



A new didactical approach to determinants

Aleksandar Savić^{1, a*}, Đorđe Dugošija^{1, b}

¹Faculty of Mathematics, University of Belgrade, Studentski trg 16, Belgrade, Serbia

^aaleks3rd@gmail.com, ^bdjdugosija47@gmail.com

Received: 2 June 2026; **Accepted:** 2 August 2026

Abstract

The determinant of a square matrix $A = (a_{ij})$, $i, j = 1, 2, \dots, n$ over a field F is usually introduced using an explicit formula

$$\det A = \sum_{p \in S_n} (-1)^{In(p)} a_{1p(1)} a_{2p(2)} \dots a_{np(n)},$$

where S_n is the set of permutations over $\{1, 2, \dots, n\}$ and $In(p)$ is the number of inversions in p . Proving properties of determinants requires some non-elementary combinatorial properties of permutations. This may be difficult for many students. The explicit formula is also impractical for computations, since it involves a huge number of computations. Therefore, some authors (see for instance Mirsky (1955), Hoffman and Kunze (1971), Strang (2005)) have introduced determinants using some characteristic properties. However, in teaching, the proof that there is a unique function satisfying these properties is usually omitted.

A third approach to define determinants is recursively over the order of the matrices. In this paper, we show that the recursive approach to introducing determinants based on Laplace expansion along the first column can be used to obtain all standard properties, including the generalized Laplace expansion and the Cauchy-Binet formula. This is done in an elementary way suitable for teaching. The only required prior knowledge is mathematical induction and simple properties of finite sums. This approach can be very useful in undergraduate courses.

Keywords: Determinants, Laplace expansion, Cauchy-Binet formula

MSC2020: 15A15, 97H60

1. Introduction

There are several approaches to defining determinants: explicit (using an explicit formula), axiomatic (using some characteristic properties) and recursive (induction by order of determinant using Laplace expansion along the first column). The first approach is mostly used (see, for instance, Rose (2002), Dolićanin and Dugošija (2016a)). The disadvantage of this approach is the use of combinatorics, which seems to be hard for many students. The second approach is elegant, but it is not easy to prove that the chosen properties are really characteristic (see Mirsky (1955), Hoffman and Kunze (1971), Strang (2005)). In the papers Janjić (2005) and Averse and de Simone (2012), it is shown, in a simple way, that the Laplace expansion along any column is equivalent to the explicit definition. However, in order to prove basic properties of determinants, one has to go back to the first approach. Some didactical approaches to the introduction and historical development of determinants can be found in Donevska-Todorova (2017). These approaches lead

* Corresponding author

E-mail address: aleks3rd@gmail.com

to extended lectures (with many examples without proofs, showing properties of determinants only for determinants of small order, etc.), which gives the impression that the theory of determinants is too complex.

In this paper, it will be shown that all fundamental properties of determinants can be presented in an elementary way without the use of an explicit definition and combinatorics. This facilitates students' understanding of the theory of determinants. We also give new elementary and direct proofs of the generalized Laplace expansion and the Cauchy-Binet formula simpler than those given in Ueno and Machara (2002), Janjić (2008), and Dolićanin and Dugošija (2016b).

One of the problems in teaching determinants is also complicated formulas that we, in this paper, try to make simpler.

We assume that the reader is familiar with standard notation and elementary matrix operations, including block matrices, such as could be found in Dolićanin and Dugošija (2016a). Columns of a square $n \times n$ matrix $A = (a_{ij})$ we denote by $a_{*j}, j = 1, 2, \dots, n$ and rows by $a_{i*}, i = 1, 2, \dots, n$. A submatrix $(a_{i_\alpha j_\beta}), \alpha, \beta = 1, 2, \dots, k, i_1 < i_2 < \dots < i_k, j_1 < j_2 < \dots < j_k$ of A will be marked with $A_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k}$, and the associated determinant $\det A_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k}$ denoted by $M_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k}$ (or $|A_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k}|$) is called the corresponding **minor**.

If we omit rows $a_{i*}, i \in \{i_1, i_2, \dots, i_k\}$ and columns $a_{*j}, j \in \{j_1, j_2, \dots, j_k\}$ from matrix A , there remains a square submatrix $A_{\{i_1 i_2 \dots i_k\}^c}^{\{j_1 j_2 \dots j_k\}^c}$, whose determinant will be denoted $M_{\{i_1 i_2 \dots i_k\}^c}^{\{j_1 j_2 \dots j_k\}^c}$ and called the **complementary minor** of the submatrix $A_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k}$. We also use the notation of a minor when some rows and columns of A are omitted. Omitted rows and columns will be put in the middle brackets.

2. Recursive definition of determinant

Let us define determinants recursively over the order of the matrices.

Definition 1. If $A = (a_{11})$ is a matrix of order 1, then: $\det A = a_{11}$.

If $A = (a_{ij})$ is a square matrix of order $n, n \geq 2$, then:

$$\det A = a_{11} \det A_{\{1\}^c}^{\{1\}^c} - a_{21} \det A_{\{2\}^c}^{\{1\}^c} + \dots + (-1)^{n+1} a_{n1} \det A_{\{n\}^c}^{\{1\}^c} = \sum_{i=1}^n (-1)^{i+1} M_i^1 M_{\{i\}^c}^{\{1\}^c},$$

or explicitly

$$\begin{aligned} \det A = & a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} & \dots & a_{1n} \\ a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} + \dots \\ & + (-1)^{n+1} a_{n1} \begin{vmatrix} a_{12} & a_{13} & \dots & a_{1n} \\ a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-12} & a_{n-13} & \dots & a_{n-1n} \end{vmatrix}. \end{aligned}$$

Our definition of the determinant of order n is usually called the **expansion of the determinant along the first column**. For example,

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -2 & 3 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (-1) \cdot \begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & 0 \\ -2 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = -2$$

When an arbitrary minor $M_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k}$ is expanded along the first column, we obtain

$$\begin{aligned} M_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k} &= M_{i_1}^{j_1} M_{[i_1] i_2 \dots i_k}^{j_2 \dots j_k} - M_{i_2}^{j_1} M_{i_1 [i_2] i_3 \dots i_k}^{j_2 \dots j_k} + \dots + (-1)^{k+1} M_{i_k}^{j_1} M_{i_1 i_2 \dots i_{k-1} [i_k]}^{j_2 \dots j_k} \\ &= \sum_{\alpha=1}^k (-1)^{\alpha+1} M_{i_\alpha}^{j_1} M_{i_1 \dots [i_\alpha] \dots i_k}^{j_2 \dots j_k} \end{aligned}$$

(indices in the middle brackets should be omitted).

Specially,

$$M_{\{i\}^c}^{\{j\}^c} = \begin{cases} \sum_{\alpha=2}^n (-1)^\alpha M_\alpha^1 M_{\{1,\alpha\}^c}^{\{1,j\}^c}, & i = 1, j \geq 2 \\ \sum_{\alpha:\alpha < i} (-1)^{\alpha+1} M_\alpha^1 M_{\{\alpha,i\}^c}^{\{1,j\}^c} + \sum_{\alpha:\alpha > i} (-1)^\alpha M_\alpha^1 M_{\{i,\alpha\}^c}^{\{1,j\}^c}, & i \geq 2, j \geq 2 \end{cases}$$

with the agreement that the sum over an empty set (if such one appears) is equal to zero.

Also

$$M_{\{i_1, i_2, \dots, i_k\}^c}^{(k+1) \dots n} = \sum_{m=0}^k \sum_{i_m < i < i_{m+1}} (-1)^{i+1-m} M_i^{k+1} M_{\{i_1, \dots, i_m, i, i_{m+1}, \dots, i_k\}^c}^{(k+2) \dots n}$$

where $1 < k < n-1$, $i_0 = 0$, $i_{k+1} = n$.

3. Expansion of determinant along the first two columns

If we expand complementary minors along the first column in a determinant of order greater than 1, the result is

$$\begin{aligned} \det A &= \sum_{i=1}^n (-1)^{i+1} a_{i1} \left(\sum_{j:j < i} (-1)^{j+1} a_{j2} M_{\{j,i\}^c}^{\{1,2\}^c} + \sum_{j:i < j} (-1)^j a_{j2} M_{\{i,j\}^c}^{\{1,2\}^c} \right) \\ &= \sum_{i=1}^n \sum_{j:j < i} (-1)^{i+j} a_{i1} a_{j2} M_{\{j,i\}^c}^{\{1,2\}^c} + \sum_{i=1}^n \sum_{j:j > i} (-1)^{i+j+1} a_{i1} a_{j2} M_{\{i,j\}^c}^{\{1,2\}^c} \\ &= \sum_{(i,j): j < i} (-1)^{i+j} a_{j2} a_{i1} M_{\{j,i\}^c}^{\{1,2\}^c} + \sum_{(i,j): i < j} (-1)^{i+j+1} a_{i1} a_{j2} M_{\{i,j\}^c}^{\{1,2\}^c} \\ &= \sum_{(\alpha,\beta): \alpha < \beta} (-1)^{\alpha+\beta} a_{\alpha 2} a_{\beta 1} M_{\{\alpha,\beta\}^c}^{\{1,2\}^c} + \sum_{(\alpha,\beta): \alpha < \beta} (-1)^{\alpha+\beta+1} a_{\alpha 1} a_{\beta 2} M_{\{\alpha,\beta\}^c}^{\{1,2\}^c} \\ &= \sum_{(\alpha,\beta): \alpha < \beta} (-1)^{\alpha+\beta+1} (a_{\alpha 1} a_{\beta 2} - a_{\alpha 2} a_{\beta 1}) M_{\{\alpha,\beta\}^c}^{\{1,2\}^c} \\ &= \sum_{(\alpha,\beta): \alpha < \beta} (-1)^{\alpha+\beta+1+2} M_{\alpha\beta}^{12} M_{\{\alpha,\beta\}^c}^{\{1,2\}^c}. \end{aligned}$$

The obtained result is called **the expansion of the determinant along the first two columns**.

Now, we are able to prove some basic properties of determinants.

4. Properties of determinants

Theorem 1. *Interchange of two adjacent columns results in a determinant of opposite value.*

Proof. If the first and the second columns interchange, then the sign of all minors $M_{\alpha\beta}^{12}$ is changed. Using expansion along the first two columns, we see that the value of the determinant will be opposite. In the remaining cases, we use the definition of determinant and induction. Let us assume that the property holds for determinants of lower order. Interchanging two adjacent columns in complementary minors, we again see that the value of the determinant becomes the opposite. ■

Theorem 2. *The determinant of a square matrix $A = (a_{ij}), i, j = 1, 2, \dots, n$, can be expanded along an arbitrary column:*

$$\det A = \sum_{i=1}^n (-1)^{i+j} a_{ij} M_{\{i\}^c}^{\{j\}^c} = \sum_{i=1}^n (-1)^{i+j} M_i^j M_{\{i\}^c}^{\{j\}^c}.$$

Proof. Let B be the matrix obtained when column a_{*j} becomes the first one by successive interchanges with the preceding columns. Then $\det B = (-1)^{j-1} \det A$. It follows

$$\det A = (-1)^{j-1} \det B = (-1)^{j-1} \sum_{i=1}^n (-1)^{i+1} a_{ij} M_{\{i\}^c}^{\{j\}^c} = \sum_{i=1}^n (-1)^{i+j} a_{ij} M_{\{i\}^c}^{\{j\}^c}.$$

Similarly, we can prove that expansion along arbitrary two columns $a_{*i}, i = 1, 2, \dots, n$ and $a_{*j}, j = 1, 2, \dots, n$ holds, notably

$$\det A = \sum_{\alpha, \beta: \alpha < \beta} (-1)^{\alpha+\beta+i+j} M_{\alpha\beta}^{ij} M_{\{\alpha, \beta\}^c}^{\{i, j\}^c}. \blacksquare$$

Proofs of the following corollaries are easy and will be left to readers.

Corollary 1. *If the determinant has two equal columns, then its value is zero.*

Remark. If the field F has characteristic 2, the conclusion follows from Laplace expansion along two equal columns and not from Theorem 1.

Corollary 2. *Determinant is a linear function in each column, while the remaining columns are fixed:*

$$\det(a_{*1} \cdots (\lambda a'_{*j} + \mu a''_{*j}) \cdots a_{*n}) = \lambda \det(a_{*1} \cdots a'_{*j} \cdots a_{*n}) + \mu \det(a_{*1} \cdots a''_{*j} \cdots a_{*n}), \lambda, \mu \in F.$$

Corollary 3. *Determinant remains unchanged when a scalar multiple of one column is added to another column, i.e.*

$$\det A = \det(\cdots, a_{*j} + \lambda a_{*k}, \cdots), \lambda \in F, k \in \{1, 2, \dots, n\} \setminus \{j\}.$$

Theorem 3. *The determinant of every square matrix admits Laplace expansion along the first row, namely,*

$$\det A = \sum_{j=1}^n (-1)^{j+1} M_1^j M_{\{1\}^c}^{\{j\}^c}.$$

Proof. This statement is obvious for determinants of the order 1 and 2. Let us assume it holds for determinants of the order $n - 1, n > 3$. Using expansion along the first column, the inductive hypothesis and a change of the order of summation, we obtain

$$\det A = M_1^1 M_{\{1\}^c}^{\{1\}^c} + \sum_{i=2}^n (-1)^{i+1} M_i^1 M_{1 \cdots [i] \cdots n}^{[1] 2 \cdots n}$$

$$\begin{aligned}
&= M_1^1 M_{\{1\}^c}^{\{1\}^c} + \sum_{i=2}^n (-1)^{i+1} M_i^1 \left(\sum_{j=2}^n (-1)^{j+1} M_1^j M_{\{1,i\}^c}^{\{1,j\}^c} \right) \\
&= M_1^1 M_{\{1\}^c}^{\{1\}^c} + \sum_{j=2}^n (-1)^{j+1} M_1^j \left(\sum_{i=2}^n (-1)^{i+1} M_i^1 M_{\{1,i\}^c}^{\{1,j\}^c} \right) = M_1^1 M_{\{1\}^c}^{\{1\}^c} + \sum_{j=2}^n (-1)^{j+1} M_1^j M_{\{1\}^c}^{\{j\}^c} \\
&= \sum_{j=1}^n (-1)^{j+1} M_1^j M_{\{1\}^c}^{\{j\}^c}. \blacksquare
\end{aligned}$$

Theorem 4. Let $A^T = (b_{ij})$, $b_{ij} = a_{ji}$ be the transposed matrix of matrix A . Then

$$\det A^T = \det A.$$

Proof. The statement trivially holds for determinants of the order 1. Let us assume it hold for determinants of the order $n - 1$, $n \geq 2$. Expanding along the first row and applying the inductive hypothesis on complementary minors, we obtain an expansion of $\det A$ along the first column, so the statement holds for determinants of order n . Hence, it holds in any case. $\blacksquare$

It is now easy to conclude that all statements about columns hold also for rows.

5. Generalized Laplace expansion

Theorem 5. Let $1 \leq j_1 < j_2 < \dots < j_k \leq n$. Then the determinant admits Laplace expansion along the columns $a_{*j_1}, a_{*j_2}, \dots, a_{*j_k}$, namely,

$$\det A = \sum_{i_1 < \dots < i_k} (-1)^{(i_1 + \dots + i_k) + (j_1 + \dots + j_k)} M_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k} M_{\{i_1, \dots, i_k\}^c}^{\{j_1, \dots, j_k\}^c}$$

(Sum is taken over all strictly increasing k -tuples of the set $\{1, 2, \dots, n\}$.)

Proof. At the beginning, we will prove Laplace expansion along the first k columns by induction over the number of columns, generalizing the proof of Theorem 2. For $k = 1$ the statement obviously holds. It also holds for $k = n - 1$ since the term is equal to the Laplace expansion along the column n . Let us assume that Laplace expansion holds along the first k columns, $k < n - 1$, i.e.

$$\det A = \sum_{i_1 < i_2 < \dots < i_k} (-1)^{(i_1 + \dots + i_k) + (1 + \dots + k)} M_{i_1 i_2 \dots i_k}^{12 \dots k} M_{\{i_1, \dots, i_k\}^c}^{k+1 \dots n}.$$

Let us prove that Laplace expansion holds for the first $k + 1$ columns. Putting $i_0 = 0$, $i_{m+1} = n$, then

$$M_{\{i_1, i_2, \dots, i_k\}^c}^{(k+1) \dots n} = \sum_{m=0}^k \left(\sum_{i_m < i < i_{m+1}} (-1)^{i+1-m} M_i^{k+1} M_{\{i_1, \dots, i_m, i, i_{m+1}, \dots, i_k\}^c}^{(k+2) \dots n} \right),$$

By changing the order of summation, it follows

$$\begin{aligned}
\det A &= \sum_{m=0}^k \left(\sum_{i_1 < i_2 < \dots < i_k} \sum_{i_m < i < i_{m+1}} (-1)^{(i_1 + \dots + i_k) + (1 + \dots + k) + i + 1 - m} M_i^{k+1} M_{i_1 \dots i_k}^{1 \dots k} M_{\{i_1, \dots, i_m, i, i_{m+1}, \dots, i_k\}^c}^{(k+2) \dots n} \right) \\
&= \sum_{m=0}^k \left(\sum_{i_1 < \dots < i_m < i < i_{m+1} < \dots < i_k} (-1)^{(i_1 + \dots + i_k) + (1 + \dots + k) + i + 1 - m} M_i^{k+1} M_{i_1 \dots i_k}^{1 \dots k} M_{\{i_1, \dots, i_m, i, i_{m+1}, \dots, i_k\}^c}^{(k+2) \dots n} \right).
\end{aligned}$$

Using new indices

$$\alpha_r = \begin{cases} i_r, & 1 \leq r \leq m \\ i, & r = m + 1 \\ i_{r-1}, & m + 2 \leq r \leq k + 1 \end{cases}$$

and then changing the order of summation and using the expansion of $M_{\alpha_1 \alpha_2 \dots \alpha_{k+1}}^{12 \dots (k+1)}$ along $(k + 1)^{th}$ column, we further obtain

$$\begin{aligned} &= \sum_{\alpha_1 < \alpha_2 < \dots < \alpha_{k+1}} (-1)^{(\alpha_1 + \dots + \alpha_{k+1}) + (1 + \dots + (k+1))} M_{\{\alpha_1, \dots, \alpha_{k+1}\}}^{(k+2) \dots n} \sum_{m=0}^k (-1)^{-k-m} M_{\alpha_{m+1}}^{k+1} M_{\alpha_1 \dots [\alpha_{m+1}] \dots \alpha_{k+1}}^{12 \dots k} \\ &= \sum_{\alpha_1 < \alpha_2 < \dots < \alpha_{k+1}} (-1)^{(\alpha_1 + \dots + \alpha_{k+1}) + (1 + \dots + (k+1))} M_{\{\alpha_1, \dots, \alpha_{k+1}\}}^{(k+2) \dots n} \sum_{m=0}^k (-1)^{(k+1) + (m+1)} M_{\alpha_{m+1}}^{k+1} M_{\alpha_1 \dots [\alpha_{m+1}] \dots \alpha_{k+1}}^{12 \dots k} \\ &= \sum_{\alpha_1 < \alpha_2 < \dots < \alpha_{k+1}} (-1)^{(\alpha_1 + \dots + \alpha_{k+1}) + (1 + \dots + (k+1))} M_{\alpha_1 \alpha_2 \dots \alpha_{k+1}}^{12 \dots (k+1)} M_{\{\alpha_1, \dots, \alpha_{k+1}\}}^{(k+2) \dots n}. \end{aligned}$$

Let us now prove Laplace expansion along arbitrary columns $a_{*j_1}, a_{*j_2}, \dots, a_{*j_k}, j_1 < j_2 < \dots < j_k$.

Let B be the matrix obtained from matrix A after moving column a_{j_1} to the first position, column a_{j_2} to the second, and so on, column a_{j_k} to the k^{th} position. Then

$$\det B = (-1)^{j_1-1} (-1)^{j_2-2} \dots (-1)^{j_k-k} \det A,$$

so

$$\begin{aligned} \det A &= (-1)^{j_1-1} (-1)^{j_2-2} \dots (-1)^{j_k-k} \det B \\ &= (-1)^{(j_1 + \dots + j_k) - (1 + \dots + k)} \sum_{i_1 < \dots < i_k} (-1)^{(i_1 + \dots + i_k) + (1 + \dots + k)} M_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k} M_{\{i_1, \dots, i_k\}}^{\{j_1, \dots, j_k\}^c} \\ &= \sum_{i_1 < \dots < i_k} (-1)^{(i_1 + \dots + i_k) + (j_1 + \dots + j_k)} M_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k} M_{\{i_1, \dots, i_k\}}^{\{j_1, \dots, j_k\}^c}. \blacksquare \end{aligned}$$

Since all properties formulated for determinants' columns also hold for rows, it holds the following.

Theorem 6. *Laplace expansion of the determinant along arbitrary rows $a_{i_1*}, a_{i_2*}, \dots, a_{i_k*}, 1 \leq i_1 < i_2 < \dots < i_k \leq n$, is given by:*

$$\det A = \sum_{j_1 < j_2 < \dots < j_k} (-1)^{i_1 + i_2 + \dots + i_k + j_1 + j_2 + \dots + j_k} M_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k} M_{\{i_1, i_2, \dots, i_k\}}^{\{j_1, j_2, \dots, j_k\}^c}.$$

6. Cauchy-Binet formula

We give an elementary proof of the Cauchy-Binet formula for the determinant of the product of two matrices $A_{m \times n}$ and $B_{n \times m}$.

Theorem 7. *Let A be an $m \times n$ matrix and B be an $n \times m$ matrix over a field F . Then*

$$\det(AB) = \begin{cases} \sum_{j_1 < j_2 < \dots < j_m} \det A_{12 \dots m}^{j_1 j_2 \dots j_m} \cdot \det B_{j_1 j_2 \dots j_m}^{12 \dots m}, & m < n \\ \det A \cdot \det B, & m = n \\ 0, & m > n \end{cases}.$$

Proof. Let I_n be the identity matrix of order n , and $C = \begin{pmatrix} A & 0 \\ -I & B \end{pmatrix}$. Multiplying $(m + 1)^{st}$, $(m + 2)^{nd}$, ..., $(m + n)^{th}$ row with $a_{i_1}, a_{i_2}, \dots, a_{i_n}$ and adding to the row a_{i_*} for $i = 1, 2, \dots, m$ we obtain the equality

$$\det \begin{pmatrix} A & O \\ -I_n & B \end{pmatrix} = \det \begin{pmatrix} O & AB \\ -I_n & B \end{pmatrix}.$$

Changing the order of columns and using Laplace expansion along the first m rows, we obtain

$$\begin{aligned} \det \begin{pmatrix} O & AB \\ -I_n & B \end{pmatrix} &= (-1)^{n(n+m-1)} \det \begin{pmatrix} AB & O \\ B & -I_n \end{pmatrix} = (-1)^{mn} \det A B \cdot \det(-I_n) \\ &= (-1)^{mn+n} \det A B. \end{aligned}$$

The case $m < n$. Laplace expansion along the first m rows gives:

$$\begin{aligned} \det C &= \sum_{1 \leq j_1 < \dots < j_m \leq m+n} C_{12\dots m}^{j_1 j_2 \dots j_m} (-1)^{(1+2+\dots+m)+(j_1+j_2+\dots+j_m)} C_{\{12\dots m\}^c}^{\{j_1 j_2 \dots j_m\}^c} \\ &= \sum_{1 \leq j_1 < \dots < j_m \leq n} A_{12\dots m}^{j_1 j_2 \dots j_m} (-1)^{(1+2+\dots+m)+(j_1+j_2+\dots+j_m)} C_{\{12\dots m\}^c}^{\{j_1 j_2 \dots j_m\}^c}. \end{aligned}$$

First $n - m$ columns of $C_{\{12\dots m\}^c}^{\{j_1 j_2 \dots j_m\}^c}$ are negative unit columns with elements equal to -1 in the $i_1, i_2, \dots, i_{n-m}$ row uniquely defined through conditions:

$$\{i_1, i_2, \dots, i_{n-m}, j_1, j_2, \dots, j_m\} = \{1, 2, \dots, n\}, i_1 < i_2 < \dots < i_{n-m}.$$

Using the Laplace expansion along these columns, we obtain

$$C_{\{12\dots m\}^c}^{\{j_1 j_2 \dots j_m\}^c} = (-1)^{n-m+(i_1+1)+(i_2+2)+\dots+(i_{n-m}+n-m)} B_{j_1 j_2 \dots j_m}^{12\dots m}.$$

Since

$$\begin{aligned} &(1+2+\dots+m) + (1+2+\dots+n) + (n-m) + (1+2+\dots+(n-m)) \\ &= m^2 + n^2 - mn + 2n - m \equiv m + n - mn - m \equiv n - mn \equiv n + mn \pmod{2}, \end{aligned}$$

it follows

$$\det C = (-1)^{n+mn} \sum_{j_1 < \dots < j_m} A_{12\dots m}^{j_1 j_2 \dots j_m} B_{12\dots m}^{j_1 j_2 \dots j_m}$$

and the Cauchy-Binet formula follows.

The case $m = n$. The proof is a simple modification of the former proof.

The case $m > n$. Using Laplace expansion along the first m rows, we obtain immediately $\det C = 0$ and therefore $\det AB = 0$. ■

7. Conclusion

As we have shown using elementary mathematics (mathematical induction, changing the order of summation of finite sums and changing indices), the recursive definition of determinants is well suited for undergraduate studies and is convenient for teaching determinants. Further extensions to some topics of linear algebra (Cramer's rule, rank of a matrix, Kronecker-Capelli's theorem, etc.) are natural for future consideration.

Acknowledgment

This work is partially supported by the Ministry of Science, Technology and Development, Republic of Serbia.

References

- AVERSE, V., & DE SIMONE, A. (2012). An elementary proof of Laplace's formula on determinants. *Int. J. Math. Education*, 43, 418–420.
- DONEVSKA-TODOROVA, A., (2017) Utilizing Technology to Facilitate the Transition from Secondary- to Tertiary Level Linear Algebra. PhD dissertation, Mathematisch-Naturwissenschaftlichen Fakultät der Humboldt-Universität zu Berlin
- DOLIĆANIN, D., & DUGOŠIJA, Đ. (2016). *Fundamentals of Mathematics*. Akademska misao, Belgrade.
- DOLIĆANIN, D., & DUGOŠIJA, Đ. (2016). A note on the generalized Laplace's expansion and Cauchy-Binet formula. *J. Adv. Math. Study*, 9(1), 117–120.
- HOFFMAN, K., & KUNZE, R. (1971). *Linear Algebra* (2nd ed.). Prentice Hall Inc., Englewood Cliffs, New Jersey.
- JANJIĆ, M. (2005). A note on Laplace's expansion theorem. *Int. J. Math. Education*, 35, 696–698.
- JANJIĆ, M. (2008). A proof of generalized Laplace's expansion theorem. *Bull. Soc. Math. Banja Luka*, 15, 5–7.
- MIRSKY, L. (1955). *An Introduction to Linear Algebra*. University Press Oxford, London.
- ROSE, H. (2002). *Linear Algebra. A Pure Mathematical Approach*. Birkhäuser Verlag, Basel.
- STRANG, G. (2005). *Linear Algebra and its Applications* (4th ed.). Brooks Cole, New York.
- UENO, Y., & MACHARA, K. (2002). An elementary proof of the generalized Laplace expansion formula. *Academic Rep. Fac. Eng. Tokyo Inst. Polytech.*, 25(1), 61–63.