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The Royal Game of Ur and mathematics

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Abstract

Most pedagogues believe that motivation is a key prerequisite for successful learning. Nowadays, especially at an older age, it is very difficult to get students interested in learning mathematical content. This paper will describe the Royal Game of Ur, the oldest board game originating from ancient Mesopotamia. We will also present several probability theory problems inspired by that game. Finally, an electronic version of the Royal Game of Ur, modeled by the author in the Geogebra software package, will be presented.

Keywords: Royal Game of Ur, Probability, Geogebra, Motivation

MSC2020: 97K50

1. Introduction

In our high school mathematics curriculum, probability theory and statistics are studied at the end of the fourth year. The students then almost secured their final grades, and preparations are in full swing for taking entrance exams at the faculties, as well as preparing graduation papers. That is why they are not sufficiently concentrated and motivated for a systematic approach to learning the mentioned field. However, this theory should not remain unnoticed and understudied. As most pedagogues believe that motivation is the key to learning and learning results in the given circumstances, the lecturer should find such an approach to teaching that would initiate motivational processes for learning the theory of probability and statistics.

Raising motivation is most often done by a historical introduction to a topic, a project approach with the integration of various school and extracurricular areas, by pointing out the practical importance and application of the predicted theory, as well as the practical implementation in real time and environment. According to Alshuler (1970), there are six stages in increasing motivation for learning:

- Drawing attention to course content
- Conducting lessons through various games that will evoke an emotional reaction to think about the issue
- Adoption of new special terms that denote different components of the subject of motivation
- The relationship of a given motive with students' value attitudes
- Practical realizations of motifs
- Independent involvement in further progress of the training subject with decreasing influence of the lecturer.

Here, the oldest board game from the ancient Mesopotamian period, dating back to 2000 BC, will be described, as well as several mathematical problems related to it that could influence the motivational potential for learning probability theory and statistics.

Original Research Article

2. The Royal game of Ur

The Royal Game of Ur is an ancient board game for two players, which appeared in Mesopotamia in the early third millennium BC. In 1926, during the excavation of the royal cemetery of the city of Ur, the English archaeologist Leonard Woolley found 4 boards (Fig. 1), which turned out to be the board for playing the oldest board game. Some time later, Irwin Finkel found a clay tablet on which the incomplete rules of the game were described. Her original name was not found, so she was given this modern name after the place where she was found.

Fig. 1. First 4 boards found by Wooley

The first complete version of the rules of the game was described by Robert Bell (1979), and Finkel (2007) wrote a separate manual especially for the shop in the British Museum (Fig. 2), where the game was sold. Other rules soon appeared, which gave the game more dynamism and uncertainty. The most disagreements were about the number of tetrahedra thrown to score points and the path the pieces take on the board. Fig. 3 shows the path proposed by Skiriuk (2021).

Fig. 2. A modern play set

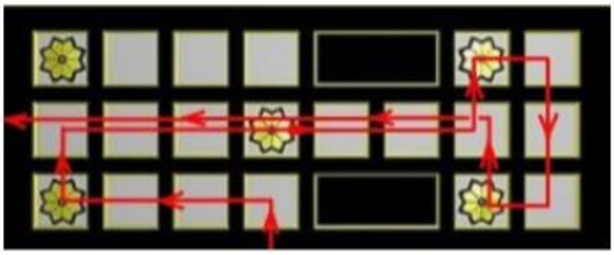


Fig. 3. The path of the figures proposed by Skiriuk

The game set next to the board contains seven figures for each player and three tetrahedrons to roll instead of dice. There is also a variant of the game in which 4 tetrahedra are thrown. The first mathematical analysis of this variant was described by Lamont (2021) and somewhat later by Raposo and Lamont (2023). The rules proposed by Skiriuk (2021) will be described here.

Figures are lozenge-shaped and different colors for players. One side is smooth, and the other has five dots. The figure is placed on the plate with the smooth side facing up, and during the return path it turns to the other side to distinguish the direction of movement. Each tetrahedron has two colored vertices, and depending on the number of fallen colored vertices, the number of fields that the figure crosses on the board is determined (Fig. 4).

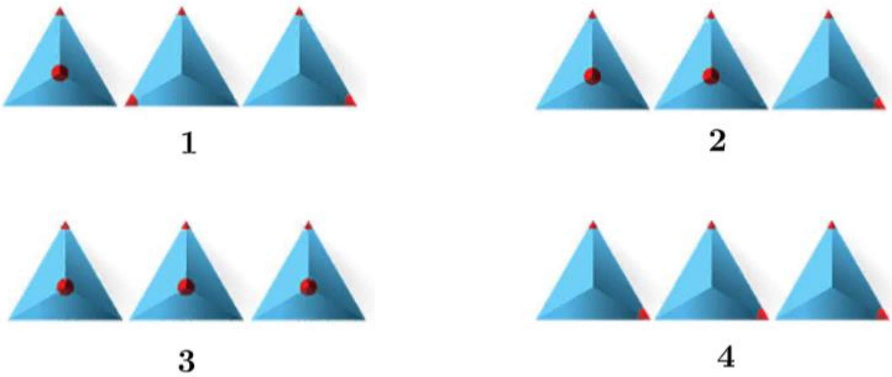
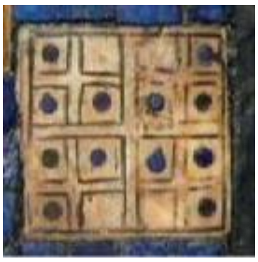


Fig. 4. Distribution of points based on the number of colored vertices of the tetrahedron

At the beginning of the game, each player throws three tetrahedrons. The advantage of the first move is given to the player who gets the least number of points. If both players get the same number of points, the roll is repeated until a different number of points is obtained. If a player gets four points during the game, he gets an additional roll to move another piece. If the player comes to the square where the opponent's piece is, he throws it off the board and the opponent starts from the beginning with it. Exceptions to this rule are fields where there can be more than one figure. If a player has nowhere to play a move with any pieces, he passes the move to the opponent. The winner is the player who first moves all the pieces across the board.

The board has 20 squares decorated with thumbnails. The dynamics of the game are increased by the rules that the player must follow when entering a certain field (Skiriuk).

Table 1. Meanings of the fields on the board

Field	Meaning	Field	Meaning
	Rosette - the player whose figure lands on this square has an additional roll for the same figure. Except if he got 4 points.		Square with eyes - a field in which only one player can place up to 4 figures.
	Square in a square – a field for turning pieces so that the returning pieces are not mixed with the original ones.		Decorative square - has no meaning other than aesthetic.
	Four squares - a field where pieces are safe and where a maximum of 4 pieces of any player can be placed.		Final square – the piece ends the game if it gets 1 point on this square, 2 on the previous one, etc.

The author of this paper described the rules and modeled this game in GeoGebra in Serbian, and it can be found and played at https://www.geogebra.org/m/aksgudv7 (Zivanovic, M. V., 2020). In this simulation, players take turns starting a move by pressing the PLAY button, represented by a small triangle in a circle in the lower corner of the desktop. This simulates the mixing of tetrahedra in a handful before throwing. Then, after a few moments, if desired, the player presses the appropriate pause button in the same place, after which he receives the result of the throw. The game then proceeds according to the rules described in Table 1. Players monitor each other's score and control the movement of pieces on the board. Instead of rotating a smooth lozenge in the "square within a square" field, the player replaces it with a lozenge with 5 dots. Therefore, 7 smooth lozenges and 7 lozenges with dots on the top side are prepared for each player. Here too, Skirjuk suggests that the "rotation" of the figure be done in the central square of the last line of the board.

Assuming that a game, especially one with historical significance, has the capacity to interest students in its mathematical dimension as well, we suggest that it be presented in an introductory probability theory class as a motivational impulse. It could also be a topic for an integrated mathematics and history project.

3. Mathematical Problems and the Royal Game Ur

As every board game with dice (here with tetrahedrons) has a great potential of combinatorial problems, so it is with the Royal Game of Ur. Primarily, the problems of number disassembly: *in how many ways can a certain space of the board be reached with a given number of moves*. Or layout problems: *how many ways a given number of pieces can be placed on the board or some prominent board spaces*. Here we will analyze several probability problems inspired by the described game.

1. A homogeneous tetrahedron with two colored vertices is thrown. Find the probability that the tetrahedron will fall so that its vertex is colored.

Solution: According to the classical definition, the probability of an event A in a random experiment, it is equal to the quotient of the number of favorable outcomes of that event $m(A)$ and the number of elements of the set of all outcomes E of that experiment $m(E)$, or formula

$$P(A) = \frac{m(A)}{m(E)}. \tag{1}$$

In our case it is $m(E) = 4$ and represents the number of vertices of the tetrahedron. Let *A* be the event that a tetrahedron with a colored vertex is dropped, then it is $m(A) = 2$. Applying formula (1) we get that $P(A) = \frac{2}{4} = \frac{1}{2}$.

2. In the Royal Game of Ur, three tetrahedra with two colored vertices are rolled. Determine the probability:

a) that exactly one of the fallen tetrahedra will have a colored vertex,

b) that among the fallen tetrahedrons, at most two will have a colored vertex

Solution: Let us first describe the set of all outcomes in a given random experiment. Let's agree that the record 010 means that a colored vertex fell on the second tetrahedron and an uncolored vertex on the other two. Then is the set of all outcomes $E = \{000, 100, 010, 001, 110, 101, 011, 111\}$ and we have that it is $m(E) = 8$.

a) Let *A* the event that a colored vertex falls on exactly one tetrahedron. That's when $m(A) = 3$ so it is $P(A) = \frac{3}{8}$.

b) Let it be B the event that among the fallen tetrahedra at most two have a colored vertex. By counting favorable outcomes for B we get $m(B)=7$, that is $P(B)=\frac{7}{8}$.

Note. Example under b) is a good opportunity to introduce the opposite event. Thus, the event is the opposite B of the event in the sign $\overline{B}$ that at least 3 colored vertices fell on the tetrahedra. Applying the definition of classical probability, we calculate $P(\overline{B})=\frac{1}{8}$. Actually this is an illustration of the claim that it is $P(\overline{B})=1-P(B)$.

Also this is a good example to illustrate the claim that the probability of the sum of disjoint* events is equal to the sum of their individual probabilities. Let them be B_i, $i \in \{0,1,2\}$ the events that during the throw, exactly i a tetradara with a colored top fell, then it is $B=B_0 \cup B_1 \cup B_2$. We have it $P(B)=P(B_0)+P(B_1)+P(B_2)=\frac{1}{8}+\frac{3}{8}+\frac{3}{8}=\frac{7}{8}$. These facts are used in practice when it is easier to calculate the probability of the opposite event or the probability of disjoint events into which the initial event can be broken down.

3. What is the probability that the player lands on the fifth square after two throws?

Solution: Each throw of the tetrad in any move is independent of each other, that is, the outcome of the subsequent throw does not depend on the outcome of the previous throw. Therefore, the probabilities of events in different throws are also independent. For two events A and B in some experiment we say that they are independent if $P(AB)=P(A)P(B)$. Let us denote by A_i, $i \in \{1,2,3,4\}$ the event that the player won i points in one throw, and by B that the player reached the fifth square after two throws. That's when $B=A_1A_4 \cup A_2A_3 \cup A_3A_2 \cup A_4A_1$. Therefore it is $P(B)=\frac{3}{8}\cdot\frac{1}{8}+\frac{3}{8}\cdot\frac{1}{8}+\frac{1}{8}\cdot\frac{3}{8}+\frac{1}{8}\cdot\frac{3}{8}=\frac{12}{64}=\frac{3}{16}$.

4. The advantage of the first move in the Royal Game of Ur is given to the player who gets fewer points in the roll. Find the probability:

a) that the first player will have the first move advantage after one throw

b) that the throw for the first move will be repeated

c) that the first player will have the first move advantage

Solution: For events of the number of points won, we use labels as in the previous example.

a) Let A us denote the event that the first player will have the advantage of the first move. That's when $A=A_1A_2 \cup A_1A_3 \cup A_1A_4 \cup A_2A_3 \cup A_2A_4 \cup A_3A_4$, so it is

$$P(A)=\frac{3}{8}\cdot\frac{3}{8}+\frac{3}{8}\cdot\frac{1}{8}+\frac{3}{8}\cdot\frac{1}{8}+\frac{3}{8}\cdot\frac{1}{8}+\frac{3}{8}\cdot\frac{1}{8}+\frac{1}{8}\cdot\frac{1}{8}=\frac{22}{64}=\frac{11}{32}.$$

The event that the second player will have an advantage in the first move is $A_2A_1 \cup A_3A_1 \cup A_3A_2 \cup A_4A_1 \cup A_4A_2 \cup A_4A_3$, so its probability is $\frac{11}{32}$. That is, players have equal chances for the starting move.

b) Let be B the event that the toss will be repeated: $B=A_1A_1 \cup A_2A_2 \cup A_3A_3 \cup A_4A_4$, that's when $B=A_1A_1 \cup A_2A_2 \cup A_3A_3 \cup A_4A_4$ so it is $P(B)=\frac{3}{8}\cdot\frac{3}{8}+\frac{3}{8}\cdot\frac{3}{8}+\frac{1}{8}\cdot\frac{1}{8}+\frac{1}{8}\cdot\frac{1}{8}=\frac{20}{64}=\frac{5}{16}$.

* By disjoint events we mean those that do not have common outcomes

c) First solution: Let C the event be that the toss will be repeated an infinite number of times. That's when $C = BBB...B...$ so it is $P(C) = \lim_{n\to\infty}\left(\frac{5}{16}\right)^n = 0$ Therefore it is $P(\overline{C}) = 1$ the probability that the number of rolls for the initial move will be finite. Since both players have equal chances, the probability that the first player has the first move advantage is $\frac{1}{2}$.

Second solution: According to the adopted notation, the requested event can be written as $\overline{C} = A \cup BA \cup BBA \cup ... \cup BB...BA \cup ...$ That's when

$$P(\overline{C}) = \frac{11}{32} + \frac{5}{16}\cdot\frac{11}{32} + \left(\frac{5}{16}\right)^2\cdot\frac{11}{32} + ... + \left(\frac{5}{16}\right)^n\cdot\left(\frac{11}{32}\right) + ... = \frac{11}{32}\left[1 + \frac{5}{16} + ... + \left(\frac{5}{16}\right)^n + ...\right]$$

$$P(\overline{C}) = \frac{11}{32}\cdot\frac{1}{1-\frac{5}{16}} = \frac{11}{32}\cdot\frac{1}{\frac{11}{16}} = \frac{16}{32} = \frac{1}{2}$$

5. Let the random variable X represent the number of points in one roll of the Royal Game Ur. Determine:

a) the distribution law of the random variable X

b) expectation of random variable X

c) the dispersion of the random variable X

Solution: a) Based on the previous results, we write down the distribution law of the random variable $X : \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$.

b) The mathematical expectation of a random variable is equal to the sum of the products of all values of the random variable and their corresponding probabilities $E(X) = 1\cdot\frac{3}{8} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{8} = \frac{16}{8} = 2$

c) We calculate the dispersion according to the formula $\sigma^2(X) = E(X^2) - E^2(X)$

$$E(X^2) = 1\cdot\frac{3}{8} + 4\cdot\frac{3}{8} + 9\cdot\frac{1}{8} + 16\cdot\frac{1}{8} = \frac{40}{8} = 5$$

$$\sigma^2(X) = 5 - 4 = 1.$$

4. Conclusion

As mentioned, the game has several variants, which certainly affects the different results of its mathematical interpretations. The first mathematical analysis of the variant played with 4 tetrahedra was described by Lemont. This text describes a variant of the game with 3 tetrahedra according to the rules of Skirjuk. An electronic version of the game simulated by the author in Geogebra is also presented. The selected examples of mathematical problems are representative of the scope and level of probability theory in high schools. The concept of increasing motivation is based on Alshuler's proposal of stages of the same. This work also has the ambition to promote the Royal Game of Ur and its potential for mathematical research that would increase students' motivation for learning. This achieves the popularization of mathematics teaching as one of the important factors in its improvement and modernization. In the future, research could be conducted with this goal during regular classes.

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