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DYNAMIC STABILITY OF A CONICAL PIPE RESTING ON A PASTERNAK ELASTIC FOUNDATION

Research paper

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Abstract : This study employs the Generalized Differential Quadrature Method (GDQM) to analyze the stability of a fluid-conveying conical pipe, clamped at both ends and supported by a Pasternak elastic foundation. The primary objective is to assess how the foundation parameters influence the critical fluid velocity—the point at which the pipe loses stability. A numerical analysis was performed on a straight pipe with specified geometric and material properties, considering both parameters of the elastic foundation as variables. The resulting data, presented through graphical correlations between critical velocity and foundation parameters, underscore the significant influence of the Pasternak foundation on the pipe's vibrational behavior and overall stability.
Key words: Conical pipe, Fluid, Critical velocity, Dynamic stability, Pasternak elastic foundation, Generalized Differential Quadrature Method

INTRODUCTION

Fluid-conveying pipes are widely utilized across various industries, including chemical, petroleum, aviation, and nuclear sectors. Their dynamic stability has long been a subject of extensive scientific investigation.
Reference (Gaith, 2021) investigates a simply supported pipe conveying fluid, whose cross-section varies linearly along its length. The study determines the natural circular frequencies of free vibrations and the critical flow velocity.
In (Khomarian et al., 2025), fluid-conveying pipes resting on a Pasternak foundation and subjected to various boundary conditions are analyzed. The study proposes a method for identifying damage locations along the pipe by combining the finite element method with the Galerkin approach. Another recent study (Jerad et al., 2025) examines the stability of a pipe resting on a Pasternak elastic foundation. The pipe material is axially functionally graded. Hamilton's principle is employed to derive the governing differential equation for the transverse displacement of points along the pipe axis, followed by the application of the Galerkin method.
The dynamics of conical fluid-conveying pipes are examined in (Rajasekaran, 2024) and (Yan et al., 2010). In (Rajasekaran, 2024), the author investigates the free and forced oscillations of pipes with internal fluid flow. In (Yan et al., 2010), the study focuses on the oscillation characteristics of conical tubes used in heat exchangers, and the critical fluid velocity is determined.
Paper (Rahmani & Petrudi, 2020) examines the dynamics of conical pipes at the nanoscale. An analytical solution is presented for the nonlinear oscillations of a nanocomposite conical tube conveying fluid.
This research investigates a conical fluid-conveying tube supported by a Pasternak elastic foundation, employing the Generalized Differential Quadrature Method (GDQM). The study reveals the influence of foundation stiffness on the critical velocity of the internal fluid.
The structure of the paper is as follows:

- First, the pipe model is introduced, including its static configuration and the differential equation governing its natural lateral vibrations.
- Next, the GDQM is applied to solve the problem, with a detailed derivation of the system's frequency equation.
- The roots of this equation are then analyzed to assess the system's dynamic stability.
- Finally, numerical results are presented and discussed, followed by a concise summary of the main findings.

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PROBLEM FORMULATION

The differential equation governing the transverse vibrations of a straight conical pipe conveying an inviscid fluid is presented in (Yuzhen et al., 2021). When the pipe is supported by a Pasternak elastic foundation, the equation takes the following form:

$$E\frac{\partial^2}{\partial x^2}\left[I(x)\frac{\partial^2 w}{\partial x^2}\right]+\left(m_f+m_p\right)\frac{\partial^2 w}{\partial t^2}+V\frac{dm_f}{dx}\frac{\partial w}{\partial t}+2Vm_f\frac{\partial^2 w}{\partial x\,\partial t}+m_f\frac{dV}{dx}\frac{\partial w}{\partial t}+$$
$$+V^2\frac{dm_f}{dx}\frac{\partial w}{\partial x}+\left(m_fV^2-k_r\right)\frac{\partial^2 w}{\partial x^2}+\left(2m_fV\frac{dV}{dx}+k_w\right)w=0, \tag{1}$$

where E is the Young's modulus, m_f is the mass of the fluid per unit length of the pipe, m_p is the mass of the pipe per unit length, k_r and k_w are the parameters of the Pasternak elastic foundation, x is the longitudinal coordinate along the pipe axis, t is the time and l is the length of the pipe. $w(x,t)$ is the transverse displacement of the pipe's cross-section and V is the velocity of the conveyed fluid.
Figure 1 illustrates the static scheme of the pipe analyzed in this paper.

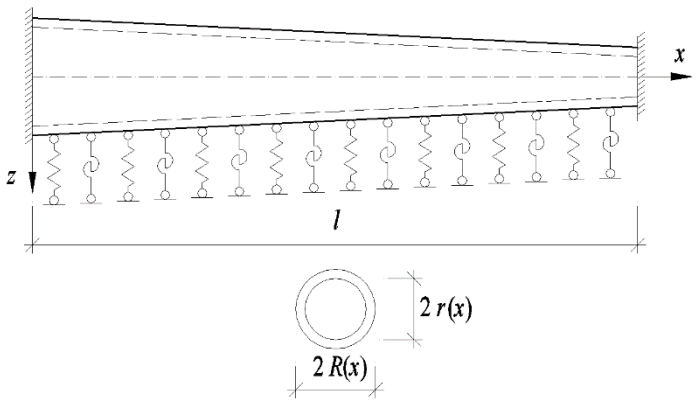


Fig. 1. *Static scheme and cross-section dimensions of the investigated pipe*

The following relation holds for the radii of an arbitrary cross-section:

$$R(x)=\left(1-\gamma\frac{x}{l}\right)R_0;\; r(x)=\left(1-\gamma\frac{x}{l}\right)r_0 \tag{2}$$

where the outer and inner radii at the inlet cross-section of the pipe are denoted by R_0 and r_0, respectively, and γ denotes the reduction coefficient.
At any given cross-section of the pipe, the fluid velocity is assumed to follow the expression below.

$$V(x)=\frac{V_0}{\left(1-\gamma\frac{x}{l}\right)^2} \tag{3}$$

with $V(0)=V_0$ representing the inlet velocity at the cross-section positioned at an abscissa of $x=0$.
At any given abscissa, the pipe's mass m_p is defined as follows.

$$m_p(x) = \left(1 - \gamma \frac{x}{l}\right)^2 \rho A(0) \tag{4}$$

Here, $A(0)$ denotes the cross-sectional area at the left end of the pipe (i.e., the section located at abscissa $x = 0$), and ρ represents the material density of the pipe.
Similarly, the fluid mass per unit length of the pipe m_f at a given position along the pipe is defined by the following relation.

$$m_f(x) = \left(1 - \gamma \frac{x}{l}\right)^2 \rho_f A_f(0) \tag{5}$$

Here, $A_f(0)$ denotes the cross-sectional area of the fluid at the left end of the pipe (i.e., the section located at abscissa $x = 0$, and ρ_f represents the density of the conveyed fluid.
The moment of inertia $I(x)$ at any cross-section of the pipe is given by the following expression.

$$I(x) = \left(1 - \gamma \frac{x}{l}\right)^4 I(0) \tag{6}$$

where $I(0)$ is the inertial moment of the cross-section at the left end of the pipe.
Differential equation (1) is solved by proposing a solution of the form:

$$w(x,t) = W(x)e^{\omega t} \tag{7}$$

In equation (7) ω denotes the circular frequency of the pipe.
Following the transformations, differential equation (1) takes the form:

$$\begin{aligned} E\frac{d^2}{dx^2}\left[I(x)\frac{d^2W}{dx^2}\right] + \left(m_f + m_p\right)\omega^2 W + V\frac{dm_f}{dx}\omega W + 2Vm_f\omega\frac{dW}{dx} + m_f\frac{dV}{dx}\omega W + \\ + V^2\frac{dm_f}{dx}\frac{dW}{dx} + \left(m_f V^2 - k_r\right)\frac{d^2W}{dx^2} + \left(2m_f V\frac{dV}{dx} + k_w\right)W = 0, \end{aligned} \tag{8}$$

Equation (8) may be reformulated as a system of algebraic equations through the application of the Generalized Differential Quadrature Method (GDQM), as presented by Soltani P. and Saadati M. (Soltani & Saadati, 2009), and Tornabene F. et al. (Tornabene et al., 2010). This method fundamentally relies on approximating the derivative of a function at any discrete point within a domain by expressing it as a weighted linear combination of the function's values at all discrete points, as described mathematically below:

$$\left.\frac{d^n W(x)}{dx^n}\right|_{x=x_i} = \sum_{j=1}^{m} \beta_{ij}^{(n)} W(x_j), \qquad i = 1, \dots, m \tag{9}$$

Here, m is the number of discrete grid points along the axis of the pipe, and $\beta_{ij}^{(n)}$ is the weighting factor for the n-th derivative evaluated at location x_i.
The Chebyshev-Gauss-Lobatto point distribution is utilized in the current work, following the approach of Tornabene F. et al. (Tornabene et al.,2010).

$$x_i = \frac{l}{2}\left[1 - cos\left(\frac{i-1}{m-1}\pi\right)\right], \qquad i = 1, \dots, m \tag{10}$$

The weighting coefficients are obtained through Lagrange interpolation functions. For the first derivative, they are calculated following the method outlined by Tornabene F. et al. (Tornabene et al., 2010):

$$\beta_{ij}^{(1)} = \frac{L^{(1)}(x_i)}{(x_i - x_j)L^{(1)}(x_j)}, \qquad i = 1, \dots, m; i \neq j \tag{11}$$

Where $L^{(1)}(x_i)$ is the first derivative of Lagrange interpolating polynomials.

$$L^{(1)}(x_k) = \prod_{l=1, l\neq k}^{m} (x_k - x_l), \qquad k = 1, \dots, m \tag{12}$$

The combination of Lagrange interpolating polynomials and Chebyshev-Gauss-Lobatto sampling points, as defined in equation (10), ensures convergence and leads to reduced error with an increasing number of sampling points.
For derivatives of higher order, the coefficients are computed through an iterative process:

$$\begin{gathered}\beta_{ij}^{(n)} = n\left(\beta_{ii}^{(n-1)}\beta_{ij}^{(1)} - \frac{\beta_{ij}^{(n-1)}}{x_i - x_j}\right), \qquad i, j = 1, \dots, m; i \neq j; n = 2, \dots, (m-1) \\ \beta_{ii}^{(n)} = -\sum_{j=1, j\neq i}^{m} \beta_{ij}^{(n)}, \qquad i, j = 1, \dots, m; n = 2, \dots, (m-1)\end{gathered} \tag{13}$$

The Generalized Differential Quadrature Method (GDQM) is applied to convert the governing equation (8) into its discrete form at specified grid points.

$$\begin{gathered}EI(x_i)\sum_{j=1}^{m}\beta_{ij}^{(4)}W(x_j) + 2E\frac{dI(x_i)}{dx}\sum_{j=1}^{m}\beta_{ij}^{(3)}W(x_j) + E\frac{d^2I(x_i)}{dx^2}\sum_{j=1}^{m}\beta_{ij}^{(2)}W(x_j) + \\ +\left[m_f(x_i) + m_p(x_i)\right]\omega^2 W(x_i) + V(x_i)\frac{dm_f(x_i)}{dx}\omega W(x_i) + \\ +2V(x_i)m_f(x_i)\omega\sum_{j=1}^{m}\beta_{ij}^{(1)}W(x_j) + m_f(x_i)\frac{dV(x_i)}{dx}\omega W(x_i) + \\ +V^2(x_i)\frac{dm_f(x_i)}{dx}\sum_{j=1}^{m}\beta_{ij}^{(1)}W(x_j) + \left[m_f(x_i)V^2(x_i) - k_r\right]\sum_{j=1}^{m}\beta_{ij}^{(2)}W(x_j) + \\ +\left(2m_f(x_i)V(x_i)\frac{dV(x_i)}{dx} + k_w\right)W(x_i) = 0,\end{gathered} \tag{14}$$

The boundary conditions at the left and right ends of the pipe (Fig. 1) are defined as follows.

$$\begin{gathered}w(0,t) = 0;\ \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0 \\ w(l,t) = 0;\ \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=l} = 0\end{gathered} \tag{15}$$

Equation (16) can be expressed in matrix form as:

$$\left[B^{(4)} + B^{(3)} + B^{(2)} + B^{(1)}\right]\delta + I\delta_d = 0 \tag{16}$$

where the matrix components in Eq. (16) are given by:

$$\begin{gathered} B_{ij}^{(4)} = EI(x_i)\beta_{ij}^{(4)}; \quad B_{ij}^{(3)} = 2E\frac{dI(x_i)}{dx}\beta_{ij}^{(3)}; \\ B_{ij}^{(2)} = \left[E\frac{d^2I(x_i)}{dx^2} + m_f(x_i)V^2(x_i) - k_r\right]\beta_{ij}^{(2)}; \\ B_{ij}^{(1)} = \left[2V(x_i)m_f(x_i)\omega + V^2(x_i)\frac{dm_f(x_i)}{dx}\right]\beta_{ij}^{(1)}; \\ I_{ij} = \left[m_f(x_i) + m_p(x_i)\right]\omega^2 + V(x_i)\frac{dm_f(x_i)}{dx}\omega + m_f(x_i)\frac{dV(x_i)}{dx}\omega + \\ + \left(2m_f(x_i)V(x_i)\frac{dV(x_i)}{dx} + k_w\right); \\ i = 3,4,\dots,(m-2), \quad j = 1,\dots,m. \end{gathered} \tag{17}$$

Here, δ and δ_d are vectors of unknown displacements at the grid points, defined as:

$$\begin{gathered} \delta = \{W(x_1),\dots,W(x_m)\}^T \\ \delta_d = \{W(x_3),\dots,W(x_{m-2})\}^T \end{gathered} \tag{18}$$

Matrix representation is also applied to the four boundary conditions outlined in equation (15).

$$K_b\delta = 0 \tag{19}$$

where

$$K_b = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ \beta_{11}^{(1)} & \beta_{12}^{(1)} & \beta_{13}^{(1)} & \dots & \beta_{1m}^{(1)} \\ 0 & 0 & 0 & \dots & 1 \\ \beta_{m1}^{(1)} & \beta_{m2}^{(1)} & \beta_{m3}^{(1)} & \dots & \beta_{mm}^{(1)} \end{vmatrix} \tag{20}$$

The discrete formulation in equation (16), combined with the boundary conditions in (19), yields algebraic equations for the unknown nodal displacements, expressed as:

$$\begin{vmatrix} K_b \\ B^{(4)} + B^{(3)} + B^{(2)} + B^{(1)} \end{vmatrix}\delta + \begin{vmatrix} 0 \\ I\delta_d \end{vmatrix} = 0 \tag{21}$$

After transformations, Eq. (21) takes the following form:

$$\begin{vmatrix} K_{bb} & K_{bd} \\ B_{db}^{(4)} + B_{db}^{(2)} + B_{db}^{(1)} & B_{db}^{(4)} + B_{db}^{(2)} + B_{db}^{(1)} \end{vmatrix}\begin{vmatrix} \delta_b \\ \delta_d \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & I \end{vmatrix}\begin{vmatrix} \delta_b \\ \delta_d \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \tag{22}$$

where:

$$\delta_b = \{W(x_1), W(x_2), W(x_{m-1}), W(x_m)\}^T \tag{23}$$

$$K_{bb} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ \beta_{11}^{(1)} & \beta_{12}^{(1)} & \beta_{1(m+1)}^{(1)} & \beta_{1m}^{(1)} \\ 0 & 0 & 0 & 1 \\ \beta_{m1}^{(1)} & \beta_{m2}^{(1)} & \beta_{m(m+1)}^{(1)} & \beta_{mm}^{(1)} \end{vmatrix} \tag{24}$$

$$K_{bd} = \begin{vmatrix} 0 & 0 & 0 & \dots & 0 \\ \beta_{13}^{(1)} & \beta_{14}^{(1)} & \beta_{15}^{(1)} & \dots & \beta_{1(m-2)}^{(1)} \\ 0 & 0 & 0 & \dots & 0 \\ \beta_{m3}^{(1)} & \beta_{m4}^{(1)} & \beta_{m5}^{(1)} & \dots & \beta_{m(m-2)}^{(1)} \end{vmatrix} \tag{25}$$

$$B_{db}^{(4)} = \begin{vmatrix} B_{31}^{(4)} & B_{32}^{(4)} & B_{3(m-1)}^{(4)} & B_{3m}^{(4)} \\ B_{41}^{(4)} & B_{42}^{(4)} & B_{4(m-1)}^{(4)} & B_{4m}^{(4)} \\ \dots & \dots & \dots & \dots \\ B_{(m-2)1}^{(4)} & B_{(m-2)2}^{(4)} & B_{(m-2)(m-1)}^{(4)} & B_{(m-2)m}^{(4)} \end{vmatrix} \tag{26}$$

$$B_{dd}^{(4)} = \begin{vmatrix} B_{33}^{(4)} & B_{34}^{(4)} & \dots & B_{3(m-2)}^{(4)} \\ B_{43}^{(4)} & B_{44}^{(4)} & \dots & B_{4(m-2)}^{(4)} \\ \dots & \dots & \dots & \dots \\ B_{(m-2)3}^{(4)} & B_{(m-2)4}^{(4)} & \dots & B_{(m-2)(m-2)}^{(4)} \end{vmatrix} \tag{27}$$

It is taken into account that:

$$\delta = -K_{bb}^{-1} K_{bd} \delta_d \tag{28}$$

As a result, the following system of $(m-4)$ equations with $(m-4)$ unknowns is obtained.

$$\left(B_{dd}^{(4)} - B_{db}^{(4)} K_{bb}^{-1} K_{bd}\right)\delta_d + \left(B_{dd}^{(2)} - B_{db}^{(2)} K_{bb}^{-1} K_{bd}\right)\delta_d + \left(B_{dd}^{(1)} - B_{db}^{(1)} K_{bb}^{-1} K_{bd}\right)\delta_d + I\delta_d = 0 \tag{29}$$

For Equation (29) to have a non-trivial solution at the onset of loss of stability, the determinant of the matrix preceding the vector δ_d must equal zero. This determinant is a function of the unknown critical velocity V_0 and the unknown circular frequency ω.
The stability of the system is classified as follows:

- When $Re\omega < 0$, the system is stable.
- When $Re\omega > 0$, the system loses stability.
- If $Re\omega = 0$, the system is at the threshold of losing stability (critical state).

To determine the critical velocity, the following computational procedure is applied: a specific value is assigned to V_0, and the circular frequency ω is calculated from the condition that the determinant in Eq. (28) is zero. Based on the calculated frequency, the stability status of the system is evaluated.

RESULTS AND DISCUSSION

Numerical simulations have been conducted for the fluid-conveying pipe illustrated in Figure 1. The pipe's geometric and material properties include: the inner and outer radii of the inlet cross-section: $r = 0.022\ m; R = 0.024\ m$, the inner and outer radii of the outlet cross-section: $r = 0.012\ m; R = 0.014\ m$, Young's modulus $E = 210\ GPa$, material density $\rho_p = 7800 kg/m^3$, and the pipe length $l = 3\ m$. The analysis was performed using a specified number of sampling points $m = 21$.
For the configuration shown in Figure 1, critical inlet velocities were determined across varying parameters of the Pasternak elastic foundation, considering density of the transported fluid ($\rho_f = 1000\ kg/m^3$). The corresponding results are presented in Figure 2.

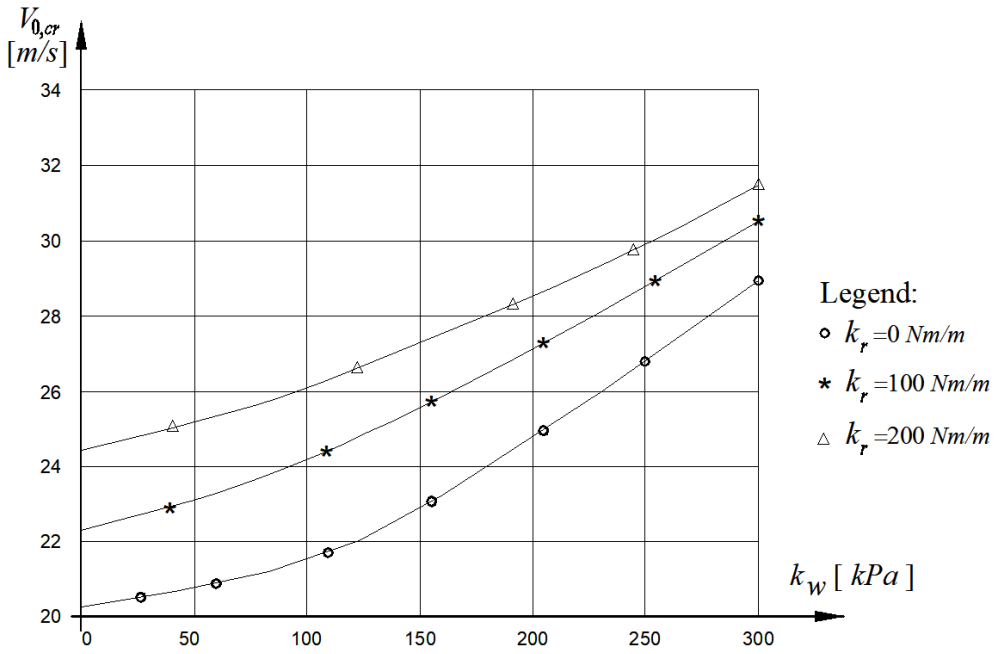


***Fig. 2**. Relationship between the critical inlet fluid velocity and the parameters of the Pasternak foundation*

CONCLUSIONS

In this paper, a conical pipe conveying fluid and resting on a Pasternak elastic foundation was investigated. The differential equation describing the lateral vibration of the pipe was solved using the Generalized Differential Quadrature Method (GDQM). A key factor in achieving a precise solution with this method is the appropriate choice of the number of sampling points. The numerical analyses presented in this article lead to several key conclusions

- The findings indicate that fluid velocity has a substantial impact on the system's dynamic behavior, which may pose risks to its structural integrity.
- Another significant observation is that increasing the parameters of the Pasternak elastic foundation leads to improved system stability.

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