Journal of Educational Studies in
Mathematics and Computer Science

Proof Without Words: The Sum of Cubes of Natural Numbers

Nenad Stojanović
Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, Serbia
nenad.stojanovic@pmf.kg.ac.rs

Received: 25 August 2025; **Accepted:** 19 November 2025

Abstract

We present a proof without words for the classical identity concerning the sum of the cubes of the first n natural numbers. The argument employs geometric visualization to reveal the underlying structure of the formula, offering both clarity and aesthetic appeal. Beyond confirming the equality, the proof highlights the pedagogical value of visual reasoning, which has been widely applied to identities, inequalities, and series convergence in mathematics.

Keywords: proof without words, visual reasoning, sum of cubes, mathematical visualization.

MSC2020: 11B57, 97G30, 00A66

1. Introduction

Visual reasoning has long occupied an important place in mathematical practice. Beyond purely symbolic manipulations, mathematicians have often sought to represent abstract relationships through diagrams, figures, and spatial intuition. This approach is most strikingly embodied in the so-called *proofs without words*, where a geometric or visual argument communicates the essence of a theorem in a direct and intuitive manner, without the need for lengthy symbolic derivations. Such visual proofs encourage insight, foster aesthetic appreciation, and often provide a deeper understanding of why a mathematical statement is true.

The technique of proof without words has been extensively documented and popularized in the mathematical literature. Nelsen's seminal collections *Proofs Without Words: Exercises in Visual Thinking* (Nelsen, 1993) and *Proofs Without Words II: More Exercises in Visual Thinking* (Nelsen, 2000) introduced a wide audience to the diversity and elegance of this mode of reasoning. Earlier examples can be traced throughout mathematical history, where geometric representations often preceded formal symbolic derivations. While initially treated as didactic curiosities, visual proofs are now recognized as a legitimate and powerful style of argumentation, particularly in contexts where intuition plays a central role.

Historically, proofs without words have been applied most frequently to identities and inequalities, where a figure can clearly encode the numerical relationship. For example, several authors have presented visual demonstrations of the classical inequality $\pi^e < e^\pi$ through ingenious geometric constructions (Chakraborty, 2019; Mukherjee & Chakraborty, 2019; Haque, 2020; Mandal & Lahiry, 2025; Farhadian, 2022). More recent contributions include visual proofs related to Heron's formula (Levrie, 2025), Pollock's conjecture on centered nonagonal numbers (Kureš, 2024), Catalan's identity for Fibonacci numbers (Heo, 2024), as well as combinatorial interpretations such as the inversion number of binary words (Kiran, 2024). Proofs without words also naturally appear in the evaluation of series and limits, such as the summation of reciprocals of odd powers of two (Mukherjee, 2022) or the classical geometric series involving powers of one quarter (Mabry, 1999).

In addition to demonstrating specific inequalities and equalities, this genre of proof has also been used to illustrate convergence phenomena. Visual approaches to infinite series emphasize the intuitive accumulation of areas or partitions, thereby reinforcing the analytic result with geometric clarity. These visual arguments often provide both elegance and pedagogical effectiveness, making them especially appealing in teaching contexts. Works by authors such as Stojanović (2021) and Jakšetić (2021) illustrate how seemingly abstract numerical relationships can be illuminated through purely visual devices.

The present manuscript situates itself within this tradition by focusing on a classical identity: the formula for the sum of cubes of natural numbers. While the algebraic derivation of this identity is well known, its visualization through a carefully designed proof without words highlights the intrinsic harmony of number patterns. By embedding the result in a geometric representation, we seek to demonstrate not only the truth of the formula but also the aesthetic power of visual reasoning in mathematics.

2. Main Result

Before introducing our visual construction, we note that the formal algebraic proof of the formula for the sum of cubes of natural numbers is classical and readily available in standard textbooks. Since this general proof is well known, we do not reproduce it here. Instead, we illustrate the identity using the specific case $n = 5$, which serves as a concrete example. The visual argument presented below applies conceptually to any natural number n, but the accompanying figure depicts the configuration explicitly for $n = 5$, both for clarity and pedagogical emphasis.

Theorem. For every positive integer n, the sum of the cubes of the first n natural numbers is given by

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Proof. Let $S = 1^3 + 2^3 + 3^3 + \cdots + n^3$. Each term in the sum $S = 1^3 + 2^3 + 3^3 + \cdots + n^3$ is represented as a cube of volume k^3, modeled visually as a solid composed of exactly k^3 unit cubes.

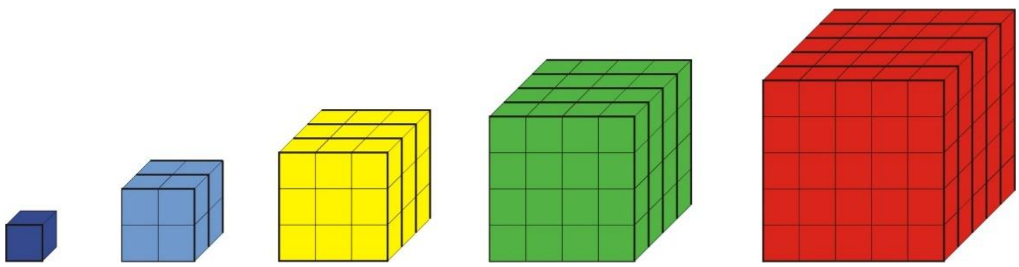

Fig. 1. $S = 1^3 + 2^3 + 3^3 + \cdots + n^3$

The cubes shown in Fig. 1 can be rearranged to obtain the shape displayed in Fig. 2.

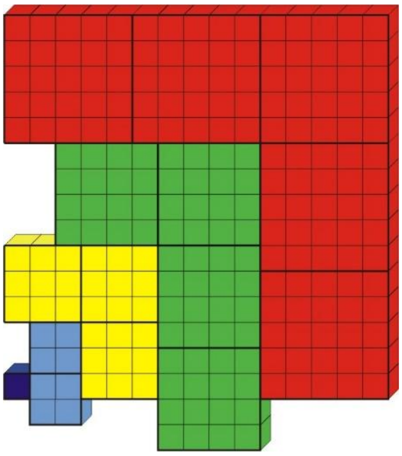

Fig. 2. The figure S obtained by rearranging the cubes

By combining the four identical figures S shown in Fig. 2, and placing them together by means of appropriate rotations, one obtains the configuration displayed in Fig. 3. In this arrangement, the four congruent copies of S fit together to form a rectangular solid whose base is a square of area $n(n+1)$ and whose height is 1.

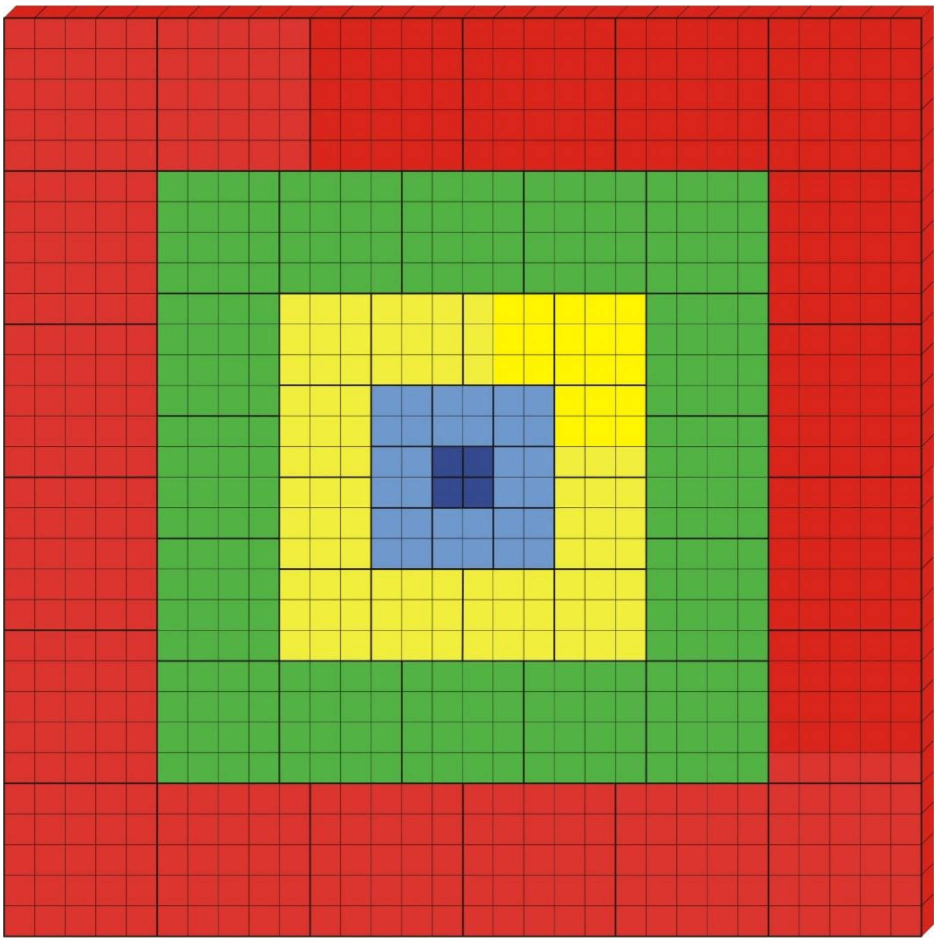

Fig. 3. The four figures S together form a square

Hence, we have $4S = \left(n(n+1)\right)^2$, which implies $S = \frac{\left(n(n+1)\right)^2}{4}$.

Finally, we conclude that

$$S = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2. \quad \blacksquare$$

3. Conclusion

The visual argument presented in this paper offers a transparent and aesthetically appealing demonstration of the classical identity

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

By reorganizing cubic blocks into a geometric configuration whose structure directly encodes the value of the sum, the proof reveals why the identity holds, rather than merely verifying that it does. This intuitive perspective highlights the power of visual reasoning in uncovering numerical patterns that might otherwise remain hidden behind symbolic manipulation. Beyond establishing the formula itself, the construction illustrates the broader pedagogical strengths of proofs without words. Such proofs allow students and readers to grasp mathematical relationships through spatial insight, promote conceptual understanding, and encourage independent exploration. As demonstrated here, geometric intuition can provide a natural bridge between discrete combinatorial expressions and continuous geometric representations. We hope that this visual proof will contribute to the growing body of literature advocating for diagrammatic reasoning in mathematics and will serve as a useful resource for teaching, learning, and appreciating the elegance inherent in classical identities.

Acknowledgment

This work was supported by the Serbian Ministry of Education, Science and Technological Development (Agreement No. 451-03-137/2025-03/ 200122).

References

CHAKRABORTY, B. (2019). A visual proof that $\pi^e < e^\pi$. *Mathematical Intelligencer*, 41, 56. https://doi.org/10.1007/s00283-018-9816-4

FARHADIAN, R. (2022). A generalized form of a visual proof of $\pi^e < e^\pi$. *Mathematical Intelligencer*, 44, 191. https://doi.org/10.1007/s00283-021-10161-y

HAQUE, N. (2020). A visual proof that $e < A \Rightarrow e^A > A^e$. *Mathematical Intelligencer*, 42, 74. https://doi.org/10.1007/s00283-019-09964-x

HEO, N. G. (2024). Visual proof of Catalan's identity for the Fibonacci numbers. *Mathematical Intelligencer*, 46, 331. https://doi.org/10.1007/s00283-024-10340-7

JAKŠETIĆ, J. (2021). Proof without words: From floor to stairs. *Mathematical Intelligencer*, 43, 29. https://doi.org/10.1007/s00283-021-10111-8

KIRAN, N. U. (2024). Proof without words: Inversion number of binary words. *Mathematical Intelligencer*, 46, 133–134. https://doi.org/10.1007/s00283-023-10276-4

KUREŠ, M. (2024). A proof of Pollock's conjecture on centered nonagonal numbers. *Mathematical Intelligencer*, 46, 234–235. https://doi.org/10.1007/s00283-023-10307-0

LEVRIE, P. (2025). A straightforward proof of Heron's formula. *Mathematical Intelligencer*, 47, 35–36. https://doi.org/10.1007/s00283-024-10383-w

MABRY, R. (1999). Proof without words: $\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}$. *Mathematics Magazine*, 72(1), 63.

MANDAL, S., & LAHIRY, S. (2025). Visual proofs of the inequality $\pi^e < e^\pi$. *Journal of Educational Studies in Mathematics and Computer Science*, 2(1), 15–17. https://doi.org/10.5937/JESMAC2501015M

MUKHERJEE, R. (2022). Visual evaluation of a geometric series: Sum of reciprocals of odd powers of two. *Mathematical Intelligencer*, 44, 44–45. https://doi.org/10.1007/s00283-021-10090-w

MUKHERJEE, A., & CHAKRABORTY, B. (2019). Yet another visual proof that $\pi^e < e^\pi$. *Mathematical Intelligencer*, 41, 60. https://doi.org/10.1007/s00283-018-09867-3

NELSEN, R. B. (1993). *Proofs without words: Exercises in visual thinking*. Washington, DC: The Mathematical Association of America.

NELSEN, R. B. (2000). *Proofs without words II: More exercises in visual thinking*. Washington, DC: The Mathematical Association of America.

STOJANOVIĆ, N. (2021). Proof without words: Inequality for the sum of the diagonals. *Mathematical Intelligencer*, 43, 32. https://doi.org/10.1007/s00283-021-10075-9