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Exploring Student Understanding of Limits and Derivatives Through Examples and Pedagogical Commentary

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Abstract

This paper explores common misconceptions and cognitive obstacles that students face when learning fundamental concepts of calculus, with particular focus on limits, derivatives, and their interpretations. Through carefully selected classroom examples and real student responses, we examine how learners often prioritize symbolic manipulation over conceptual understanding. Special attention is given to the geometric interpretation of the derivative, the importance of continuity, and the distinction between average and instantaneous rates of change. Using classical problems, we highlight both the strengths and gaps in students' reasoning. The goal is not only to identify these difficulties, but also to propose strategies for fostering deeper mathematical thinking (emphasizing active engagement, graphical reasoning, and real-world contexts). We argue that meaningful learning in calculus emerges when students are encouraged to discover, justify, and reflect rather than memorize and reproduce procedures.

Keywords: Calculus Education, Conceptual Understanding, Derivative, Tangent Line Misconceptions, Student Misconceptions.

MSC2020: 97I40, 97D70

1. Introduction

Students' understanding of core calculus concepts such as limits, derivatives, and tangent lines is usually incoherent, grounded more in procedural steps than in meaningful insight. Numerous studies have documented persistent misconceptions that hinder the transition from mechanical manipulation to deeper mathematical reasoning. For example, students frequently confuse the derivative with the average rate of change, use the limit definition solely as a mechanical calculation, or misinterpret the tangent line as any line touching a curve at a point, without understanding the underlying limiting process (Bivens, 1986; Meel, 1998; Murphy, 2000; Zandieh, 2000).

As Meel (1998) demonstrated in a comparative study between traditional and technology-enhanced calculus courses, a narrow focus on procedures often undermines conceptual understanding. Bivens (1986) likewise found that students tend to overlook the dynamic and locally linear nature of the derivative. These findings align with broader calls by authors such as Thurston (1994) and Zandieh (2000), who emphasize the need to coordinate multiple representations – symbolic, graphical, verbal, and physical – to support meaningful mathematical learning. More recent contributions further underscore the value of dynamic geometry environments and locally linear reasoning in helping students understand the concept of the derivative (Santos-Trigo et al., 2024; Trigueros, 2023).

Original Research Article

Beyond conceptual difficulties, research highlights the importance of language in mathematics instruction. Mathematics is a language in itself – precise, structured, and symbolic, but often separated from its true meaning within classroom communication. As Kelley and Carifio (2001) argue, mistakes in calculus often arise not from insufficient knowledge but from incorrect reasoning approaches, worsened by unclear communication. Similar findings emerge from research on students' reasoning about slope and rate of change (Tyne, 2016). Mkhatshwa (2024a, 2024b) underlines that language and representation are fundamental to fostering understanding, particularly in problem-based learning environments.

In this context, the mathematics classroom must be more than a site of knowledge transmission. It should become a space for inquiry, dialogue, and exploration. Using dialogue as the foundation of instruction, with carefully formulated questions and a focus on students' reasoning, creates an effective pathway to develop conceptual understanding. It transforms passive reception into active construction of meaning. When students articulate their thinking, defend their answers, and revise their reasoning, they are not only learning mathematics; they are doing mathematics.

Therefore, effective teaching should balance symbolic accuracy with deep conceptual understanding. Placing too much emphasis on either formalism or intuition results in a flawed understanding. The challenge lies in harmonizing these components through tasks that are not only solvable but intellectually rich: problems that provoke reflection, reveal misconceptions, and lead to generalization. This pedagogical vision aligns with Descartes's assertion that mathematical maturity comes not from memorizing proofs, but from training the mind to solve problems independently.

In this paper, we explore how students approach foundational calculus ideas – limits, derivatives, and tangent lines – through real classroom examples and problem-based tasks. By analyzing typical errors and difficulties in understanding, we seek to reveal the thinking and language challenges inherent in learning differential calculus. Grounded in educational theory and classroom experience, this work offers methodological reflections and concrete examples that illustrate how carefully designed tasks can foster mathematical thinking, connect representations, and encourage persistent understanding.

2. The foundational role of limits in the conceptual structure of calculus

A rigorous understanding of the concept of a limit is foundational to the development of two central ideas in calculus: the derivative and the definite integral. The derivative of a function at a point is defined as the limit of the average rate of change over an interval that becomes infinitesimally small, while the definite integral is formally the limit of Riemann sums as the partition becomes increasingly fine. In both cases, the concept of a limit provides the mathematical foundation that gives precise meaning to intuitive notions of instantaneous change and accumulation. Without a strong conceptual grasp of limits, these key calculus constructs risk remaining simple procedures rather than deeply comprehended concepts.

As Schoenfeld (1992) emphasizes, mathematics involves more than applying learned algorithms to familiar problems; it requires engaging with mathematical structures and ideas through abstraction, generalization, and the articulation and testing of conjectures. In this spirit, Halmos famously remarked that the heart of mathematics consists of concrete examples and concrete problems, while Einstein insisted that the important thing is not to stop questioning. These insights emphasize the need for instructional approaches that combine formal precision with the development of conceptual understanding.

The notion of a limit exemplifies the gap that often arises between procedural fluency and conceptual understanding. When asked to explain the meaning of the notation $\lim_{x \to a} f(x) = b$ in their own words, many students struggle to provide an intuitive or graphical interpretation, even though they may confidently reproduce the formal definition:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathcal{D}_f)(0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon).$$

This disconnect suggests that students often learn formal definitions by rote, without truly understanding the underlying ideas. When learners cannot articulate the meaning of a limit in natural language or connect it to graphical and numerical representations, the symbolic expression risks becoming superficial and devoid of meaning.

This issue is not only cognitive but fundamentally pedagogical. It highlights the importance of instructional strategies that explicitly bridge formalism with intuition. In particular, rich classroom dialogue and instruction that moves from intuitive reasoning to formalization, rather than the reverse, can help learners construct lasting and meaningful understanding. Instead of starting with abstract definitions, instruction should move through multiple representations: numerical tables, graphical trends, symbolic manipulation, and verbal explanation.

Carefully chosen questions can support this process, such as:

1. What does it mean for a function to have a limit at a point?
2. Can a function have a limit at a point where it is not defined?
3. If two functions agree near a point, can they still have different types of discontinuities at that point?

These types of open-ended questions encourage students to reflect, explore, engage in mathematical reasoning, and articulate their thinking. They invite students into authentic mathematical reasoning and promote ownership of learning. By valuing students' voices and allowing room for inquiry, we create space for developing mathematical agency. Let us now consider an illustrative example that highlights different types of discontinuities and clarifies the limiting behavior of functions near a point.

Example 3.1. Consider the rational function $f(x) = \frac{x^2-1}{x-1} = x + 1, x \neq 1$, and define two related functions:

$$g(x) = \begin{cases} x+1, & x \neq 1, \\ 1, & x = 1, \end{cases} \qquad h(x) = x + 1.$$

While all three functions share identical behavior in a neighborhood of $x = 1$, their values at $x = 1$ distinguish them:

1. f is undefined at $x = 1$;
2. g has a removable discontinuity at $x = 1$;
3. h is continuous at $x = 1$.

The table of functions' values when x is approaching 1 from both sides illustrates the convergence of outputs:

Table 1. Behavior of $f(x)$, $g(x)$ and $h(x)$ near $x = 1$

x	$f(x), g(x), h(x)$
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.9999

x	$f(x), g(x), h(x)$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001

The values in Table 1 confirm that

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x) = \lim_{x\to 1} h(x) = 2,$$

regardless of whether the function is defined at that point. This reinforces a central insight: the limit concerns behavior near a point, not the value at that point.

This example illustrates a key distinction: the existence of a limit at a point depends exclusively on the behavior of the function arbitrarily close to that point, regardless of whether the function is defined at that point. Such examples help to concretize the notion of a limit and assist students in distinguishing between the concepts of limit and continuity.

Teaching methods begin with intuitive reasoning, such as examining numerical data or interpreting graphs, enabling students to formulate conjectures before formal definitions are introduced. These definitions should be introduced only after an intuitive foundation has been established, allowing symbols to acquire meaning through students' own reasoning. When students are later introduced to the epsilon-delta definition, they are more likely to view it not as an abstract symbolic construct, but as a formal expression of ideas they have already explored.

Asking students to express the meaning of a limit in their own words should be viewed not as a supplementary activity but as an essential part of teaching. The ability to move fluently between formal, graphical, numerical, and verbal representations characterizes a deep and meaningful understanding of mathematics. Supporting the growth of this ability is fundamental to building reasoning skills, improving mathematical discourse, and promoting overall mathematical maturity.

3. Conceptualizing derivatives

The derivative of a real-valued function $f\colon E \to \mathbb{R}$ at a point $x_0 \in E$ is typically introduced via the limit expression:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

While students are often able to recall and reproduce this formal definition, their conceptual understanding of the derivative as a measure of local variation or instantaneous rate of change is frequently limited or imprecise.

A particularly revealing example involves the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

When asked to compute $f'(0)$, students often hastily assert that the derivative is zero, citing the fact that $f(0) = 0$, without engaging in any calculation or considering how the function behaves in a neighborhood of zero. This behavior is illustrated in Figure 1.

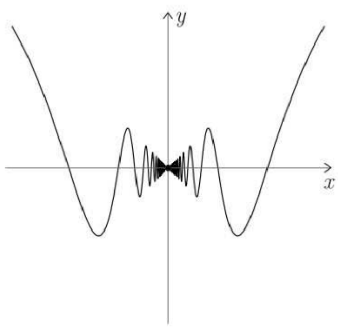

Fig. 1. Continuity but not differentiability at the origin due to infinite oscillations

The function is not differentiable at $x_0 = 0$, because the limit defining the derivative does not exist. The error in the students' reasoning lies in equating the function's value at a point with its rate of change near that point. They neglect to analyze the variation of the function in a neighborhood of zero, incorrectly assuming that since the function "flattens" at the origin (i.e., $f(0) = 0$), it must also have zero derivative there. This misconception highlights a fundamental confusion between value and behavior of a function, a confusion that must be actively addressed in calculus instruction.

Such reasoning overlooks the fundamental idea that the derivative expresses local behavior, not pointwise value. In contrast, when asked to compute the derivative of the simpler function $f(x) = x$ at $x_0 = 0$, students typically recognize that the derivative is 1, as the function increases linearly and predictably in every neighborhood of 0. This comparison highlights a crucial conceptual distinction: the derivative is determined not by the function's value at a point, but by its behavior in an arbitrarily small interval surrounding that point.

To determine whether a derivative exists at a point, one must examine how the function behaves in its immediate neighborhood. This focus on local behavior is fundamental to the concept of differentiability. This idea becomes clearer to students when they examine functions that have the same value at a point but differ significantly in their local behavior.

In contrast to limits, which concern only the behavior of a function arbitrarily close to a point, regardless of whether the function is defined there, differentiability imposes stricter conditions. A function must not only approach a predictable form near the point but also be defined and continuous at that point. Without continuity, differentiability is not just undefined; it is conceptually meaningless. A function cannot have a finite derivative at a point where it is discontinuous. However, examples such as $f(x) = \operatorname{sgn} x$, which exhibits a jump discontinuity at $x_0 = 0$, show that although no finite derivative exists there, the sudden break in the graph can still evoke an intuitive impression of an "infinite" or instantaneous rate of change, a notion that lies beyond the bounds of classical differentiability.

This hierarchy of ideas (pointwise value, continuity, and differentiability) must be explicitly addressed in instruction. Targeted examples that isolate and contrast these concepts can help students move beyond memorization toward true mathematical understanding. Through carefully selected comparisons and guided classroom discussions, learners can develop a greater understanding of the fine balance between function values, continuity, and local variation and begin to grasp the subtle yet essential nature of the derivative.

3.1. Tangents and graphical misconceptions

Geometrically, students often understand tangent lines as lines that touch a curve at exactly one point, a concept usually introduced through conic sections. However, when faced with the graph of the function $f(x) = |x|$, many students develop a common misconception: they intuitively believe that there are infinitely many tangents at the point $x_0 = 0$. In particular, they tend to consider the entire set of lines lying below the graph and intersecting it only at the origin, most notably the x-axis, as tangent lines. This arises from a visual interpretation that any line 'just touching' the graph at a point qualifies as a tangent, without recognizing the stricter mathematical requirement that the tangent line corresponds to a unique limiting slope defined by the derivative.

Intuition plays an essential role in understanding abstract mathematical concepts. However, regarding the notion of the tangent line to a function at a point, intuitive reasoning alone may lead students to incorrect conclusions, particularly if they rely on visual impressions or everyday notions of 'touching' rather than the limiting process that defines tangency. This is especially problematic in cases like $f(x) = |x|$, where the function is continuous at zero but not differentiable, highlighting the need to move beyond visual heuristics and toward a more precise analytical understanding. Fig. 2 shows the graph of $f(x) = |x|$, highlighting the cusp at the origin where the derivative does not exist.

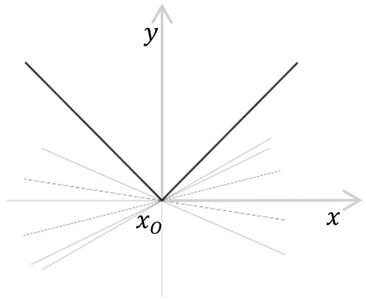

Fig. 2. Visualization of the function $f(x) = |x|$ highlighting the non-differentiability at the origin

Interestingly, these students often consider the x-axis as a tangent line at zero, yet they do not apply the same reasoning to the y-axis, which also intersects the graph of $f(x) = |x|$ at that single point. Despite the symmetric roles of these axes at the origin, students' intuitive notion fails to treat the y-axis as a tangent. This inconsistency reveals that their conceptual model of tangents is not fully developed; it is based more on visual familiarity and less on the precise definition involving local linearity and uniqueness of the tangent.

From the formal calculus perspective, the function $f(x) = |x|$ is continuous at zero, but not differentiable there because the left-hand and right-hand limits of the difference quotient differ: the slope approaches -1 from the left and 1 from the right. Consequently, no single tangent line exists at $x_0 = 0$. Recognizing this requires students to move beyond their initial, intuitive impressions to understand that a tangent line is defined by the unique slope given by the derivative, reflecting the function's local behavior.

To correct such misconceptions, it is helpful to guide students through graphical representations showing how secant lines approach the tangent line in differentiable cases and fail to converge to a unique line in non-differentiable cases such as $f(x) = |x|$. Encouraging students to compare $f(x) = |x|$ with smoother functions like $f(x) = x^2$ or oscillatory functions such as $f(x) = x \sin \frac{1}{x}$ near zero can highlight the role of local behavior in defining tangents. Moreover, challenging students to explain why their reasoning about the x-axis as a tangent does not apply symmetrically to the y-axis reveals gaps in their intuitive model and promotes deeper understanding.

Ultimately, this refined understanding of tangents as unique lines determined by local linear approximation is crucial for grasping the meaning of the derivative. It helps students move away from superficial, visual-based notions toward a more rigorous analytic perspective. Through targeted examples, active exploration, and focused discussion, mathematics teachers can support students in overcoming common graphical misconceptions and develop a coherent conceptual framework for differentiability and tangency.

3.2. Conceptual errors in understanding velocity

One of the key challenges in mathematics education, especially when dealing with applied concepts like derivatives, is ensuring that students not only master formal procedures, but also develop a sound conceptual understanding. A particularly illustrative example comes from classical mechanics: the motion of a falling object. Consider a stone dropped from the top of the Eiffel Tower, approximately 300 meters high, where the vertical displacement is modeled by the function $s(t) = 12t^2$, with $s(t)$ measured in meters and t in seconds (see Aleksić, 2021).

When students are asked to determine the total time the stone takes to hit the ground, or to compute the average velocity during specific intervals, such as the first or last two seconds of the fall, they usually respond correctly. These tasks align well with procedural competencies and familiar kinematic formulas. However, a striking difficulty emerges when students are asked to determine the instantaneous velocity at the moment of impact. Many of them mistakenly claim that the velocity is zero at that instant, reasoning that the stone "stops moving" when it touches the ground. Only when prompted with the additional question: Does the object really come to an immediate stop at the moment of impact?, do they begin to reconsider their understanding. They then realize that, according to the laws of physics, the object does not stop instantaneously; rather, the impact marks the end of the motion within the given model, after which new forces and physical processes come into play. This reflection encourages them to think differently and to understand that the instantaneous velocity does not describe the state after the impact, but the velocity just before the object touches the ground. This misconception highlights a fundamental confusion between a physical event and its mathematical representation.

The misconception also points to a deeper pedagogical challenge: bridging the gap between symbolic manipulation and physical interpretation. Students may correctly differentiate a function and evaluate it at a given value, but without an understanding of what the derivative signifies in context, such calculations remain disconnected from meaning. In this case, they fail to recognize that velocity at the instant of impact refers to how fast the object is moving just before hitting the ground, not the state after impact, which is governed by a different set of physical laws.

This example illustrates how crucial it is to guide students through the interpretation of mathematical results within real-life examples. The derivative is not merely a number computed through a limit; it expresses a dynamic property of change, and its meaning depends on how and where it is applied. By emphasizing that the instantaneous velocity is obtained by analyzing the behavior of the function as the input approaches a given time, rather than at or after a physical discontinuity, mathematics teachers can help students correct their misconceptions and develop a more robust understanding of motion.

Such contexts not only enhance comprehension but also demonstrate to students that mathematics is not a set of arbitrary procedures, but a language for describing and predicting real phenomena. Engaging with these subtleties helps shift student thinking from static to dynamic reasoning, a transition essential for mastering the conceptual foundations of calculus.

4. Pedagogical implications and future directions

Teaching calculus effectively requires more than presenting definitions and procedural techniques. It demands thoughtful selection and sequencing of examples, questions, and representations that guide students toward conceptual understanding. The example of a falling object, discussed earlier, reveals a common misconception among students: the belief that instantaneous velocity at the moment of impact is zero, simply because the object "stops" upon hitting the ground. This confusion between physical intuition and mathematical formalism highlights a fundamental pedagogical challenge: helping students understand and grasp the meaning of mathematical concepts beyond their symbolic representations.

When students mistake the derivative for a final state instead of a rate of change approaching a boundary point, it becomes clear that instruction must make the concept of limits and the role of local behavior more explicit. The instantaneous velocity at the moment of impact, in this context, does not describe the state after impact, as the model no longer applies beyond that point, but rather by the behavior immediately prior to impact. Emphasizing that the derivative at a point is computed as a limit from within the domain, particularly from the left in this example, helps students separate physical intuitions from mathematical reasoning.

To support student learning, several pedagogical strategies emerge as particularly effective. Graphical representation of both the function and its derivative, with key moments (such as impact) clearly marked, enables students to visualize how velocity changes continuously with time. Coupled with animations or dynamic simulations using software like GeoGebra, this approach strengthens the link between formal calculus and intuitive motion. Equally important is the use of guided classroom discourse: before providing the correct answer, mathematics teachers can invite students to debate whether the velocity at impact is zero. Such questioning promotes active engagement, surfaces misconceptions, and creates moments of cognitive conflict that encourage deeper thinking.

Another crucial instructional focus is the clarification of limits. Students often interpret derivatives as mere formulas, losing sight of their foundational definition. Reinforcing the idea that instantaneous rates of change arise from a limit process, not from evaluating a function at a single point, can shift the classroom culture from mechanical computation toward conceptual inquiry. In doing so, students begin to understand that derivatives describe behavior "just before" or "just after" a moment, not necessarily at the moment itself.

Furthermore, the inclusion of examples designed to highlight and confront misconceptions, such as assuming zero velocity at impact, can create important learning opportunities. These are not merely exercises but tools for encouraging critical thinking. When students are asked not only to compute but also to explain, justify, and challenge claims, they engage with mathematics as a language for describing the world, rather than as a collection of rules to memorize.

An effective calculus curriculum balances accessibility and challenge. It begins with concrete, physically motivated problems that ground student thinking, and gradually moves toward more abstract analytical tasks that require generalization and formal proof. Through this progression, students develop both confidence and competence, learning not only how to calculate but also how to think mathematically. Examples should be chosen not solely to test skills, but to invite reflection, expose common errors, and support the transition from intuition to abstraction. For instance, graph-based tasks that require matching a function with its derivative or second derivative develop students' ability to coordinate symbolic and visual information. Similarly, problems that ask for reasoning rather than numerical answers, such as determining whether a function is increasing or whether a conclusion follows from given assumptions, build mathematical argumentation skills.

To guide this development, mathematics teachers must be intentional in selecting and sequencing tasks. A good example in calculus is one that prompts students to:

1. move fluidly between graphical, algebraic, and verbal representations;
2. reason about limits, continuity, and change;
3. identify and correct misconceptions;
4. relate mathematical results to real-world phenomena.

Such examples do more than assess understanding; they structure it. They shape how students perceive the subject, what kinds of questions they ask, and how they connect mathematics to the world around them.

Mathematical learning becomes deeper and more meaningful when students are actively engaged in making their own discoveries. In the calculus classroom, this means fostering curiosity, promoting exploration, and providing opportunities for students to confront and resolve the complexities of mathematical change. It must be made engaging through thoughtful presentation and the purposeful use of problems that illuminate key ideas.

Ultimately, the goal of differential calculus instruction is not merely to teach students how to compute derivatives, but to help them see why these concepts matter, how they describe, explain, and predict change in both theoretical and applied contexts. When students understand derivatives not just as tools for calculating, but as tools for thinking, they begin to appreciate the beauty and power of mathematics as a discipline rooted in meaning, insight, and discovery.

5. Conclusion

If we aim to develop mathematical thinkers rather than rote performers, then we must place reasoning, communication, and exploration at the center of our teaching. The concept of limit offers a particularly rich opportunity for this shift. By integrating multiple representations, supporting students in verbalizing their understanding, and placing conceptual clarity at the center, we help students not only to learn mathematics but to become independent, critical, and creative problem-solvers.

The examples discussed in this paper, such as the motion of a falling object, reveal that many student misconceptions are not the result of insufficient procedural knowledge, but of cognitive conflict between intuitive interpretations and formal mathematical ideas. Rather than treating such errors as failures, we can use them as entry points for meaningful learning. When students are encouraged to express and justify their reasoning, especially when faced with unexpected outcomes, they engage in deeper cognitive processing and develop an understanding of how mathematical arguments are constructed.

Moreover, the conscious use of varied representations (graphical, symbolic, verbal, and dynamic) enables students to build flexible understanding and to move fluidly across different modes of thinking. Well-designed examples serve not just as exercises, but as cognitive supports, guiding learners from concrete experience to abstract generalization. Through them, students not only learn how to compute limits or derivatives but also why these concepts matter and how they model change in both mathematical and real-world contexts.

As mathematics teachers, our role is to create a learning environment that inspires curiosity and invites students to engage with mathematics as a process of discovery. By identifying common misconceptions, encouraging mathematical dialogue, and selecting tasks that balance accessibility with conceptual depth, we equip students not only to succeed in calculus but to appreciate its intellectual beauty and power. In doing so, we reaffirm that mathematics is not a static body of knowledge to be transferred, but a living discipline to be explored, one that, at its best, transforms both thought and thinker.

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