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Visual Proofs of the Inequality $\pi^e < e^\pi$

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Abstract

We provide two simple and visual geometric proofs of the $\pi^e < e^\pi$. These proofs are based on comparing areas under exponential curves and basic geometric shapes. Our approach builds on earlier visual arguments but introduces new geometric constructions that make the result more accessible for students and teachers. By highlighting these relationships visually, we hope to encourage further exploration and pedagogical use of similar inequalities involving transcendental numbers.

Keywords: Visual Proof, Exponential Inequality, Transcendental Numbers, e, π, $e^\pi > \pi^e$.

MSC2020: 26D07, 26A06

1. Introduction

The inequality $\pi^e < e^\pi$ is a beautiful and surprising fact that has caught the attention of many mathematicians. It compares two expressions involving the famous constants π and e and the result is not what most people would guess at first glance. At first, it might seem that since π is bigger than e, raising π to the power of e should give a bigger number than raising e to the power of π. However, surprisingly, the opposite is true! Nakhli (1987), Nelsen (2009), Chakraborty (2019), and many others (Mukherjee at al. 2019; Haque, 2020; Gallant, 1991; Nam, 2023; Farhadian, 2022) have given some elegant proofs of this inequality that are very visual and intuitive. In this article, we offer two new geometric proofs of the inequality using simple area arguments.

2. Main Result

Theorem: For the transcendental numbers e and π, the following inequality holds $\pi^e < e^\pi$.

Proof: Let us consider the function $\psi(x) = \frac{x}{1+x}$. Note that, $\psi(x) > 0$ for all $x > 0$. It follows that

$$0 < \int_0^{\frac{\pi}{e}-1} \frac{x}{1+x}dx = \int_0^{\frac{\pi}{e}-1} \left(1 - \frac{1}{1+x}\right) dx = \left(\frac{\pi}{e} - 1\right) - \ln\left(\frac{\pi}{e}\right) = \frac{\pi}{e} - \ln \pi$$

which leads to the conclusion that $\pi^e < e^\pi$ (Fig. 1).

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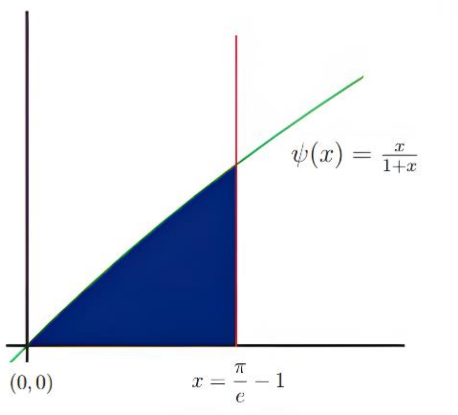

Fig. 1. Graphical representation of the area under $\psi(x) = \frac{x}{1+x}$ from $x = 0$ to $x = \frac{\pi}{e} - 1$

We now present a second proof based on comparing the area of the rectangle with the area under the graph of $\psi(x) = \frac{1}{1+x}$.

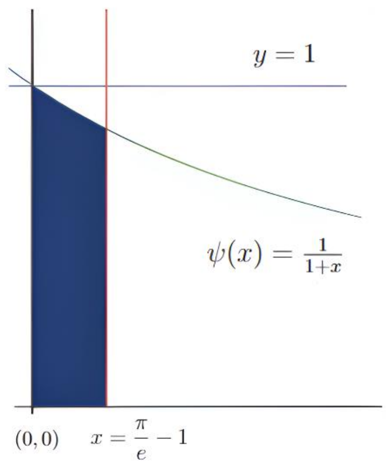

Fig. 2. The rectangular area compared to the area under $\psi(x) = \frac{1}{1+x}$ from $x = 0$ to $x = \frac{\pi}{e} - 1$

Looking at Fig. 2 it is clear that the area of the rectangle with vertices at $(0, 0)$, $(0, 1)$, $\left(\frac{\pi}{e} - 1, 1\right)$, $\left(\frac{\pi}{e} - 1, 0\right)$ is greater than the area under the curve $\psi(x) = \frac{1}{1+x}$ from $x = 0$ to $x = \frac{\pi}{e} - 1$. Therefore

$$\frac{\pi}{e} - 1 > \int_0^{\frac{\pi}{e} - 1} \frac{1}{1 + x} dx = \ln\left(\frac{\pi}{e}\right) = \ln \pi - 1$$

thus, it follows that $\pi^e < e^\pi$. □

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