty} \frac{m_2 - m_1}{m_1 + m_2} g \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{m_2 \to \infty} \frac{(m_2 - m_1)'}{(m_1 + m_2)'} g = \frac{1}{1} g = g \tag{6}$$

Students conclude that the horizontal asymptote represents the limit, the maximum acceleration in the considered example. Also, the acceleration will never reach the value of 9.81 m/s^2 because the weight m_1 is non-zero, and the mass m_2 is finite. It should be emphasized that the graphical presentation of function 4 (Fig. 4) is an effective tool for estimating acceleration for certain values of mass m_2 that are outside the range of masses used in the measurement. So, for example, when the mass is 6 kg, the acceleration is about 9.33 m/s^2. We leave it to the students to think about what the acceleration of the system will be if the mass of the weight is $m_2 \to m_1$.

4. Conclusion

The paper first describes the classic Atwood machine, a device that is mentioned in almost all physics textbooks. The main reason for dealing with this topic was the need to visualize the concept of the limit value of a function through an experiment for which we used a modified classic Atwood machine. The classic Atwood machine can be digitized using an Arduino card, where precise timing sensors can easily track and measure time and show the validity of the law of accelerated motion. Based on the minimum mass of the weight that causes movement, we can conclude that for a more precise description of the operation of Atwood's machine, a weight of significantly greater mass must be used so that the force of friction during movement has a negligible effect. Students first conclude about the maximum value of the acceleration through the experiment, which they later confirm by applying the limit value of the function. This is just one way to introduce students to the concept of the horizontal asymptote of a function. The proposed experimental setup provides the possibility of converting a physical formula into a function and its further mathematical analysis. Furthermore, by applying differential calculus and taking the first derivative of acceleration, students learned how to avoid indeterminacy of the ∞/∞ type. The introduction of the function provides multiple possibilities in the analysis of the physical system, especially in borderline cases, which are the focus of this paper. Students find the application of limits in physics intriguing and thought-provoking, yet the transition from formulas to algebraic functions generally needs more instructional attention, not only in physics but also in math classes. Perhaps the idea of widely propagated STEM education will encourage more action for such changes (Ковачевић & Јовановић, 2023; Ковачевић, 2024; Kovacevic et al.,2021). In all these concrete problems, no matter how simple they may seem, one feels the uniqueness of physical laws, sees the importance of experiments in physics class, the necessity of mathematical and numerical analysis, and the requirement of a logical approach to approximations.

References

Atwood, G. (1784). *A treatise on the rectilinear motion and rotation of bodies: with a description of original experiments relative to the subject*. J. & J. Merrill.

A. de Sousa C. (2012). Atwood's machine as a tool to introduce variable mass systems. *Physics Education 47(2)*, 169. https://doi.org/10.1088/0031-9120/47/2/169.

Beeken, P. (2011). Atwood's heavy chain. *The Physics Teacher*, *49*(8), 470-472. https://doi.org/10.1119/1.3651724.

Brown P. (2013). *Limits and continuity – A guide for teachers*. Education Services Australia AMSI.

Connolly, W. (1987). Atwood's machine using moment of inertia pulley. *The Physics Teacher*, *25*(8), 514-515. https://doi.org/10.1119/1.2342354.

Dominguez, A., De la Garza, J., Quezada-Espinoza, M., & Zavala, G. (2023). Integration of Physics and Mathematics in STEM Education: Use of Modeling. *Education Sciences*, *14*(1), 20. https://doi.org/10.3390/educsci14010020.

Greenslade Jr, T. B. (1985). Atwood's machine. *Phys. Teach*, *23*, 24. https://doi.org/10.1119/1.2341703.

Hansson, Ö., Juter, K., & Redfors, A. (2023). On Mathematics and Physics Teaching in Upper-Secondary School. *Education Sciences*, *13*(6), 564. https://doi.org/10.3390/educsci13060564.

Kovačević S. M. (2012). Oscilatorna Atvudova mašina, *Zbornik radova sa Republičkog seminara o nastavi fizike*, 240-245.

Ковачевић С.М. (2024). Експерименти са Атвудовом машином, *Настава физике*, *13*, 1-12.

KOVAČEVIĆ, M., JOVANOVIC, M., & MILOSEVIC, M. (2021). On the calculus of Dirac delta function with some applications in classical electrodynamics. *Revista Mexicana de Fısica E 18*, 020205 1-7 https://doi.org/10.31349/RevMexFisE.18.020205.

КОВАЧЕВИЋ С. М., ЈОВАНОВИЋ М. (2023). Визуализација у процесу решавања рчунских задатака, *Зборник радова са 10. међународне конференције о настави физике у средњим чколама*, Алексинац, 78-84.

KOVAČEVIĆ S.M., MARKOVIĆ V., MILOŠEVIĆ M. M. (2024). Digitalizovana Atvudova mašina. *Zbornik radova sa 11. Međunarodne konferencije o nastavi fizike u srednjoj školi*, 65-68.

KRAUS, S. F., & KRAUSE, E. (2020). *Comparison of mathematics and physics education I*. Springer Spektrum.

LOPRESTO, M. C. (1999). Another look at Atwood's machine. *The Physics Teacher*, *37*(2), 82-83. https://doi.org/10.1119/1.880182.

LOVATT I. (2019). The physics of $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, *The Physics Teacher, 57*, 558-559. https://doi.org/10.1119/1.5131126.

MATEHKOLAEE, M. J. (2011). Introduction of Atwood's machines as Series and Parallel networks. *Latin-American Journal of Physics Education*, *5*(2), 2.

MARTELL, E. C., & MARTELL, V. B. (2013). The effect of friction in pulleys on the tension in cables and strings. *The Physics Teacher*, *51*(2), 98-100. https://doi.org/10.1119/1.4775531.

WANG, C. T. (1973). The improved determination of acceleration in Atwood's machine. *American Journal of Physics*, *41*(7), 917-919. https://doi.org/10.1119/1.1987416.

WEST, J. O., & WELIVER, B. N. (1999). The Atwood machine: Two special cases. *The Physics Teacher*, *37*(2), 83-85.